Mémoire d'Introduction au Domaine de Recherche

Heegaard Floer Homology and Link Spectral Invariants

Ibrahim Trifa

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École Normale Supérieure
Département de Mathématiques et Applications

Abstract. In this memoir, we will present a short introduction to symplectic geometry, and some interesting questions that arise from its study. In particular, we will focus on C^0 symplectic geometry and the study of the algebraic structure of area-preserving homeomorphism groups on surfaces. Various flavours of Floer homology offer powerful tools, such as spectral invariants, to answer many of our questions. We will give a concise construction of Lagrangian Floer Homology, from which we will define Heegaard Floer Homology and link spectral invariants. We will use them to show some results of Cristofaro-Gardiner, Humilière, Mak, Seyfaddini and Smith ([CGHM+21] and [CHM+22]), and eventually present some open problems we hope to solve using similar techniques.

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Contents

1. Introduction to Symplectic Geometry

1.1. **A brief history of symplectic geometry.** Symplectic geometry was born when physicists were trying to formalize classical mechanics in a natural mathematical way. This formulation slowly started with contributions of eminent mathematicians and physicists such as Newton, Lagrange, Poisson, Hamilton and Poincaré.

However it was only in the 1960s that the modern formalization of the tools of symplectic geometry was introduced by Arnold. Since then, symplectic geometry has become a more and more active field of research, with applications in physics, Riemannian geometry, Algebraic geometry and dynamical systems.

For instance, Arnold enunciated a conjecture bounding from below the number of periodic orbits of Hamiltonian dynamical systems on compact symplectic manifolds.

This conjecture was proved for a large class of manifolds by Floer, who introduced Floer Homology, which would become one of the most powerful tools of modern symplectic geometry.

1.2. **Definitions.** We start with basic definitions of symplectic geometry.

Definition 1. *A symplectic manifold* (M, ω) *is a smooth manifold* M *equipped with a symplectic form* ω*, that is a non-degenerate closed 2-form on* M*.*

Remark 2. *Since there are no non-degenerate 2-forms when the dimension of* M *is odd, a symplectic manifolds is always of even dimension.*

Example 3. The following are symplectic manifolds :

- $\mathbb{R}^{2n}$ equipped with the standard symplectic form $\omega_{\text{std}} = \sum dx_i \wedge dy_i$, where $(x_1, ..., x_n, y_1, ..., y_n)$ is a basis of $\mathbb{R}^{2n}$. This example is important since by Darboux's theorem, any symplectic manifold is locally 'symplectomorphic' (cf. Definition 6) to an open subset of $(\mathbb{R}^{2n}, \omega_{\text{std}})$;
- Any oriented surface equipped with an area form;
- For M any smooth manifold, T^*M equipped with the symplectic form given locally by $\sum dp_i \wedge dq_i$, where $(p_1, ..., p_n, q_1, ..., q_n)$ are local coordinates on T^*M, with $(p_1, ..., p_n)$ parallel to M,and $(q_1, ..., q_n)$ parallel to the fiber. This example is particularly important for physicists, since the phase space of a mechanical system can be described as a cotangent bundle;
- $\mathbb{C}P^n$ equipped with the Fubini-Study form ω_{FS}.

Definition 4. *Let* (M^{2n}, ω) *be a symplectic manifold, and* $L \subset M$ *a submanifold.* L *is called:*

- *isotropic if* $\omega|_L = 0$.
- *Lagrangian if it is isotropic and of maximal dimension, that is of dimension* n*.*

Example 5. The following are Lagrangian submanifolds:

- The vector space generated by $x_1, ..., x_n$ in $(\mathbb{R}^{2n}, \sum dx_i \wedge dy_i)$;
- Any simple closed curve on an oriented surface;
- The zero section in T^*M;
- For any symplectic manifold (M, ω), the diagonal $\Delta = \{(x, x) | x \in M\}$ in $(M \times M, \omega \oplus (-\omega))$.

Definition 6. *A diffeomorphism* φ *from a symplectic manifold* (M, ω) *to another one* (M', ω') *is called a symplectomorphism if* $\varphi^*\omega' = \omega$

Remark 7.

