Pluripotential Theory: Complex and Non-Archimedean

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1 Complex Pluripotential Theory

1.0 Notations and Conventions

- For an open subset U in $\mathbb{C}^n$, we take the complex coordinate $(z_1, z_2, \cdots, z_n)$, and the real coordinate $z_k = x_k + iy_k$.
- Write the 1-forms $dz_k = dx_k + idy_k$, and $d\overline{z}_k = dx_k - idy_k$.
- Write the partial differential operator $\partial^k = \frac{\partial}{\partial z_k} = \frac{1}{2}(\frac{\partial}{\partial x_k} - i\frac{\partial}{\partial y_k})$, and $\partial^{\overline{k}} = \frac{\partial}{\partial \overline{z}_k} = \frac{1}{2}(\frac{\partial}{\partial x_k} + i\frac{\partial}{\partial y_k})$.
- Let $\omega = \sum_{I,J} \omega_{I,J} dz_I \wedge d\overline{z}_J$ be a (p,q)-form. We define Dolbeault operators

$$\partial\omega = \sum_{I,J}\sum_{l} \frac{\partial \omega_{I,J}}{\partial z_l} dz_l \wedge dz_I \wedge d\overline{z}_J, \qquad \overline{\partial}\omega = \sum_{I,J}\sum_{l} \frac{\partial \omega_{I,J}}{\partial \overline{z}_l} d\overline{z}_l \wedge dz_I \wedge d\overline{z}_J$$

- Write $d = \partial + \overline{\partial}$, and $d^c = \frac{i}{2\pi}(-\partial + \overline{\partial})$, so $dd^c = \frac{i}{\pi}\partial\overline{\partial}$.
- We take the Lebesgue measure $\mu_{\text{leb}} := dx_1 \wedge dx_2 \wedge \cdots \wedge dx_n \wedge dy_1 \wedge dy_2 \wedge \cdots \wedge dy_n$. We have

$$dz_1 \wedge dz_2 \wedge \cdots \wedge dz_n \wedge d\overline{z}_1 \wedge d\overline{z}_2 \wedge \cdots \wedge d\overline{z}_n = (-2i)^n \mu_{\text{leb}}$$

1.1 Potential Theory on Complex Plane

The study of potential theory partly originates from electrostatics. Recall what we learned from physics classes. Let $U \subset \mathbb{C}$ be a bounded domain in vacuum, and suppose we have a static electric field $\mathbf{E}$ inside U. Then the electric potential ϕ satisfying

$$\nabla\phi = -\mathbf{E}$$

Let ρ be the charge density, and ϵ_0 be the electric constant, we also have Gauss's law

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0}$$

Therefore, we have Poisson's equation

$$\Delta\phi = -\frac{\rho}{\epsilon_0}$$

where we recall that $\Delta = \nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$ is the Laplace operator. The potential theory solve the electric potential from a given charge density and boundary condition. Here we consider the Dirichlet problem

$$\begin{cases} \Delta\phi = g \\ \phi|_{\partial U} = \phi_0 \end{cases}$$

where $g \in C^{\infty}(\overline{U})$, ϕ_0 is a bounded function on ∂U, and we search for a solution function ϕ on $\overline{U}$.

It is easy to just produce a function ϕ' satisfying $\Delta\phi' = g$. In fact, let $\Gamma = -\log|z|$ on $\mathbb{C}$, and we define $\phi' = \Gamma * g$, then it's easy to check that ϕ' is a bounded smooth function on U, and $\Delta\phi' = g$. So the problem is reduced to the case where $g = 0$, that is to find a harmonic function satisfying the Dirichlet boundary condition. To do this, we use Perron's method, which involves the study of subharmonic functions.

Definition 1.1. We call $f : U \to \mathbb{R} \cup \{-\infty\}$ subharmonic if f is upper semicontinuous (usc for short), and for any $x \in U$ and any disc $\overline{B(x,r)} \subset U$, we have

$$f(x) \leq \int_0^{2\pi} f(x + re^{i\theta})d\theta$$

We can define superharmonic function in similar way, and we say a function is harmonic if it's subharmonic and superharmonic. We have the following properties for subharmonic functions.

Lemma 1.2. *1. When $f \in C^2(U)$, f is subharmonic iff $\Delta f \leq 0$ on U.*

2. The set of subharmonic functions on U is closed under finite sum and maximum.

3. (Maximum Principle) Let $f, f' \in C^0(\overline{U})$. Assume that f is a subharmonic function on U and f' is a superharmonic function on U. Assume further that $f'|_{\partial U} \geq f|_{\partial U}$. Then either $f' > f$ on U or $f' \equiv f$ on U.

