Fontaine-Mazur Conjecture and Differential Operators on Modular Curves

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1 Fontaine-Mazur Conjecture

We begin with Fontaine-Mazur conjecture. This is a special case of Langlands correspondence, which predicts a correspondence between cuspidal automorphic representations for $\mathrm{GL}_n(\mathbb{A}_F)$ and n-dimensional "geometric" Galois representations of Gal_F.

Conjecture 1.1 (Fontaine-Mazur)**.** *If* $\rho : \mathrm{Gal}_{\mathbb{Q}} \to \mathrm{GL}_2(V)$ *is an odd absolutely irreducible continuous* ℓ*-adic representation which is unramified at all except finitely many places, and* $\rho|_{\mathrm{Gal}_{\mathbb{Q}_p}}$ *is de Rham of Hodge-Tate* $0, k$*, then there exists a cuspidal new form* f *of weight* $k+1$ *such that* $\rho_f \cong \rho$*.*

Remark 1.2. The conditions imposed here are all necessary. Moreover, by [Cal12], in the generic case, oddness follows from the other conditions.

The Fontaine-Mazur conjecture was first proven in [Eme11] with generic condition on residue representation $\bar{\rho}$. The proof is separated into two steps:

(1) (Promodularity / $R = T$) [Eme11, Theorem 1.2.3] Assume $\rho : \mathrm{Gal}_{\mathbb{Q}} \to \mathrm{GL}_2(V)$ is an odd absolutely irreducible continuous ℓ-adic representation which is unramified at all except finitely many places. With generic condition on (the semisimplification) of the residue representation $\bar{\rho}^{\mathrm{ss}}$, ρ will appear in the "completed cohomology" introduced in [Eme06].

(2) (Classicality) [Eme11, Corollary 1.2.2] If ρ is in addition de Rham at p with distinct Hodge-Tate weight, then ρ is attached to some f.

The idea is the following: if we consider both sides of the Langlands correspondence, i.e. cuspidal automorphic representations and geometric Galois representations, they only form a discrete set. For example, if we consider GL_2, the first one corresponds to modular forms, which forms a finite set once we fix the level and the weight. Therefore, it is hard to apply geometric method to both sides.

The idea of p-adic interpolation is to expand both sides, so that both sides form a geometric family. For the automorphic side, we consider p-adic automorphic forms, which is a much bigger space containing the classical automorphic forms. For the Galois side, we drop the condition of being de Rham at p. Both sides then form a family, which locally are represented by some affine formal schemes. Promodularity Conjecture claims that both sides are roughly the same.

The automorphic side is controled by the Hecke algebra T, and the right hand is described using Galois deformation theory R. It is know that in the generic case, we have a natural map $R \to T$, which is equivalent to the existence of Galois representations. Then the promodularity conjecture will follow if the map $R \to T$ is an isomorphism, or at least, if it has nilpotent kernel. Its proof relies on the work of Böckle, Diamond-Flach-Guo, Khare-Wintenberger and Kisin, which uses Taylor-Wiles methods and description of Galois deformation rings.

We will focus on the second step in this survey. If ρ is promodular, then we know $\rho \cong \rho_f$ for some "p-adic modular forms" f, and then we would like to know when f is classical. This step is called "classicality". Roughly, we hope that de Rham condition on ρ_f corresponds to classicality of f. This kind of result is first proven in [Kis03] for finite slope overconvergent modular forms. In [Eme11], Emerton prove a similar result for completed cohomology using the strong version of local-global compatibility proven in his paper, which involves p-adic local Langlands Correpondence ([Col10]). This is a bit unsatisfactory, because p-adic local Langlands involves heavy representation theoretical argument, is only available for $\mathrm{GL}_2(\mathbb{Q}_p)$, and becomes so much more complicated even for $\mathrm{GL}_2(\mathbb{Q}_{p^2})$

(see for example [BP12]).

In the recent of work of Lue Pan ([Pan22a],[Pan22b]), a complete new proof of classicality is given, which uses the geometry of Shimura varieties. He constructs certain differential operators on O^{la} over the (infinite level) Shimura varieties, and relates directly the "de Rhamness" to the kernel of these differential operators. In the classical work of [Col96], Coleman proves that the classicality is related to the cokernel of the theta operator θ^k. Pan's result can be taken as a vast generalization of this result. We will give more details in the following sections.