- $\varphi : M \to M$ *is a symplectomorphism if and only if* $\text{graph}(\varphi) \subset M \times M$ *is a Lagrangian submanifold.*
- *If* (M^{2n}, ω) *is a symplectic manifold, then* $\omega^{\wedge n}$ *is a non-vanishing volume form. Any symplectomorphism of* (M, ω) *is then a volume-preserving diffeomorphism.*

We now define an important class of symplectomorphisms, called Hamiltonian diffeomorphisms.

Definition 8. *Let* (M, ω) *be a symplectic manifold. Let* $H : S^1 \times M \to \mathbb{R}$ *be a smooth map. Then, since* ω *is non-degenerate, for every* t *in* S^1*, there exists a vector field* X_{H_t} *such that* $\omega(X_{H_t}, \cdot) = -dH_t$ *(where* H_t *stands for* $H(t, \cdot)$*).* X_H *is called the Hamiltonian vector field associated to the Hamiltonian* H*.*

Definition 9. *A diffeomorphism* φ *of a symplectic manifold* (M, ω) *is called a Hamiltonian diffeomorphism if there exists a Hamiltonian* H *on* M *such that* $\varphi = \varphi_H^1$*, where* φ_H *denotes the flow of* X_H*. We say that* H *is a Hamiltonian generating* φ*. The set of all Hamiltonian diffeomorphisms of* M *is denoted by* $Ham(M)$*.*

Remark 10.

- *There could exist several Hamiltonians generating the same Hamiltonian diffeomorphism.*

- *Let X_t be a vector field on (M,ω). Then:*
$$\begin{aligned}(\forall t, \varphi^t_{X_t} \text{ is symplectic}) &\Leftrightarrow \frac{d}{dt}\left(\varphi^t_H\right)^*\omega = 0\\ &\Leftrightarrow \forall t, \mathcal{L}_{X_t}\omega = 0\\ &\Leftrightarrow \forall t, i_{X_t}d\omega + di_{X_t}\omega = 0\\ &\Leftrightarrow \forall t, i_{X_t}\omega \text{ is closed (since } \omega \text{ is closed)}\end{aligned}$$
and
$$(\forall t, \varphi^t_{X_t} \text{ is Hamiltonian}) \Leftrightarrow i_{X_t}\omega \text{ is exact.}$$
- *In particular, Hamiltonian diffeomorphisms are symplectomorphisms, but the converse is not necessarily true when $H^1_{dR}(M) \neq 0$.*
- *One can check that* $\mathrm{Ham}(M)$ *is a group for the composition operation:*
$$\varphi^1_H\varphi^1_K = \varphi^1_{H\#K}$$
with $H\#K(t,x) = H(t,x) + K\left(t, (\varphi^t_H)^{-1}(x)\right)$, and
$$\left(\varphi^1_H\right)^{-1} = \varphi^1_{\overline{H}}$$
where $\overline{H}(t,x) = -H\left(t, \varphi^t_H(x)\right)$.

We will now define a distance on $\mathrm{Ham}(M)$. In order to do that, we first define the following norm on Hamiltonian maps :
$$\|H\| := \int_{S^1} \operatorname{osc} H_t dt = \int_{S^1} (\max H_t - \min H_t)dt$$

Definition 11. *Let φ be a Hamiltonian diffeomorphism. Then, the Hofer norm of φ, denoted $\|\varphi\|_H$ is defined as the infimum of $\|H\|$, taken over all H that generate φ.*

Proposition 12. *The Hofer norm is a norm on* $\mathrm{Ham}(M)$*, and $(\varphi,\psi) \mapsto \|\varphi\psi^{-1}\|_H$ is a well-defined distance called the Hofer distance.*

When Σ has non-empty boundary, one can define the Calabi invariant $\mathrm{Cal} : \mathrm{Ham}(\Sigma) \to \mathbb{R}$: Let $\varphi \in \mathrm{Ham}(\Sigma)$, and H_t be a Hamiltonian supported in the interior of Σ such that $\varphi = \varphi^1_{H_t}$. Then,
$$\mathrm{Cal}(\varphi) = \int_0^1 \int_\Sigma H_t \omega dt$$
This definition does not depend on the choice of the Hamiltonian H_t, and Cal is a group homomorphism.

1.3. C^0 Symplectic Geometry.

For a long time, symplectic geometers have asked themselves if being a symplectomorphism is a 'soft' or 'rigid' condition among diffeomorphisms of a manifold (M,ω). For instance, can any volume-preserving diffeomorphism be written as a C^0-limit of symplectomorphisms? Or is the set of symplectomorphisms C^0-closed among C^1-diffeomorphisms of M?