Now We define a subset of subharmonic functions

$$S_{\phi_0} := \{\varphi \in C^0(\overline{U}) \mid \varphi \text{ is subharmonic on } U, \text{ and } \varphi|_{\partial U} \leq \phi_0\}$$

Theorem 1.3 (Perron's method)**.** *The function $\phi = \sup_{\varphi \in S_{\phi_0}} \varphi$ is harmonic.*

Remark 1.4. Perron's method cannot say anything about the regularity property of *phi*, but we can use other methods to show that a harmonic function is always smooth.

Therefore, we get $\phi \in C^{\infty}(U)$ satisfying $\Delta\phi = 0$ and $\phi|_{\partial U} \leq \phi_0$. Suppose the solution to the Dirichlet's problem exists, and we assume that this solution is ϕ_1. Then by definition, $\phi_1 \in S_{\phi_0}$, so $\phi \geq \phi_1$. And by maximal principle, $\phi = \phi_1$. Therefore, the solution to the Dirichlet's problem exists iff $\phi|_{\partial U} = \phi_0$. And it turns out that the latter property relies on the regularity of the boundary.

For $z_0 \in \partial U$, we say that $w \in C^0(U)$ is a barrier function at z_0 if w is superharmonic on U and $w > 0$ on $\overline{U} \backslash z_0$ and $w(z_0) = 0$. z_0 is called a regular boundary point if there exists a barrier function at z_0.

Theorem 1.5. *1. The Dirichlet problem has a solution iff all the boundary points of U is regular.*

2. For a bounded domain $U \subset \mathbb{C}$, if any connected component of $\mathbb{C} \backslash U$ contains more than one points, then all the boundary points of U is regular.

1.2 Local Theory in Higher Complex Dimension

One can easily generalize the potential theory on complex plane to higher real dimension. The definition of Poisson equations, subharmonic functions and Perron's method can be generalized mostly words by words. But there is also another direction of generalization. Observe that on complex plane, $\partial\overline{\partial} f = \frac{1}{4}\Delta f$. So we can also generalize the potential theory to higher complex dimension.

We first generalize subharmonic function to higher dimension. We assume that U is a bounded domain in $\mathbb{C}^n$.

Definition 1.6. We call $f : U \to \mathbb{R} \cup \{-\infty\}$ plurisubharmonic (psh for short) if f is upper semicontinuous (usc for short), and for any complex line $l \subset \mathbb{C}^n$, $f|_{l\cap U}$ is subharmonic.

Pluripotential functions have the similar properties in Lemma 1.2.

Proposition 1.7. *1. When $f \in C^2(U)$, f is psh iff the complex Hessian $\left(\frac{\partial^2 f}{\partial z_i \partial \bar{z}_j}\right)_{ij}$ is positive semidefinite on U.*

2. $\mathrm{PSH}(U)$ is closed under finite maximum and finite sum.

3. If $\varphi_1, \varphi_2, \cdots, \varphi_k \in \mathcal{O}(U)$, then $|\varphi_i|, \log(\sum_{i=1}^n |\varphi_i|^2) \in \mathrm{PSH}(U)$.

4. More generally, suppose $\varphi : U_1 \to U_2$ is a holomorphic map between bounded domain and $f \in \mathrm{PSH}(U_2)$, then $f \circ \varphi \in \mathrm{PSH}(U_1)$.

Plurisubharmonic functions on $\mathbb{C}^n$ are the analogue of convex functions on $\mathbb{R}^n$. In fact it's easy to see

Lemma 1.8. *Suppose $f(z) = g(\mathrm{Re}\ z)$, then f is psh iff g is convex.*

Now we define the generalization of Laplace operator in higher complex dimension. For $f \in \mathrm{PSH}(U) \cap C^2(U)$, we define its Monge-Ampère measure

$$MA(f) := 2^{2n} \det \left(\frac{\partial^2 f}{\partial z_i \partial \bar{z}_j}\right)_{ij} \mu_{\mathrm{leb}}$$

where μ_{leb} is the Lebesgue measure on U. By Proposition 1.7, $MA(f)$ is a positive measure when $f \in C^2(U) \cap \mathrm{PSH}(U)$.

We notice that when $n = 1$, the complex Monge-Ampère operator is exactly $\frac{1}{4}\partial\overline{\partial}$. The Laplace operator is a second-order linear partial differential operator. However, when $n \geq 2$, Monge-Ampère operator is fully non-linear. Therefore, the complex Monge-Ampère equation in higher complex dimension is of completely different nature, and the study of it is much more difficult. A classical result in the Dirichlet problem of complex Monge-Ampère equation is the following.