2 Infinite Level Modular Curve

Let $G := \mathrm{GL}_2$, and given an open compact subgroup $K^p \subset G(\mathbb{A}_f^p)$ and $K_p \subset G(\mathbb{Q}_p)$, we can consider (the analytification of) the compactified modular curve $\mathcal{X}_{K_pK^p}$ of level K^pK_p over $\mathbb{C}_p$, where $\mathbb{C}_p$ denote the p-adic completion of $\bar{\mathbb{Q}}_p$. We have the universal family of (generalized) elliptic curves $\mathcal{E}_{K_pK^p}$ over $\mathcal{X}_{K_pK^p}$, and denote $\omega := e^*\Omega^1_{\mathcal{E}_{K_pK^p}/\mathcal{X}_{K_pK^p}}$. We have the following Hodge-Tate sequence:

$$0 \to \omega^{-1} \to T_p\mathcal{E}_{K_pK^p} \otimes_{\mathbb{Z}_p} O_{\mathcal{X}_{K_pK^p}} \to \omega \to 0.$$

The idea of [Sch15] is that if one consider the inverse limit

$$\mathcal{X}_{K^p} \sim \varprojlim_{K_p} \mathcal{X}_{K_pK^p}$$

in a suitable sense, the *infinite level modular curve* $\mathcal{X}_{K^p}$ will turn out to be a good geometric object, i.e. a perfectoid space. The geometry of $\mathcal{X}_{K^p}$ is in some sense easier to understand than $\mathcal{X}_{K^pK_p}$. It has a so-called Hodge-Tate period map π_{HT} to the flag variety $\mathscr{F}\ell \cong \mathbb{P}^1$, and the fiber of this map is the so-called Igusa variety.

More precisely, over $\mathcal{X}_{K_pK^p}$, $T_p\mathcal{E}_{K_pK^p}$ has a trivialization up to K_p. Passing to infinite level, $T_p\mathcal{E}_{K^p}$ has a trivialization $T_p\mathcal{E}_{K^p} \cong \underline{\mathbb{Z}_p}^{\oplus 2}$, and hence the Hodge-Tate sequence gives

$$0 \to \omega^{-1} \to \hat{O}^{\oplus 2} \to \omega \to 0.$$

This gives rise to a point in the flag variety. We define this map to be the Hodge-Tate period map

$$\pi_{\mathrm{HT}} : \mathcal{X}_{K^p} \to \mathscr{F}\ell.$$

π_{HT} has many good properties; for example, it is affinoid and partially proper, and in particular, it behaves like a finite morphism. This is critically used for proving the vanishing of torsion cohomology in [CS17].

Moreover, $\mathscr{F}\ell \cong \mathbb{P}^1$ admits a Newton stratification. In our case, there are two strata, $\mathbb{P}^1(\mathbb{Q}_p)$ and $\Omega := \mathscr{F}\ell \backslash \mathbb{P}^1(\mathbb{Q}_p)$, where the first stratum is a profinite set, and the second stratum is the Drinfeld's upper half plane. We call $\mathbb{P}^1(\mathbb{Q}_p)$ the ordinary locus, and Ω the supersingular locus. As we will see, these strata corresponds respectively to principal series and to supercuspidal representations.

We can consider the restriction of π_{HT} to each stratum. For any $x \in \mathbb{P}^1(\mathbb{Q}_p)$, $\pi_{\mathrm{HT}}^{-1}(\{x\})$ is the infinite level Igusa varieties Ig_{K^p}. On the other hand, over Ω, π_{HT} is proétale, and $\pi_{\mathrm{HT}}^{-1}(\Omega)$ has a uniformization by Lubin-Tate space. More precisely, we have a $G(\mathbb{Q}_p)$-equivariant isomorphism

$$\pi_{\mathrm{HT}}^{-1}(\Omega) \cong D^\times \backslash [M_{\mathrm{LT},\infty} \times (D \otimes \mathbb{A}_f^p)^\times / K^p],$$

identify π_{HT} with the Hodge-Tate period map $\pi_{\mathrm{HT,LT}}$ as defined in [SW13]. This is a special case for general PEL type Shimura varieties that the preimage of Newton strata can be described as certain "product" of Rapoport-Zink spaces and Igusa varieties, each of which is relatively simpler and admits a canonical integral structure.

3 p-adic Interpolation and Completed Cohomology

Another motivation of introducing the infinite level modular curve is to realize p-adic interpolation in a geometric way. As we will see, completed cohomology can be realized as certain cohomology of infinite level Shimura varieties.