This question was known as the Gromov alternative, which was later answered in [Eli87]:

Theorem 13 (Gromov-Eliashberg, 1987)**.** *If a sequence of symplectic diffeomorphisms C^0-converges to a diffeomorphism, then this diffeomorphism is also symplectic.*

Let φ be a homeomorphism between symplectic manifolds (M,ω) and (M',ω'). Even though $\varphi^*\omega$ is not well defined when φ is not smooth, we can still define a notion symplectic homeomorphism:

Definition 14. *We say that φ is a symplectic homeomorphism if it is a C^0-limit of symplectic diffeomorphism.*

This definition makes sense since by the Gromov-Eliashberg rigidity theorem, a smooth symplectic homeomorphism is a symplectic diffeomorphism.

2. Volume-preserving homeomorphisms

In the 1970s, Albert Fathi has studied the algebraic structure of volume-preserving homeomorphisms (cf. [Fat80]).

Let (M, v) be an oriented compact manifold (possibly with boundary) equipped with a non-vanishing volume form v.

Denote by $\mathrm{Homeo}(M, v)$ the group.of volume-preserving homeomorphisms of M that coincide with the identity near the boundary of M, and by $\mathrm{Homeo}_0(M, v)$ the connected component of the identity.

Then, $\mathrm{Homeo}_0(M, v)$ is a normal subgroup of $\mathrm{Homeo}(M, v)$, and Fathi showed that there exists a surjective group homomorphism, called the 'mass-flow':

$$\mathcal{F} : \mathrm{Homeo}_0(M, v) \to H_1(M; \mathbb{R})/\Gamma$$

where Γ is a discrete subgroup of $H_1(M; \mathbb{R})$.

Then, $\mathrm{Ker}(\mathcal{F})$ is a normal subgroup of $\mathrm{Homeo}_0(M, v)$ (which is proper when $H_1(M; \mathbb{R}) \neq 0$), and:

Theorem 15 (Fathi, 1980)**.** *If M has dimension at least 3, then $\mathrm{Ker}(\mathcal{F})$ is simple.*

However, the case of surfaces has remained an open question for forty years.

In what follows we will focus on compact oriented surfaces (possibly with boundary). Let (Σ, ω) be such a surface.

Then, the identity component of the group of symplectic homeomorphisms of Σ, $\overline{\mathrm{Diff}}_0(\Sigma, \omega)$, coincides with the identity component of the group of area and orientation preserving homeomorphisms $\mathrm{Homeo}_0(\Sigma, \omega)$.

In order to study the algebraic structure of this group, we define the following subgroup:

Denote by $\overline{\mathrm{Ham}}(\Sigma)$ the C^0-closure of $\mathrm{Ham}_c(\Sigma)$ inside $\mathrm{Homeo}(\Sigma, \omega)$ (where $\mathrm{Ham}_c(\Sigma)$ denotes Hamiltonian diffeomorphisms generated by Hamiltonians supported in the interior of Σ when it has non-empty boundary).

Then, $\overline{\mathrm{Ham}}(\Sigma) = \mathrm{Ker}(\mathcal{F})$ is a normal subgroup of $\mathrm{Homeo}(\Sigma, \omega)$.

In the smooth case, it was already known since 1978 that for a closed symplectic manifold (M, ω) of any dimension, $\mathrm{Ham}(M)$ is simple (cf. [Ban78]).

In 2021, Cristofaro-Gardiner, Humilière, Mak, Seyfaddini and Smith proved the following theorem (cf. [CGHM+21]), completing the picture of Fathi's theorem in dimension 2:

Theorem 16 (CG-H-M-S-S, 2021)**.** *For any compact surface (Σ, ω), the group $\overline{\mathrm{Ham}}(\Sigma)$ is not simple.*

They proved this theorem by exhibiting a normal subgroup. They showed this subgroup is proper using link spectral invariants defined with Heegaard Floer homology.

Using similar techniques, they managed to describe more precisely the structure of $\mathrm{Homeo}_0(\Sigma)$, especially when Σ is the sphere or the disk.