Theorem 1.9 (Caffarelli-Kohn-Nirenberg-Spruck [CKNS85])**.** *Let $U \subset \mathbb{C}^n$ be a bounded pseudo-convex domain. Assume $g \in C^\infty(\overline{U})$ and $g > 0$. Assume further that $\phi_0 \in C^\infty(\partial U)$. Then the Dirichlet problem to the complex Monge-Ampère equation:*

$$\begin{cases} MA(\phi) = g \\ \phi|_{\partial U} = \phi_0 \end{cases}$$

has a unique solution $\phi \in \mathrm{PSH}(U) \cap C^\infty(\overline{U})$.

We can extend the definition of Monge-Ampère operator to any psh function.

Definition-Proposition 1.10 (Bedford-Taylor [BT82] [BT87])**.** *The Monge-Ampère operator defined above on* $\mathrm{PSH}(U) \cap C^2(U)$ *extends uniquely to an operator*

$$\mathrm{PSH}(U) \cap C^0(U) \to \mathcal{M}(U)$$

where $\mathcal{M}(U)$ *is the space of Radon measures on* U*.*

The proof of the proposition relies on the following regularization property.

Proposition 1.11. *Let* $f \in \mathrm{PSH}(U)$*, then for any* $x \in U$*, there exists an open neighborhood* V *of* x *and a decreasing sequence* $\{f_i\}_{i=1}^{\infty} \in C^{\infty}(V) \cap \mathrm{PSH}(V)$ *such that* f *is the pointwise limit of* $\{f_i\}_{i=1}^{\infty}$*.*

The basic idea of the proof is the following. The above proposition allows us to approximate any $f \in \mathrm{PSH}(U) \cap C^0(U)$ by a decreasing sequence of smooth psh functions $(f_i)_1^{\infty}$ locally. By Dini's lemma, $(f_i)_1^{\infty}$ converges to f uniformly on the closure of some open neighborhood. Then we may define $MA(f) := \lim_{m \to \infty} MA(f_i)$. The key point is proving the limit exists and independent of the sequence we choose.

1.3 Global Complex Pluripotential Theory and Complex Geometry

Let X be a smooth projective complex variety. We want to study the pluripotential theory on X. Notice that from the definition of the pluripotential function, it satisfies the maximum property. So any pluripotential function on X is constant. For this reason, we study pluripotential metric on a line bundle over X instead.

Definition 1.12. Suppose L is a line bundle over X. A C^0-metric (resp. C^{∞}-metric) ϕ on L is the following datum:

For any open subset $U \subset X$, any section $s \in H^0(U, L|_U)$, we assign a C^0 function (resp. C^{∞} function) $||s||_{\phi} : U \to \mathbb{R}_{\geq 0}$, such that:

1. $||s||_{\phi}$ is compatible with restrictions.
2. $||fs||_{\phi} = |f| ||s||_{\phi}$ for any $f \in \mathcal{O}(U)$ and section $s \in H^0(U, L|_U)$.
3. For $x \in U$, $||s||_{\phi}(x) = 0$ iff $s(x) = 0$.

Locally we have $L|_U \cong U \times \mathbb{C}$, then we have $||1||_{\phi} = e^{-\phi_1}$ for some $\phi \in C^0(U)$, and $||s||_{\phi} = |s| e^{-\phi_1}$. And a change of local trivialization will replace $e^{-\phi_1}$ by $e^{-\phi_2}$ where $\phi_2 = \phi_2 + \log |f|$ for some $f \in \mathcal{O}(U)$.

Definition 1.13. We say a metric ϕ is psh if ϕ_1 is psh for any local trivialization.

Example 1.14. 1. Let $X = \mathbb{P}^n$, $L = \mathcal{O}(1)$, we define Fubini-Study metric by

$$||s||_{\phi_{FS}} = \frac{|s(z)|}{(\sum_{i=0}^{n} z_i^2)^{\frac{1}{2}}}$$

ϕ_{FS} is psh since $\log(\sum_{i=0}^{n} z_i^2)$ is psh.

2. Suppose in general, (X, L) is a pair of smooth projective variety X with ample line L, and $L^{\otimes m}$ is a very ample line bundle which is the pullback of $\mathcal{O}(1)$ via the closed embedding τ. We define $\phi = m^{-1}\tau^*\phi_{FS}$. Then ϕ is also psh. We call ϕ a Fubini-Study metric (FS metric in short) of (X, L).

We have the following global version of the regularization property. (Compared to Proposition 1.11)

Theorem 1.15 (Demailly [Dem92]). *Suppose L is ample, and ϕ is a psh metric on L. Then ϕ is the pointwise limit of a decreasing sequence of FS metric. In particular, if $\phi \in C^0 \cap \mathrm{PSH}(X, L)$, then by Dini's lemma, ϕ is the uniform limit of a decreasing sequence of FS metric.*

Now we define the global Monge-Ampère operator.