There are many ways to p-adically interpolate the classical modular forms. There is the classical definition of overconvergent modular forms of [Kat73], which is later geometrically reconstructed by [Pil13] and [AIS14]. The space of overconvergent modular forms is very big, infinite dimensional, and contains the classical modular forms as a subspace. The classical result of classicality is given by [Col96] and by [Kas06]. In this setting, one usually look at the finite slope part of the space with respect to U_p-operator, and the classicality theorem of Coleman claims that when valuation of the U_p-eigenvalue is smaller (resp. greater) than the weight minus 1, the forms is classical (resp. non-classical). However, the finite slope part only captures those automorphic representations that are principal series at p, and thus is not useful enough for our purpose.

Hence we are lead to consider the larger space of completed cohomology.

Definition 3.1 ([Eme06])**.** Fix K^p as above. We define the completed cohomology as

$$H^i(K^p, \mathbb{Z}/p^n) := \varinjlim_{K_p} H^i(\mathcal{X}_{K_pK^p}(\mathbb{C})^{\mathrm{an}}, \mathbb{Z}/p^n),$$

and

$$\tilde{H}^i(K^p, \mathbb{Z}_p) := \varprojlim_{n} H^i(K^p, \mathbb{Z}/p^n), \ \tilde{H}^i(K^p, \mathbb{Q}_p) := \tilde{H}^i(K^p, \mathbb{Z}_p)[1/p].$$

Then $\tilde{H}^i(K^p, \mathbb{Q}_p)$ carries natually the commuting continuous actions of $\mathrm{GL}_2(\mathbb{Q}_p)$, $\mathrm{Gal}_{\mathbb{Q}_p}$ and the Hecke algebra $\mathbb{T}(K^p)$. Moreover, $\tilde{H}^i(K^p, \mathbb{Q}_p)$ is admissible as a Banach representation of $\mathrm{GL}_2(\mathbb{Q}_p)$. By [ST03], for any admissible representations of p-adic Lie groups, the locally analytic vectors are dense. Hence we can consider $\tilde{H}^i(K^p, \mathbb{Q}_p)^{\mathrm{la}}$, the subspace of $\mathrm{GL}_2(\mathbb{Q}_p)$-locally analytic vectors.

By the work of [Pan22a], the space of overconvergent forms is part of completed cohomology. More precisely, $\tilde{H}^i(K^p, \mathbb{C}_p)^{\mathrm{la}} := \tilde{H}^i(K^p, \mathbb{Q}_p)^{\mathrm{la}} \hat{\otimes}_{\mathbb{Q}_p} \mathbb{C}_p$, then we have that the space of overconvergent modular forms of weight k, $M_k^\dagger(K^p)$ is a (Hecke equivariant) direct summand of $\tilde{H}^1(K^p, \mathbb{C}_p)^{\mathrm{la}, \mathfrak{b}=(0,2-k)}$. Note that the completed cohomology is better behaved than the overconvergent modular forms as it carries the action of $\mathrm{Gal}_{\mathbb{Q}_p}$ as well as the action of $\mathrm{GL}_2(\mathbb{Q}_p)$.

Finally, we note that by primitive comparison of [Sch13], $H^i(K^p, \mathbb{Z}/p^n) \otimes$

$\mathcal{O}_{\mathbb{C}_p} \cong^a H^i(\mathcal{X}_{K^p}, \mathbb{Z}/p^n)$, and taking limits, we have

$$\tilde{H}^i(K^p, \mathcal{O}_{\mathbb{C}_p}) \cong^a H^i(\mathcal{X}_{K^p}, \mathcal{O}^+_{\mathcal{X}_{K^p}}),\ \tilde{H}^i(K^p, \mathbb{Q}_p)\hat{\otimes}_{\mathbb{Q}_p}\mathbb{C}_p \cong H^i(\mathcal{X}_{K^p}, \mathcal{O}_{\mathcal{X}_{K^p}}),$$

where the cohomology on $\mathcal{X}_{K^p}$ is taken with respect to the analytic topology. This gives a geometric realization of the completed cohomology.