3. Lagrangian Floer Homology

In order to define spectral invariants, we need to start by presenting the Lagrangian Floer Homology theory. Its definition is similar to Morse homology on an infinite dimensional space. For this section, I will follow [Lec08]'s construction (one can also look at [Aur14] for another construction). We consider a compact symplectic manifold (M, ω), and two closed Hamiltonian isotopic Lagrangian submanifolds L and L', such that

$$\omega|_{\pi_2(M,L)} = \mu|_{\pi_2(M,L)} = 0$$

where μ is the Maslov index defined in [Lec08].

Remark 17. *Leclercq and Zapolsky have shown in* [LZ18] *that the construction of Lagrangian Floer homology still holds in the more general setting of monotone Lagrangians, which satisfy* $\omega|_{\pi_2(M,L)} = \tau\mu|_{\pi_2(M,L)}$ *for some constant* τ. *We will have to use this more general setting in section 4, but for simplicity of exposition we only present this simpler case.*

Let $\mathcal{P}(L,L') := \{\gamma \in C^\infty([0,1],M), \gamma(0) \in L, \gamma(1) \in L'\}$. Fix η in $\mathcal{P}(L,L')$, and let $\widetilde{P}_\eta(L,L')$ be the universal cover of the connected component of η (with base point η). Given a Hamiltonian H, we can define an action functional on $\widetilde{P}_\eta(L,L')$, by :

$$\mathcal{A}_H([\gamma,w]) := -\int w^*\omega + \int H\,(t,\gamma(t))\,dt$$

(here, w is a homotopy from η to γ in $\mathcal{P}(L,L')$).

We define the Lagrangian Floer complex $CF_*(L,L',H)$ as the complex generated by critical points of the action, which are trajectories of H from L to L' that lie in the same connected component as η. Those are in one-to-one correspondence with a finite subset of $\varphi_H^1(L) \cap L'$. There exists a graduation on this complex given by the *Maslov index* (cf. [Aur14]).

To define the Floer differential, we fix an ω-compatible almost complex structure J on M, i.e. an almost complex structure such that $g(v,w) := \omega(v,Jw)$ is a Riemannian metric. Then, we can define a Riemannian metric on the set of paths given by integrating g along the path.

As in Morse homology, the differential is given by counting trajectories of the negative gradient flow of the action connecting two critical points.

Those trajectories are smooth maps $u : \mathbb{R} \times [0,1] \to M$ satisfying:

- $\frac{\partial u}{\partial s} + J_t\left(\frac{\partial u}{\partial t} - X_H(u)\right) = 0$ (where $(s,t) \in \mathbb{R} \times [0,1]$);
- u has finite energy, i.e. $\int u^*\omega < \infty$;
- $u(\mathbb{R},0) \subset L$ and $u(\mathbb{R},1) \subset L'$;
- u is asymptotic to a critical point γ at $s = -\infty$, and to another critical point γ' at $s = +\infty$.

Given a relative homotopy class $\beta \in H_2(M, L \cup L')$, denote by $\hat{\mathcal{M}}_{J,\beta}(\gamma,\gamma')$ the set of such trajectories that satisfy $[u] = \beta$, and let $\mathcal{M}_{J,\beta}(\gamma,\gamma')$ be the quotient of this moduli space by the action of $\mathbb{R}$ by translation.

Using techniques from the study of pseudo-holomorphic curves developped by Gromov (cf. [Gro85]), one can show that when β has Maslov index 1, for a generic choice of H and J, $\mathcal{M}_{J,\beta}(\gamma,\gamma')$ is a zero dimensional compact manifold. Hence, it contains a finite number of points and one can define the Floer differential:

$$\partial[\gamma,w] := \sum_{[\gamma,w\#\beta]\in Crit(\mathcal{A}_H)} \#\mathcal{M}_{J,\beta}(\gamma,\gamma')[\gamma',w\#\beta]$$

where we define $\#\mathcal{M}_{J,\beta}(\gamma,\gamma')$ to be 0 when the Maslov index of β is not 1.