Definition 1.16. Let ϕ be a smooth psh metric on L.

1. We define the real closed $(1,1)$-form $i\partial\overline{\partial}\phi$. Notice that this is well-defined since $\partial\overline{\partial}\log|f| = 0$ for $f \in \mathcal{O}^*(U)$.

2. The Monge-Ampère measure of ϕ is defined to be

$$MA(\phi) := 2^n(i\partial\overline{\partial}\phi)^n$$

By definition, $MA(\phi)$ is a positive measure on X for ϕ smooth psh. Similar to the local case, we have the following:

Proposition 1.17. *The Monge-Ampère operator can be extended uniquely to the class of continuous psh metrics.*

We can also solve the Monge-Ampère equation globally.

Theorem 1.18. *Let (X, L) be a polarized smooth projective variety over $\mathbb{C}$ of complex dimension n, and μ be a positive measure on X such that $\mu(X) = (L)^n$. Consider the Monge-Ampère equation $MA_L(\phi) = \mu$. Then*

1. *(Calabi-Yau Theorem) If μ comes from a smooth volume form, then there exists a smooth psh metric ϕ, unique up scaling, solving the equation.*

2. *If μ is absolutely continuous with respect to the Lebesgue measure, and the density of μ is in L^p for some $p > 1$. Then there exists a Holder continuous metric ϕ, unique up scaling, solving the equation.*

The uniqueness part in the smooth setting was proven by Calabi in 1950s, who also conjectured the existence part. This conjecture was solved by Yau in 1970s by PDE techniques. The more general setting was proved by Kolodzidj by using the methods in complex pluripotential to essentially reduce to smooth setting.

Now we recall:

Definition 1.19. Let X be a complex smooth projective variety. A Kähler metric ω is a smooth real closed positive $(1,1)$-form.Locally ω has the form $\omega = i\sum_{i,j} h_{i\bar{j}} dz_i \wedge d\bar{z}_j$. ω is real implies that $(h_{i\bar{j}})_{ij}$ is Hermitian, and positive amounts to say that this matrix is positively definite.

By local $\partial\overline{\partial}$-lemma, locally there always exists a smooth and real-valued potential function u satisfying $i\partial\overline{\partial}u = \omega$. A (real-valued) function φ is called ω-psh if $\varphi + u$ is psh locally.

ω-psh functions and psh metrics are related as follows. For a smooth psh metric ϕ on (X, L), we fix a Kähler form ω in $c_1(L)$. Since $i\partial\overline{\partial}\phi \in c_1(L)$, by $\partial\overline{\partial}$-lemma, there exists a real-valued function φ such that $i\partial\overline{\partial}\phi = \omega + i\partial\overline{\partial}\varphi$. By definition φ is ω-psh.

Proposition 1.20. *The psh metric over an ample line bundle L is one-to-one corresponding to ω-psh functions on X, where ω is a Kähler form in $c_1(L)$.*

Therefore, solving the Monge-Ampère equation in Theorem 1.18 amounts to find some canonical Kähler metric. The following is a classical example and in fact the original question arose by Calabi.

Definition 1.21. Suppose X is a smooth projective variety over $\mathbb{C}$. X is called Calabi-Yau (CY manifold in short) if the canonical bundle K_X is trivial, that is, there exists a holomorphic volume form on X.

Corollary 1.22. *Let X be a smooth projective Calabi-Yau variety. For any ample line bundle L over X, there exists a Ricci-flat Kähler form ω such that $\omega \in c_1(L)$.*

Definition 1.23. A polarized CY manifold (X, L, ω, Ω) is a smooth projective CY variety X, equipped with an ample line bundle L over X, a nowhere vanishing holomorphic volume form Ω on X, and the Ricci-flat Kähler form ω with $\omega \in c_1(L)$ and $\omega^n = \Omega$. Notice that the existence of such a Kähler form is guaranteed by the above corollary.

2 Non-archimedean Pluripotential Theory

2.0 Introduction

In this part, we introduce the pluripotential theory on varieties over non-archimedean fields. The non-archimedean pluripotential theory originates from [KT02], where the authors proposed an ambitious project, which aims to build a "non-archimedean Kähler geometry" on Berkovich spaces by mimicking the notions in the classical Kähler geometry. This non-archimedean pluripotential theory has been largely developed by Boucksom, Favre, Jonsson etc. during the last decade. ([BFJ11] [BFJ15] [BJ17] [BJ22]).