4 Locally Analytic Vectors

Furthermore, we can also describe the locally analytic part $\tilde{H}^i(K^p, \mathbb{C}_p)$ in a similar way. The functor $(-)^{\mathrm{la}}$ can be defined for $\mathrm{GL}_2(\mathbb{Q}_p)$-equivariant sheaves, and we can consider its derived functor $(-)^{R-\mathrm{la}}$. One can prove that $\mathcal{O}_{\mathcal{X}_{K^p}}$ is $(-)^{R-\mathrm{la}}$-acyclic, i.e. $\mathcal{O}^{R-\mathrm{la}}_{\mathcal{X}_{K^p}} \cong \mathcal{O}^{\mathrm{la}}_{\mathcal{X}_{K^p}}[0]$. Then a formal argument shows that

$$\tilde{H}^i(K^p, \mathbb{C}_p)^{\mathrm{la}} \cong H^i(\mathcal{X}_{K^p}, \mathcal{O}_{\mathcal{X}_{K^p}})^{\mathrm{la}} \cong H^i(\mathcal{X}_{K^p}, \mathcal{O}^{\mathrm{la}}_{\mathcal{X}_{K^p}}),$$

where $\mathcal{O}^{\mathrm{la}}_{\mathcal{X}_{K^p}}$ is the subsheaf of $\mathcal{O}_{\mathcal{X}_{K^p}}$ consisting of locally analytic vectors with respect to $\mathrm{GL}_2(\mathbb{Q}_p)$-actions. Thus we are led to understand the sheaf $\mathcal{O}^{\mathrm{la}}_{\mathcal{X}_{K^p}}$. It turns out that this sheaf has quite simple description (Section 4.3 of [Pan22a]) as a colimit of power series rings over the functions on the finite level Shimura varieties. More precisely, fix $U \subset \mathscr{F}\ell$ open affinoid, with $\pi_{\mathrm{HT}}^{-1}(U)$ open affinoid, and $\pi_{\mathrm{HT}}^{-1}(U) = \pi_{K_p}^{-1}(U_{K_pK^p})$ for some open affinoid $U_{K_pK^p} \subset \mathcal{X}_{K_pK^p}$. Assume $\infty \notin U$ or $0 \notin U$, then $\mathcal{O}^{\mathrm{la}}_{\mathcal{X}_{K^p}}(\pi_{\mathrm{HT}^{-1}}(U))$ can be described as

$$\mathcal{O}^{\mathrm{la}}_{\mathcal{X}_{K^p}}(\pi_{\mathrm{HT}^{-1}}(U)) \cong \varinjlim_n \mathcal{O}_{\mathcal{X}_{K_p^{(n)}K^p}}(U_{K_p^{(n)}K^p})[[p^{-n}X_1, p^{-n}X_2, p^{-n}X_3]],$$

where $K_p^{(n)}$ is a decreasing system of open compact subgroups of K_p, and $U_{K_p^{(n)}K^p}$ is the preimage of $U_{K_pK^p}$ along the natural map $\mathcal{X}_{K_p^{(n)}K^p} \to \mathcal{X}_{K_pK^p}$.

$\mathcal{O}^{\mathrm{la}}_{\mathcal{X}_{K^p}}$ is quite large, and we have an action of $\mathfrak{g}$ given by deriving the group action. Denote $\mathcal{O}^{\mathrm{la}}_{K^p} := \pi_{\mathrm{HT},*}\mathcal{O}^{\mathrm{la}}_{\mathcal{X}_{K^p}}$. Using π_{HT}, we can pull-back the function $\mathcal{O}_{\mathscr{F}\ell}$ on the flag varieties. We obtain an action of $\mathfrak{g}^0 := \mathfrak{g} \otimes_{\mathbb{Q}_p} \mathcal{O}^{\mathscr{F}\ell}$ on $\mathcal{O}^{\mathrm{la}}_{K^p}$. As defined in [BB83], $\mathfrak{g}^0$ contains $\mathfrak{n}^0 \subset \mathfrak{b}^0 \subset \mathfrak{g}^0$, and we can consider the action of these subsheaves on $\mathcal{O}^{\mathrm{la}}$. The conclusion is quite surprising:

Theorem 4.1 (Theorem 1.0.10 & 1.0.14 of [Pan22a])**.** $\mathfrak{n}^0$ *acts trivially on* $\mathcal{O}^{\mathrm{la}}$*, inducing a* horizontal action *of* $\mathfrak{h} \otimes \mathcal{O}_{\mathscr{F}\ell} \cong \mathfrak{b}^0/\mathfrak{n}^0$ *on* $\mathcal{O}^{\mathrm{la}}$*. Denote the corresponding action of* $\mathfrak{h}$ *by* $\theta_{\mathfrak{h}}$*, then* $\theta_{\mathfrak{h}}(0,1)$ *give rise to the arithmetic Sen operator of* $\mathcal{O}^{\mathrm{la}}$*.*