Using Gromov compactification to count broken trajectories, one can show that for a generic choice of (H,J), $\partial^2 = 0$, so we can define the Lagrangian Floer Homology $HF(L,L') = HF(L,L',H,J)$, which has the following properties :

Proposition 18.
- *$HF(L,L')$ does not depend on the choice of the generic pair (H,J): a homotopy between two Hamiltonian functions H and K, and two almost complex structures J and J' induces a comparison isomorphism*

$$HF(L,L',H,J) \cong HF(L,L',K,J')$$

- *There is a natural isomorphism* $HF(L,L',H,J) \cong HF\,(L,L",0,J')$ *with* $L" = \left(\varphi_H^1\right)^{-1}(L')$ *and* $J' = \varphi_H^* J$

Now, we introduce a filtration on the Floer complex: for $\lambda \in \mathbb{R}$, let $CF_{\leqslant\lambda}(L,L',H)$ be the vector subspace generated by critical points of action less than λ. The Floer differential is action decreasing, so ∂ is well defined on $CF_{\leqslant\lambda}(L, L', H)$, and one can define its homology $HF_{\leqslant\lambda}(L, L', H)$. The inclusion map $CF_{\leqslant\lambda}(L, L', H) \to CF_*(L, L', H)$ induces a map:

$$i_\lambda : HF_*^\lambda(L, L', H) \to HF_*(L, L', H)$$

Then, for any non-zero homology class $a \in HF_*(L, L', H)$, one can define a Lagrangian spectral invariant:

$$c(a, L, L', H) := \inf\{\lambda \in \mathbb{R}, a \in \operatorname{Im}(i_\lambda)\}$$

When $HF_*(L, L, H)$ is isomorphic to the Morse homology of L (which will be the case for the Lagrangians we will consider in the following sections), denote by $[L]$ the image of the fundamental class in $HF_*(L, L, H)$.

Then, we define $c_L(H) := c([L], L, L, H)$.

This invariant satisfies the following properties:

Proposition 19.

- *(Hofer Lipschitz)* $|c_L(H) - c_L(K)| \leqslant ||H - K||_{Hof}$;
- *(Spectrality)* $c_L(H) \in \operatorname{Spec}(L, L, H) := \{\mathcal{A}_H([\gamma, w]), [\gamma, w] \in \operatorname{Crit}(\mathcal{A}_H)\}$;
- *(Monotonicity) If* $H \leqslant K$, *then* $c_L(H) \leqslant c_L(K)$;
- *(Lagrangian control) If* $H_t|_L \equiv s(t)$, *then* $c_L(H) = \int_0^1 s(t)dt$. *Moreover, for any Hamiltonian* H,
$$\int_0^1 \min_L H_t dt \leqslant c_L(H) \leqslant \int_0^1 \max_L H_t dt$$
- *(Shift control)* $c_L(H_t + s(t)) = c_L(H) + \int_0^1 s(t)dt$;
- *(Triangle inequality)* $c_L(H\#K) \leqslant c_L(H) + c_L(K)$;
- *(Homotopy invariance) if* H *and* K *are mean-normalized Hamiltonians that generate homotopic isotopies* $\{\varphi_H^t\}$ *and* $\{\varphi_K^t\}$, *then* $c_L(H) = c_L(K)$. *Therefore,* c_L *induces a well-defined map* $\widetilde{\operatorname{Ham}}(M) \to \mathbb{R}$ *given by* $c_L([\{\varphi_H^t\}]) = c_L(H)$ *for a mean-normalized* H.

4. Heegaard Floer Homology and Link Spectral Invariants

In this section, using the work of the previous section and following [CGHM+21], we define new spectral invariants for Lagrangian links on surfaces.

Definition 20.

- *A Lagrangian link is a finite disjoint union* $\underline{L} = \bigcup_{i=1,\dots,k} L_i$ *of Lagrangian circles on* Σ.
- *The diameter* $diam(\underline{L})$ *of a Lagrangian link is the maximum of the diameters of its contractible components.*
- *Let* $\underline{L} = \bigcup_{i=1,\dots,k} L_i$ *be a Lagrangian link, and denote by* B_j, $1 \leqslant j \leqslant s$, *the connected components of* $\Sigma \setminus \underline{L}$. *Let* k_j *be the number of boundary components of* B_j, *and* A_j *the* ω*-area of* B_j. *Let* $\eta > 0$. *We say that* $\underline{L}$ *is* η*-monotone if all the* $\overline{B_j}$ *are planar domains, and*
$$\lambda := 2\eta(k_j - 1) + A_j$$
does not depend on j. λ *is called the monotonicity constant of* $\underline{L}$.
- *A sequence of Lagrangian links* $(\underline{L}^m)$ *is called equidistributed if :*
  - (i) $diam(\underline{L}^m) \to 0$
  - (ii) *There exists* $N \in \mathbb{N}$ *such that for all* m, $\underline{L}^m$ *has no more than* N *non-contractible component*
  - (iii) *Each contractible components of* $\underline{L}^m$ *bounds a unique disk of area less than* $diam(\underline{L}^m)$, *and those disks are disjoint. (We say that the contractible components are not nested.)*
  - (iv) *There exists a sequence* (η_m) *such that for all* m, $\underline{L}^m$ *is* η_m*-monotone.*