While the construction of the non-archimedean pluripotential theory follows the lines of the complex pluripotential theory, the notions and techniques in non-archimedean theory are mostly purely algebro-geometric, and does not heavily rely on the analysis like the complex theory. Also, the non-archimedean theory presented here is always a global theory. And in fact, in contrast to the complex theory, the non-archimedean theory does not have a satisfying local theory so far. At last, we would like to emphasize that the complex theory has been studied intensively over the past half century, but the non-archimedean theory only emerges in the beginning of this century. So we know much less about the non-archimedean theory than the complex theory.

The last decades has seen two main applications (or promising applications) of non-archimedean pluripotential theory in complex algebraic geometry. The first is on the study of K-stability and Kähler-Einstein metric ([BBJ15] [BJ18]). And the second is the non-archimedean approach to SYZ fibration ([BJ17] [Li19] [Li20]).

We first introduce non-archimedean fields and Berkovich spaces. Then we introduce the basic notions and techniques in non-archimedean pluripotential theory. Finally, we say a little about the application of non-archimedean Monge-Ampère equation to SYZ conjecture.

2.1 Non-archimedean fields

In this part we first define what is a non-archimedean field.

Definition 2.1. Suppose A is a unital commutative ring.

1. A semi-norm $|\cdot|$ is a function $|\cdot|: A \to \mathbb{R}_{\geq 0}$ satisfying for any $a, b \in A$, we have

$$|ab| = |a||b|, |a+b| \leq |a| + |b|$$

2. The semi-norm $|\cdot|$ is called a norm if it further satisfies $|a| = 0$ iff $a = 0$.

3. The norm $\|\cdot\|$ induce a metric on the ring A by $d(a,b) = \|a-b\|$. We say the norm is complete if the induced metric is complete.

Example 2.2. 1. $(\mathbb{C}, \|\cdot\|_\infty)$ with $\|a+bi\|_\infty = (a^2+b^2)^{\frac{1}{2}}$ the usual norm on $\mathbb{C}$ is a normed field in the sense of Definition 2.1.

2. For any unital commutative ring A, we define the trivial norm $\|\cdot\|_0$ on A by $\|a\|_0 = 1$ if $a \neq 0$, and $\|a\|_0 = 0$.

3. Suppose $A((t))$ is the algebra of Laurent series over A, and $A[[t]]$ is the subalgebra of formal series over A. For $f \in A((t))$, we define $\text{ord}(f)$ to be the maximal integer such that $f \in t^{\text{ord}(f)} A[[t]]$. The t-adic norm on $A((t))$ is defined by $\|f\| = e^{-\text{ord}(f)}$.

4. Any rational number $r \in \mathbb{Q}$ can be written in a unique way $r = \prod_{p \text{ prime}} p^{\mathrm{ord}_p(r)}$, where $\mathrm{ord}_p(r) \in \mathbb{Z}$. For any prime p, we define the p-adic norm on $\mathbb{Q}$ by $\|r\|_p = p^{-\mathrm{ord}(f)}$. And there exists a complete normed field $\mathbb{Q}_p$ which contains $\mathbb{Q}$ as a dense subfield and extends the p-adic norm on $\mathbb{Q}$. $\mathbb{Q}_p$ is called p-adic field. Ostrowski's theorem provides a complete list of norms on $\mathbb{Q}$: the trivial norm $\|\cdot\|_0$, $\|\cdot\|_p^r$ for $r \in (0, +\infty)$, and $\|\cdot\|_\infty^r$ for $r \in (0, 1]$.

Definition 2.3. A complete normed field $(K, \|\cdot\|)$ is called non-archimedean if it satisfies

$$\|a + b\| = \max \|a\|, \|b\|$$

Otherwise it is called archimedean.

Lemma 2.4. *If $(K, \|\cdot\|)$ is a complete archimedean field, then $(K, \|\cdot\|)$ is isomorphic to either $(\mathbb{R}, \|\cdot\|_\infty)$ or $(\mathbb{C}, \|\cdot\|_\infty)$*

The aim of the remaining of the section is to introduce the a pluripotential theory on smooth projective varieties over non-archimedean fields in a parallel way to the global complex pluripotential theory.

Suppose K is a complete non-archimedean field, and $X \hookrightarrow \mathbb{P}^n_K$ is a smooth projective variety over K. We can endow the projective space $\mathbb{P}^n_K$ by patching the product topology on K^n, and endow X with the subspace topology. But the following proposition indicates that this topology on X is too bad to do reasonable analysis.

Proposition 2.5. *X is totally disconnected, i.e. a subset $U \subset X$ is connected iff U is a single point.*

Therefore, we need to introduce Berkovich space which has a better topology.