More precisely, we consider the semilinear action of $\mathrm{Gal}_{\mathbb{Q}_p}$ *on* $\mathcal{O}^{\mathrm{la}}$*. Consider* $\Gamma := \mathrm{Gal}(\mathbb{Q}_p(\zeta_{p^\infty})/\mathbb{Q}_p) \cong \mathbb{Z}_p^\times$ *and denote by* $H := \mathrm{Ker}(\mathrm{Gal}_{\mathbb{Q}_p} \to \Gamma)$*. Then*

$$\mathcal{O}^{\mathrm{la},H-\mathrm{sm},\Gamma-\mathrm{la}} \hat{\otimes}_{\mathbb{Q}_p(\zeta_{p^\infty})} \mathbb{C}_p \cong \mathcal{O}^{\mathrm{la}}.$$

Moreover, the restriction of $\theta_{\mathfrak{h}}(0,1)$ *to* $\mathcal{O}^{\mathrm{la},H-\mathrm{sm},\Gamma-\mathrm{la}}$ *coincides with the action of* $\mathrm{Lie}\,\Gamma$*, where the latter comes from deriving the action of* Γ*.*

The theorem shows that Galois-theoretical information can be seen using the group action. Using this result, Pan is able to prove classicality of weight 1 overconvergent modular forms in [Pan22a]:

Theorem 4.2 (Theorem 6.2.2 of [Pan22a])**.** *If* $\rho = \rho_\lambda$ *is pro-modular, and* $M_1^\dagger(K^p)^{N_0}[\mathfrak{p}_\lambda]$ *has a non-zero* U_p*-eigenvector, and* $\rho|_{\mathrm{Gal}_{\mathbb{Q}_p}}$ *is Hodge-Tate of weight* $0,0$*, then* $M_1(K^p)[\mathfrak{p}_\lambda] \neq 0$*, and there exists a classical weight* 1 *eigenform* $f \in M_1(K^p)[\mathfrak{p}_\lambda]$ *such that* $\rho \cong \rho_f$*.*

The idea is that the condition for a Galois representation to be "Hodge-Tate" is equivalent to the vanishing of the Sen operator. In [Pan22a], Pan compute explicitly the kernel of $\theta_{\mathfrak{h}}(0,1)$ using the explicit description of $\mathcal{O}^{\mathrm{la}}_{K^p}$ as above. Combining this classicality theorem with representation theory of $\mathrm{GL}_2(\mathbb{Q}_p)$, [Pan22a] reproves the Fontaine-Mazur conjecture under generic condition in the weight 1 case.

5 Fontaine's Operator

We now want to attack the Fontaine-Mazur conjecture in the weight ≥ 2-case The condition of being Hodge-Tate with weight $0,0$ is a special case of being de Rham. The condition of being de Rham involves the period ring B_{dR} defined by Fontaine.

In this survey, we instead give an equivalent definition that is better suited for our purpose. Recall that we denote $\Gamma := \mathrm{Gal}(\mathbb{Q}_p(\zeta_{p^\infty}/\mathbb{Q}_p)$ and $H := \mathrm{Ker}(\mathrm{Gal}_{\mathbb{Q}_p} \to \Gamma)$.

Definition 5.1. Let ρ be a Banach semilinear representation of $\mathrm{Gal}_{\mathbb{Q}_p}$ over $\mathbb{C}_p$. We say ρ is Hodge-Tate of weight $-a, -b$ (for $a, b \in \mathbb{Z}$) if

$$\rho^{H-\mathrm{sm}, \Gamma-\mathrm{la}} \hat{\otimes}_{\mathbb{Q}_p(\zeta_{p^\infty})} \mathbb{C}_p \cong \rho,$$

and if we let Θ_{arith} denote the action of a fixed generator of $\mathrm{Lie}\,\Gamma$ on $\rho^{H-\mathrm{sm}, \Gamma-\mathrm{la}}$, then the action of Θ_{arith} is semisimple with eigenvalues in $\{a, b\}$. We denote $\rho^{H-\mathrm{sm}, \Gamma-\mathrm{la}}$ as $D_{\mathrm{sen}}(\rho)$.