Given a monotone Lagrangian link $\underline{L} = \bigcup_{i=1,\dots,k} L_i$, one can look at the Lagrangian $L_1 \times \dots \times L_k$ inside Σ^k. We would like to define a link spectral invariant based on the Lagrangian spectral invariant of this new Lagrangian. However, when $diam(\underline{L})$ is small, $L_1 \times \dots \times L_k$ is displaceable in Σ^k, so this gives us something trivial. This is why we consider $\mathrm{Sym}(\underline{L})$, the image of $L_1 \times \dots \times L_k$ in $\mathrm{Sym}^k(\Sigma) := \Sigma^k/\mathfrak{S}_k$, where the permutation group $\mathfrak{S}_k$ acts on Σ^k by exchanging coordinates. Let Δ denote the *big diagonal* of $\mathrm{Sym}^k(\Sigma)$, i.e. the image of

$$\{(x_1, \dots, x_k) \in \Sigma^k | \exists i \neq j, x_i = x_j\}$$

Then, $\mathrm{Sym}^k(\Sigma)$ inherits of an orbifold structure, and a 2-form ω_{orb} that is a symplectic form on $\mathrm{Sym}^k(\Sigma) \setminus \Delta$ (see [CGHM+21]). For any open neighbourhood U of Δ, we can fix a symplectic form ω_U on $\mathrm{Sym}^k(\Sigma)$ that coincides with ω_{orb} on $\mathrm{Sym}^k(\Sigma) \setminus U$. If $\mathrm{Sym}(\underline{L})$ does not intersect U, then it is a Lagrangian submanifold of $(\mathrm{Sym}^k(\Sigma), \omega_U)$.

We can now define the Lagrangian link spectral invariant :

Definition 21. *Let $\underline{L}_k$ be a monotone Lagrangian link, and H a Hamiltonian function on Σ. We define :*

$$\begin{aligned} \mathrm{Sym}^k(H) : & S^1 \times \Sigma^k \to \mathbb{R} \\ & (t, (x_1, \dots, x_k)) \mapsto H_t(x_1) + \dots + H_t(x_k) \end{aligned}$$

Then, $\mathrm{Sym}^k(H)$ descends to a Hamiltonian on $\mathrm{Sym}^k(\Sigma)$. Since $\mathrm{Sym}(\underline{L})$ does not intersect Δ (the components of the link are disjoint), we can pick a neighbourhood U of Δ such that $\mathrm{Sym}(\underline{L})$ and its image by $\varphi_{\mathrm{Sym}^k(H)}$ do not intersect U. With the corresponding symplectic structure ω_U, we define the link spectral invariant :

$$c_{\underline{L}_k}(H) := \frac{1}{k} c_{\mathrm{Sym}(\underline{L})}(\mathrm{Sym}^k(H))$$

This definition does not depend on the choice of the neighbourhood U or the symplectic form ω_U.

This invariant has the following properties, inherited from proposition 19:

Proposition 22.

- *(Spectrality) $c_{\underline{L}}(H)$ lies in* $\mathrm{Spec}(H, \underline{L})$
- *(Hofer Lipschitz)* $\left|c_{\underline{L}}(H) - c_{\underline{L}}(K)\right| \leqslant \|H - K\|$
- *(Monotonicity) If $H \leqslant K$ then $c_{\underline{L}}(H) \leqslant c_{\underline{L}}(K)$*
- *(Lagrangian control) If $H_t|_{L_i} = s_i(t)$ for each i, then*

$$c_{\underline{L}}(H) = \frac{1}{k} \sum_{i=1}^{k} \int s_i(t) dt$$

- *(Shift) $c_{\underline{L}}(H + s(t)) = c_{\underline{L}}(H) + \int_0^1 s(t)dt$*
- *(Support control) If $supp(\varphi) \subset \Sigma \setminus \underline{L}$ then $c_{\underline{L}}(\varphi) = -\mathrm{Cal}(\varphi)$*
- *(Triangle inequality) $c_{\underline{L}}(H \sharp K) \leqslant c_{\underline{L}}(H) + c_{\underline{L}}(K)$*
- *(Homotopy invariance) If H, K are mean normalized and φ_H is homotopic to φ_K, then $c_{\underline{L}}(H) = c_{\underline{L}}(K)$.*