2.2 Berkovich Spaces

Definition 2.6. 1. Let $(A, \|\cdot\|)$ be a complete normed ring, we define its spectrum as:

$$\mathcal{M}(A) := \{\text{semi-norm } |\cdot| \text{ on A such that } |\cdot| \leq \|\cdot\|\}$$

The topology on $\mathcal{M}(A)$ is defined to be the weakest topology such that for any $a \in A$,

$$\begin{aligned} v_a : \mathcal{M}(A) &\longrightarrow \mathbb{R}_{\geq 0} \\ |\cdot| &\longmapsto |f| \end{aligned}$$

is continuous.

2. In general, if B is an algebra over A, then we define the spectrum $\mathcal{M}(V)$ as

$$\mathcal{M}(B) := \{\text{multiplicative semi-norm } |\cdot|_B \text{ on } B \text{ such that } |\cdot|_B \leq \|\cdot\|_A \text{ on } A\}$$

The topology on $\mathcal{M}(B)$ is defined in a similar way to $\mathcal{M}(A)$.

Lemma 2.7. *$\mathcal{M}(B)$ is an non-empty compact Hausdorff space for any complete normed ring A and any finitely generated algebra B over A.*

For the rest of the section, we assume that $(K, \|\cdot\|)$ is a complete non-archimedean field, and X is an irreducible variety over K. Let $R := \{a \in K \mid \|a\| \leq 1\}$ be the valuation ring. A model of X over R is an integral scheme $\mathcal{X}$ over R such that $\mathcal{X} \times_R K \cong X$.

We would like to define a Berkovich space X^{an} associated to X which is called the analytification of X. Suppose first $X = \text{Spec } A_0$ is an affine variety, where A_0 is a finitely generated algebra over K. Then we just define $X^{an} := \mathcal{M}(A_0)$. For a general variety over X, we construct the analytification of X by gluing the analytifications of affine covering of X:

Proposition 2.8. *There exists an analytification functor*

$$\begin{aligned} \text{An} : \text{Var}_{/K} &\longrightarrow \text{Berk}_{/K} \\ X &\longmapsto X^{an} \end{aligned}$$

extending the analytifications of affine varieties defined above.

We illustrate how the Berkovich analytification of curves look like.

Example 2.9. 1. Let K be the trivially valued field. When X is a smooth projective curve. Then as a set, $X^{an} = \{v_{triv}\} \sqcup \bigsqcup_{p \in X(\mathbb{C}), t \in (0,+\infty)} v_{t \cdot \text{ord}_p} \sqcup X(\mathbb{C})$. When X is a projective curve with a nodal singularity, suppose $x \in X$ be the node point, $\pi : \widetilde{X} \to X$ is the resolution of singularity, and $\pi^{-1}(x) = \{x_1, x_2\}$. Then

$$X^{an} = (\widetilde{X})^{val} \sqcup X(\mathbb{C}) = \{v_{triv}\} \sqcup \bigsqcup_{p \in \widetilde{X}(\mathbb{C}), t \in (0,+\infty)} v_{t \cdot \text{ord}_p} \sqcup X(\mathbb{C})$$

So X^{an} is obtained by gluing $v_{x_1,\text{triv}}$ and $v_{x_2,\text{triv}}$ in $\widetilde{X}^{an}$. The visualizations of X^{an} are as follows

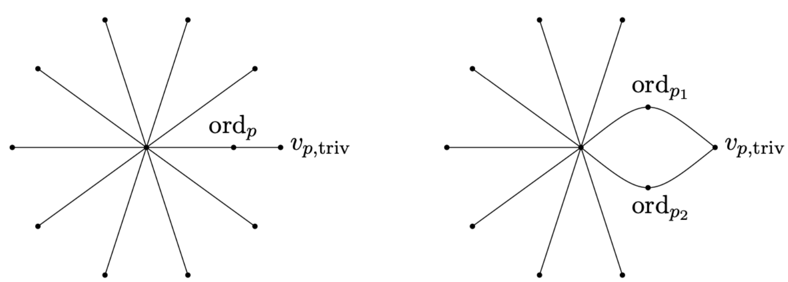
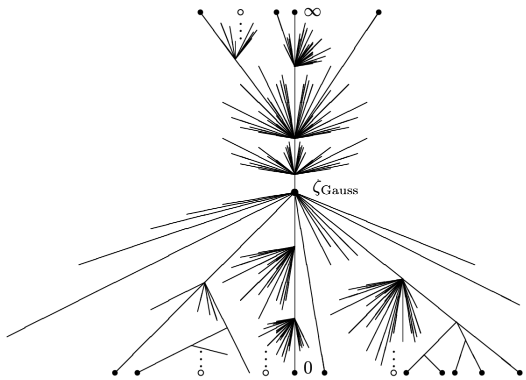


Figure 1: The Berkovich analytification of a smooth curve (left) and a nodal curve (right).