Let ρ be a Banach linear representation of $\mathrm{Gal}_{\mathbb{Q}_p}$ over $\mathbb{Q}_p$, and assume $\rho \hat{\otimes}_{\mathbb{Q}_p} \mathbb{C}_p$ *is Hodge-Tate of weight* $0, k$ *with* $k \in \mathbb{Z}_{\geq 1}$, then we can consider

$$D_{\mathrm{diff}, k}(\rho) := (\rho \hat{\otimes} B_{\mathrm{dR}}^+ / t^{k+1})^{H-\mathrm{sm}, \Gamma-\mathrm{an}}.$$

Then $D_{\mathrm{diff}, k}(\rho)$ is a finite free module over $\mathbb{Q}_p(\zeta_{p^\infty})[[t]]/t^{k+1}$. The fact is that

$$D_{\mathrm{diff}, k}(\rho) \hat{\otimes}_{\mathbb{Q}_p(\zeta_{p^\infty})[[t]]/t^{k+1}} B_{\mathrm{dR}}^+ / t^{k+1} \cong \rho \hat{\otimes}_{\mathbb{Q}_p} B_{\mathrm{dR}}^+ / t^{k+1},$$

and the action of a fixed generator of $\mathrm{Lie}\,\Gamma$ gives the arithmetic Sen operator acting on $D_{\mathrm{diff}, k}(\rho)$, which we still denotes as Θ_{arith}. Note that $D_{\mathrm{diff}, k}(\rho)$ is filtered by the Tate twist of $D_{\mathrm{sen}}(\rho)$, and the action of Θ_{arith} on $D_{\mathrm{sen}}(\rho)$ is semisimple of eigenvalue $0, k$ by our assumption on k. In particular, $D_{\mathrm{diff}, k}(\rho)$ will have a canonical decomposition into generalized eigenspaces of Θ_{arith}. We will be interested in the generalized eigenspace of $\Theta_{\mathrm{arith}} = 0$. Let us denote by $E_0(-)$ the functor taking the generalized eigenspace of Θ_{arith}. Then we have a canonical (non-split) Θ_{arith}-equivariant sequence

$$0 \to D_{\mathrm{sen}}(\rho)^{\Theta=-k}(k) \to E_0(D_{\mathrm{diff}, k}(\rho)) \to D_{\mathrm{sen}}(\rho)^{\Theta=0} \to 0.$$

We note that $D_{\mathrm{sen}}(\rho) = D_{\mathrm{sen}}(\rho)^{\Theta=-k} \oplus D_{\mathrm{sen}}(\rho)^{\Theta=0}$, with the decomposition given by different Hodge-Tate weights.

Now we consider the Θ_{arith} acting on the sequence. Since it is zero on $D_{\mathrm{sen}}(\rho)^{\Theta=0}$ and on $D_{\mathrm{sen}}(\rho)^{\Theta=k}$, it induces a unique map

$$N_k : D_{\mathrm{sen}}(\rho)^{\Theta=0} \to D_{\mathrm{sen}}(\rho)^{\Theta=k}(k).$$

This map is called the *Fontaine's map*.

Definition 5.2. Let ρ be a Banach representation of $\mathrm{Gal}_{\mathbb{Q}_p}$ over $\mathbb{Q}_p$ such that $\rho\hat{\otimes}\mathbb{C}_p$ is Hodge-Tate of weight $0, k$ for $k \in \mathbb{Z}_{\geq 1}$, and then we say ρ is *de Rham* if the Fontaine's operator $N_k = 0$.

Given this, understanding de Rhamness is reduced to understanding the kernel of the Fontaine's operator. In [Pan22b], Pan construct the Fontaine's operator on the level of sheaves on $\mathcal{X}_{K^p}$, and gives an explicit formula realizing it as certain differentials.

More precisely, in [Sch13], Scholze introduces a sheaf version of the period ring B_{dR}^+, the so-called period sheaf $\mathbb{B}_{\mathrm{dR}}^+$ over the proétale site over rigid varieties. We are then led to consider to apply the construction $(- \otimes \mathbb{B}_{\mathrm{dR}}^+/t^{k+1})^{H-\mathrm{sm},\Gamma-\mathrm{la}}$ to $\mathcal{O}^{\mathrm{la}}$.