Proposition 23. *(Calabi property)Let $(\underline{L}^m)$ be an equidistributed sequence of Lagrangian links on a closed symplectic surface (Σ, ω). Then for every Hamiltonian H,*

$$\lim_{m \to \infty} c_{\underline{L}^m}(H) = \int_0^1 \int_\Sigma H_t \omega dt$$

Let us assume $\Sigma \neq S^2$. Since $\mathrm{Ham}(\Sigma)$ is simply connected if $\Sigma \neq S^2$, the homotopy invariance property implies that $c_{\underline{L}}$ descends to $\mathrm{Ham}(\Sigma)$.

Proposition 24. *Let* $\underline{L}, \underline{L}'$ *be two monotone Lagrangian links in* Σ*. Then,*

$$\varphi \mapsto c_{\underline{L}}(\varphi) - c_{\underline{L}'}(\varphi)$$

is uniformly continuous on $\mathrm{Ham}(\Sigma)$*, with respect to the* C^0 *distance. Hence, this map extends continuously to* $\overline{\mathrm{Ham}}(\Sigma)$*.*

If Σ_0 is a compact surface with non-empty boundary, then we can define the link invariants by embedding Σ_0 into a closed surface Σ, and then restricting $c_{\underline{L}}$ to $\mathrm{Ham}(\Sigma_0) \subset \mathrm{Ham}(\Sigma)$.

Proposition 25. *Let* $\underline{L}$ *be a monotone Lagrangian link in* Σ_0*. Then*

$$f_{\underline{L}} : \varphi \mapsto c_{\underline{L}}(\varphi) + \mathrm{Cal}(\varphi)$$

is uniformly continuous on $\mathrm{Ham}(\Sigma_0)$*, with respect to the* C^0 *distance. Hence,* $f_{\underline{L}}$ *extends continuously to* $\overline{\mathrm{Ham}}(\Sigma_0)$*.*

5. The algebraic structure of $\mathrm{Homeo}_0(\Sigma)$

5.1. **Non-simplicity of** $\overline{\mathrm{Ham}}(\Sigma)$**.** With the link spectral invariants we just defined, we can give proof of theorem 16.

Proof of theorem 16. We define the following subset of $\overline{\mathrm{Ham}}(\Sigma)$:

Definition 26. $\varphi \in \overline{\mathrm{Ham}}(\Sigma, \omega)$ *is called a finite energy homeomorphism if there exists a sequence of smooth Hamiltonians* H_i *such that :*

- $\varphi^1_{H_i} \xrightarrow{C^0} \varphi$
- *There exists* $C \geqslant 0$ *such that for every* i*,*

$$||H_i||_{Hof} := \int_0^1 \mathrm{osc}(H_{i,t})dt \leqslant C$$

We denote the set of finite energy homeomorphisms by $\mathrm{FHomeo}(\Sigma, \omega)$

It is easy to check that this defines a normal subgroup of $\overline{\mathrm{Ham}}(\Sigma)$. Then, it remains to show it is proper. This subgroup contains $\mathrm{Ham}(\Sigma)$, so it is non-trivial. To conclude the proof, we only need to find a Hamiltonian homeomorphism that is not in $\mathrm{FHomeo}(\Sigma)$.

We fix an equidistributed sequence of links $(\underline{L}^k)$. Pick a point z_0 in $\Sigma \setminus \underline{L}^1$, and fix an open neighborhood D of z_0 that does not intersect $\underline{L}^1$, and symplectomorphic to the disk $D_R = \{z \in \mathbb{C}, |z| < R\}$ (with z_0 sent to the origin).

Using polar coordinates on $D \setminus \{z_0\} \cong (0, R) \times S^1$, we define a radial Hamiltonian $H(r, \theta) = h(\pi r^2)$, where $h : (0, \pi R^2) \to R$ is a smooth, non-negative function, that coincides with $r \mapsto \frac{1}{r}$ near 0, and that vanishes near πR^2. Note that $\int_0^1 \int_\Sigma H\omega dt = \int_0^{\pi R^2} h(r)dr = +\infty$.