2. Figure 2 shows how the Berkovich space of projective lines over $\mathbb{C}((t))$ and elliptic curves over $\mathbb{C}((t))$ with multiplicative reduction look like.

In general, we have

Theorem 2.10 (Berkovich, Ducro)**.** *Any non-archimedean analytic curves are graphs.*



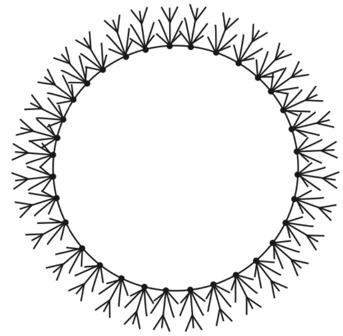


Figure 2: Berkovich analytifications over $\mathbb{C}((t))$ of projective lines (left) and elliptic curves with multiplicative reduction (right).

For analytification of higher dimensional varieties, we have a similar picture. For simplicity, we assume $K = \mathbb{C}((t))$. We say that a model $\mathcal{X}$ of X is SNC if the special fibre $\mathcal{X}_0$ is a simple normal crossing divisor in $\mathcal{X}$, i.e. $\mathcal{X}_0$ is locally defined by an equation $\prod_{i\in I} x_i = 0$ where I is a subset of $\{0, 1, 2 \cdots n\}$.

To any SNC model X, we can associate a simplicial complex called dual complex $\Delta_{\mathcal{X}}$ to it. And there are two maps relate $\Delta_{\mathcal{X}}$ and the Berkovich space X^{an}: an inclusion map $i_{\mathcal{X}} : \Delta_{\mathcal{X}} \hookrightarrow X^{an}$ and a retraction map $r_{\mathcal{X}} : X^{an} \to \Delta_{\mathcal{X}}$ satisfying $r_{\mathcal{X}} \circ i_{\mathcal{X}} = \mathrm{id}$. For any two SNC models $\mathcal{X}$ and $\mathcal{X}'$ such that there is an R-morphism $\mathcal{X}' \to \mathcal{X}$, there also exists the inclusion map $i_{\mathcal{X},\mathcal{X}'}$ and the retraction map $r_{\mathcal{X}',\mathcal{X}}$. The SNC models of X forms a poset, and we have the following isomorphism homeomorphism induced by $r_{\mathcal{X}}$ and $r_{\mathcal{X}',\mathcal{X}}$:

$$X^{an} \cong \varprojlim_{\mathcal{X}} \Delta_{\mathcal{X}}$$

For the details of the above constructions and conclusions, refer to [NX16].

2.3 Non-archimedean PSH metrics and Monge-Ampère Equations

From now on, we assume that $K \cong k((t))$, where k is an algebraically closed field of characteristic 0, and K is equipped with the standard t-adic norm. And we assume $R \cong k[[t]]$. We assume further that (X, L) is a polarized smooth projective variety of dimension n over K. Let (X^{an}, L^{an}) be the analytification of (X, L). We can define metrics on (X^{an}, L^{an}) in similar ways to Definition 1.12.

We start from defining model metrics, which is the analogue of smooth metric in the complex setting. Suppose $(\mathcal{X}, \mathcal{L})$ is a model of (X, L), i.e. $(\mathcal{X}, \mathcal{L})$ is a polarized scheme over Spec R. Let $\mathcal{X} = \cup_{i\in I}\mathcal{U}_i$ be a Zariski open covering of $\mathcal{X}$ such that $\mathcal{L}|_{\mathcal{U}_i}$ is trivial for any $i \in I$. Let $s_i \in H^0(\mathcal{X}, \mathcal{L}|_{\mathcal{U}_i})$ be the local trivialization. $X^{an} = \cup_{i\in I} V_i$. Then $\phi_{\mathcal{L}}$ is defined by $\|s_i\| \equiv 1$ on V_i. This is well-defined since for any $u_i \in \mathcal{O}^*(\mathcal{U}_i)$, $|u_i| \equiv 1$ on V_i.

We call the model metric $\phi_{\mathcal{L}}$ attached to the model $(\mathcal{X}, \mathcal{L})$ semipositive if $\mathcal{L}$ is relatively nef over $\mathcal{X}$, that is, its degree is nonnegative on any proper curve contained in the special fibre $\mathcal{X}_0$.

Now we would like to define psh metric on (X^{an}, L^{an}). For simplicity, we only give the definition of continuous psh metric. Different from the complex cases, we do not define psh metrics (functions) locally, whilst we use the regularization property in the spirit of Theorem 1.15 as the definition.