More precisely, we fix $k \in \mathbb{Z}_{\geq 1}$, and we first fix certain infinitesimal character $\tilde{\chi}_k$, with $\tilde{\chi}_k$ being the infinitesimal character of $\mathrm{Sym}^{k-1}\mathrm{Std}^\vee$. This will forces the horizontal action $(\mathfrak{h}, \theta_\mathfrak{h})$ to be given by either $(1-k, 0)$ or $(1, -k)$. In other words, we have

$$\mathcal{O}^{\mathrm{la},\tilde{\chi}_k} \cong \mathcal{O}^{\mathrm{la},(1-k,0)} \oplus \mathcal{O}^{\mathrm{la},(1,-k)},$$

with $\mathcal{O}^{\mathrm{la},(a,b)}$ denotes the subsheaf where the horizontal action is given by (a, b). Now $\mathcal{O}^{\mathrm{la},(1-k,0)}$ and $\mathcal{O}^{\mathrm{la},(1,-k)}$ is Hodge-Tate of weight 0 and k respectively. We can then apply the analogous construction above to $\mathbb{B}_{\mathrm{dR}}^+/t^{k+1}$, and consider $(\mathbb{B}_{\mathrm{dR}}^+/t^{k+1})^{\mathrm{la},\tilde{\chi}_k}$, and consider the arithmetic Sen operator Θ_{arith} in this setting. The same argument induces a *geometric Fontaine's operator*

$$\mathcal{N}_k : \mathcal{O}^{\mathrm{la},(1-k,0)} \to \mathcal{O}^{\mathrm{la},(1,-k)}(k).$$

Using primitive comparison of Scholze, we are eventually reduced to understand the kernel of $\mathcal{N}_k$.

6 Differential Operators

The description of $\mathcal{N}_k$ uses certain differential operators. We focus on the case when $k = 1$, then $\mathcal{O}^{\mathrm{la},(0,0)}$ has a simple description using power series rings. Denote $\mathcal{O}^{\mathrm{sm}} := \varinjlim_{K_p} \pi_{K_p}^{-1}\mathcal{O}_{\mathcal{X}_{K_pK^p}}$, then there is a natural embedding

$$\mathcal{O}^{\mathrm{sm}} \otimes_{\mathbb{C}_p} \mathcal{O}_{\mathscr{F}\ell} \subset \mathcal{O}^{\mathrm{la},(0,0)}.$$

We have the connection coming from the modular curve $d : \mathcal{O}^{\mathrm{sm}} \to \Omega^{1,\mathrm{sm}}$, and the connection coming from the flag variety $\bar{d} : \mathcal{O}_{\mathscr{F}\ell} \to \Omega_{1\mathscr{F}\ell}$. Pan proves that both d and $\bar{d}$ has a canonical extension to $\mathcal{O}^{\mathrm{la},(0,0)}$ in a suitable sense. So we have

$$d^1 : \mathcal{O}^{\mathrm{la},(0,0)} \to \mathcal{O}^{\mathrm{la},(0,0)} \otimes_{\mathcal{O}^{\mathrm{sm}}} \Omega^{1,\mathrm{sm}},\ \bar{d}^1 : \mathcal{O}^{\mathrm{la},(0,0)} \to \mathcal{O}^{\mathrm{la},(0,0)} \otimes_{\mathcal{O}_{\mathscr{F}\ell}} \Omega^1_{\mathscr{F}\ell}.$$

Similarly, using BGG argument, one can define

$$d^k : \mathcal{O}^{\mathrm{la},(1-k,0)} \to \mathcal{O}^{\mathrm{la},(1-k,0)} \otimes_{\mathcal{O}^{\mathrm{sm}}} (\Omega^{1,\mathrm{sm}})^{\otimes k},\ \bar{d}^k : \mathcal{O}^{\mathrm{la},(1-k,0)} \to \mathcal{O}^{\mathrm{la},(1-k,0)} \otimes_{\mathcal{O}_{\mathscr{F}\ell}} (\Omega^1_{\mathscr{F}\ell})^{\otimes k}.$$

Using the natural isomorphism $\Omega^{1,\mathrm{sm}} \otimes_{\mathcal{O}^{\mathrm{sm}}} \hat{\mathcal{O}} \cong \Omega^{1,\vee}_{\mathscr{F}\ell} \otimes_{\mathcal{O}_{\mathscr{F}\ell}} \hat{\mathcal{O}}(1)$, we have a natural map $\Omega^{1,\mathrm{sm}} \otimes \Omega^1_{\mathscr{F}\ell} \to \mathcal{O}^{\mathrm{la},(1,-1)}(1)$, and thus we can consider the composition of d^k and $\bar{d}^k$ with the map $(\Omega^{1,\mathrm{sm}})^{\otimes k} \otimes (\Omega^1_{\mathscr{F}\ell})^{\otimes k} \to \mathcal{O}^{\mathrm{la},(k,-k)}(k)$, to obtain a map $\mathcal{O}^{\mathrm{la},(1-k,0)} \to \mathcal{O}^{\mathrm{la},(1,-k)}(k)$. Then the main theorem of [Pan22b] (Theorem 1.2.10) proves that