Then, φ^1_H acts on $D \setminus \{z_0\}$ as a rotation around z_0 (whose speed depends on the radius), therefore it extends continuously to the whole disk D (by setting $\varphi^1_H(z_0) = z_0$), and to the entire surface Σ since $H(r, \theta)$ vanishes when r is close to R.

Now, let (F_i) be an increasing sequence of smooth Hamiltonians on Σ that coincide with H outside of the disk of radius $\frac{1}{n}$ centered at z_0, and such that $\lim_{i\to\infty} \int_0^1 \int_\Sigma F_i\omega dt = +\infty$.

Then, $\varphi^1_{F_i}$ C^0-converges to φ^1_H, so $\varphi^1_H \in \overline{\mathrm{Ham}}(\Sigma)$.

Suppose φ^1_H was in $\mathrm{FHomeo}(\Sigma)$. Then, there would exist a sequence (H_i), whose Hofer norm is bounded by some constant C, such that φ_{H_i} C^0-converges to φ^1_H.

By proposition 24, $\zeta_k := c_{\underline{L}^k} - c_{\underline{L}^1}$ extends continuously to $\overline{\mathrm{Ham}}(\Sigma)$, so we would get $\zeta_k(\varphi^1_H) = \lim_{i\to\infty} \zeta_k(\varphi^1_{H_i}) \leqslant 2C$ by the Hofer-Lipschitz property of the invariants.

However:

$$\begin{aligned}\zeta_k(\varphi^1_H) &= \lim_{i\to\infty} \zeta_k(\varphi^1_{F_i}) \text{ by continuity}\\ &= \lim_{i\to\infty} c_{\underline{L}^k}(\varphi^1_{F_i}) \text{ by support control, since } D \text{ is away from } \underline{L}^1\\ &\geqslant c_{\underline{L}^k}(\varphi^1_{F_i}) \text{ for any } i, \text{ by monotonicity}\end{aligned}$$

Moreover, by the Calabi property, for all i, $\lim\limits_{k\to\infty} c_{\underline{L}^k}(\varphi^1_{F_i}) = \int_0^1\int_\Sigma F_i\omega dt$, therefore, for all i,

$$\lim_{k\to\infty} \zeta_k(\varphi^1_H) \geqslant \int_0^1\int_\Sigma F_i\omega dt$$

Since $\lim\limits_{i\to\infty}\int_0^1\int_\Sigma F_i\omega dt = +\infty$, we get that $(\zeta_k(\varphi^1_H))$ is unbounded, which shows that φ^1_H is not in FHomeo(Σ). □

5.2. **Other results and open questions.** We can define the following subgroup of FHomeo(Σ,ω):

Definition 27. *φ is called a hameomorphism if there exists a continuous map $H : S^1\times M \to \mathbb{R}$ and a sequence of smooth Hamiltonians H_i such that :*

$$\varphi^1_{H_i} \to \varphi$$

and $\|H - H_i\| \to 0$.

We denote the set of hameomorphisms by Hameo(Σ,ω).

This set is a normal subgroup of FHomeo(Σ,ω). In the case of the sphere, Buhovsky proved in 2022 that it is proper (cf. [Buh22]), but the case of other surfaces is still an open question.

It was recently shown (cf. [CGHM+21]) that:

Theorem 28 (CG-H-M-S-S,2021)**.** *Let (Σ,ω) be a compact symplectic surface with non-empty boundary. Then,* Cal : Ham$(\Sigma) \to \mathbb{R}$ *extends to a group homomorphism* Hameo$(\Sigma,\omega) \to \mathbb{R}$.

Therefore, when Σ has non-empty boundary, Hameo(Σ,ω) is not simple since it admits the kernel of the Calabi homomorphism as a proper normal subgroup.

Moreover, in the case of genus zero surfaces, the link spectral invariants satisfy a 'quasi-morphism property' that enables to prove the following (cf. [CHM+22]):

Theorem 29 (CG-H-M-S-S, 2022)**.**
- Cal *extends (in a non-canonical way) to a group homomorphism from* $\overline{\text{Ham}}(\mathbb{D})$ *to* $\mathbb{R}$;
- Hameo$(\mathbb{D}) \cap$ Ker(Cal) *is not simple;*
- Hameo(S^2) *is not simple.*

The case of higher genus surfaces remains open, but it seems to be possible to generalize those results by using some fragmentation techniques, and trying to show a 'local quasimorphism property' for link spectral invariants in higher genus.

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