$$\mathcal{N}_k \cong d^k \circ \bar{d}^k \cong \bar{d}^k \circ d^k.$$

Hence we are reduced to understand the kernel and the cokernel of d^k and $\bar{d}^k$. Here we are going to use the geometry of $\mathcal{X}_{K^p}$ and compute d^k and $\bar{d}^k$ on each Newton strata.

First, $\bar{d}^k$ is relatively simpler: $\bar{d}^k$ is surjective, and $\mathrm{Ker}(\bar{d}^k)$ is given by the subsheaf of the algebraic functions $\mathcal{O}^{\mathrm{lalg},(0,k)} \cong \mathrm{Sym}^k\mathrm{Std}(k) \otimes_{\mathbb{Q}_p} (\Omega^1_{\mathscr{F}\ell})^{\otimes k}$.

We now go on to compute d^k. We first restricted to $\mathbb{P}^1(\mathbb{Q}_p)$, we have

$$\operatorname{Coker} d^k|_{\mathbb{P}^1(\mathbb{Q}_p)} \cong \omega_{\mathscr{F}\ell}^{k-1}|_{\mathbb{P}^1(\mathbb{Q}_p)} \otimes_{\mathbb{C}_p} \mathcal{H}^1_{\mathrm{ord}}(K^p, k-1),$$

where $\mathcal{H}^1_{\mathrm{ord}}(K^p, k)$ is the constant sheaf $\underline{H^1_{\mathrm{rig}}(\mathrm{Ig}(K^p, \mathrm{Sym}^{k-1}))}$, with the latter equipped with the discrete topology. Hence we know

$$H^0(\mathbb{P}^1(\mathbb{Q}_p), \operatorname{Coker} d^k) \cong \operatorname{Ind}_B^{\mathrm{GL}_2(\mathbb{Q}_p)}(H^1_{\mathrm{rig}}\mathrm{Ig}(K^p), \mathrm{Sym}^{k-1})(k) \cdot (e_2')^{k-1} t^{1-k},$$

where $(e_2')^{k-1}t^{1-k}$ denotes a suitable twist of various actions. Moreover, $\operatorname{Ker} d^k = 0$ if $k > 1$, and is given by the constant functions if $k = 1$. In particular, all the Hecke eigenvalues that appear here are classical by Coleman's results ([Col96]).

Now if we restrict to the supersingular locus Ω, we have the p-adic uniformization given by the Lubin-Tate spaces as described in the previous section. In this case, we use the duality between the Lubin-Tate tower and the Drinfeld tower proven in [SW13], to switch the role of d^k and $\bar{d}^k$, and then d^k_{LT} for the Lubin-Tate tower, coincides with the restriction of d^k for the modular curve to Ω, and also coincides with $\bar{d}^k_{\mathrm{Dr}}$ for the Drinfeld tower. In this way, the computation of $d^k|_\Omega$ reduces to the computation of $\bar{d}^k_{\mathrm{Dr}}$, which is similar to the previous computation of $\bar{d}^k$. In particular, we know $d^k|_\Omega$ is surjective, and for $k > 1$,

$$\operatorname{Ker} d^k|_\Omega \cong (\mathcal{A}^{K^p}_{D,(k-1,0)} \otimes \omega_{\mathrm{Dr}}^{-k, D_p^\times-\mathrm{sm}})^{O_{D_p}^\times} \cdot (\epsilon_p')^{1-k}.$$

Here $\mathcal{A}^{K^p}_{D,(k-1,0)}$ denotes the automorphic forms for the quaternion algebra $D^\times$. For $k = 1$, we again have an additional contribution from the constant function. Again, we see that when taking cohomology, all the Hecke eigenvalues that appear will be classical, and corresponds to automorphic representations that are supercuspidal at p via Jacquet-Langlands correspondence.

Eventually, via a complicated but direct cohomological computation, one can prove that the kernel of the Fontaine operator for $\tilde{H}^1(K^p, \mathbb{Q}_p)^{\mathrm{la}, \tilde{\chi}_k}$ is filtered by the cohomologies above, which is in particular classical, which concludes the argument.

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