Definition 2.11. Let ϕ be a continuous metric on L. We say ϕ is psh if it is a uniform limit of a sequence of semipositive model metric.

We would like to define the non-archimedean Monge-Ampère operator. As what we did in the complex cases, we first define Monge-Ampère operator for semipositive model metric $\phi_{\mathcal{L}}$ associated to a model $(\mathcal{X}, \mathcal{L})$ of (X, L) over Spec R. We view the special fibre $\mathcal{X}_0$ as a divisor in $\mathcal{X}$, and write $\mathcal{X}_0 = \sum_{i \in I} b_i E_i$, where E_i are the irreducible components of $\mathcal{X}_0$, and $b_i \in \mathbb{Z}_{>0}$ are the multiplicities. An irreducible closed subvariety E_i gives a valuation on the function field of $\mathcal{X}$ extending the valuation of $k((t))$, that is, E_i is associated to a divisorial point $x_i \in X^{an}$. Now we define the Monge-Ampère measure of $\phi_{\mathcal{L}}$ by

$$MA(\phi_{\mathcal{L}}) := \sum_{i \in I} b_i (\mathcal{L}_{E_i})^n \delta_{x_i}$$

The domain of Monge-Ampère operator can be extended to continuous psh metric in similar ways to Proposition 1.10, Proposition 1.17.

Proposition 2.12 ([CL06]). *The non-archimedean Monge-Ampère operator defined over semipostive model metric on (X, L) can be extended uniquely to continuous psh metric on (X, L).*

We also have the following non-archimedean Calabi-Yau theorem.

Theorem 2.13 ([YZ17], [BFJ15], [BGGJ+20]). *Let $K \cong k((t))$, and (X, L) be a polarized smooth projective variety of dimension n. Let L be an ample line bundle over X, μ be a positive probability measure supported on the dual complex $\Delta_{\mathcal{X}}$ of some SNC model $\mathcal{X}$. Then there exists a continuous psh metric ϕ, and unique up to scaling, solving the NA MA equation $MA_L(\phi) = (L^n)\mu$.*

2.4 Applications in SYZ Conjecture

We start this part from introducing the SYZ picture of mirror symmetry. In string theory, mirror symmetry is a phenomenon that two different string theory model gives the same physics. Mathematically, this predicts that many CY manifold (X, ω, J) admits a mirror CY manifold $(\check{X}, \check{\omega}, \check{J})$, such that symplectic structure on $\check{X}$ can be read off from the complex structure on X, and similarly the complex structure on $\check{X}$ can be read off from the symplectic structure on X. (Refer to for example [Kon95].)

SYZ picture is a proposal and heuristic to construct and study the mirror CY manifold $\check{X}$ for a given CY manifold X. To state it more precisely, we introduce the notion of special Lagrangian torus fibration.

Definition 2.14. Let (X, L, ω, Ω) be a polarized CY manifold, where we recall that ω is the Ricci-flat Kähler form with $\omega \in c_1(L)$, and Ω is the nowhere vanishing holomorphic volume form on X.

1. A (real) submanifold M in X is called Lagrangian if M is of real dimension n, and the restriction of symplectic form to M $\omega|_M = 0$. It is called special Lagrangian if it further satisfies Im $\Omega|_M = 0$.

2. A special Lagrangian torus fibration is a continuous surjection $\pi : X \to B$ for some topological space B, such that $\pi^{-1}(b)$ is a special Lagrangian submanifold in X for any $b \in B$, and there exists an open dense subset B_0 of B, and $\pi^{-1}(b)$ is a (real) torus for any $b \in B_0$.

SYZ picture propose the following recipe

1. Given a CY manifold X which we expect has a mirror, construct a special Lagrangian torus fibration $\pi : X \to B$.

2. Dualize the smooth locus $\pi_0 : \pi^{-1}(B_0) \to B_0$ by dualizing the tori fibrewisely to obtain a semi-flat mirror $\check{\pi}_0 : \check{X}_0 \to B_0$.

3. Compactify $\check{\pi}_0$ to obtain the dual torus fibration $\check{\pi} : \check{X} \to B$.

4. Prove $(X, \check{X})$ satisfies duality properties (for example homological mirror symmetry).

However, each step above except Step 2 faces many obstacles. And the whole picture has only been realized for very limited classes of CY manifolds.

SYZ conjecture is a more rigorous conjectural formulation of the step 1.

Definition 2.15. 1. An algebraic degeneration family is given by a smooth projective morphism $\pi : \mathcal{X} \to \mathbb{C}^*$, such that for any $t \in \mathbb{C}^*$, the fibre $\mathcal{X}_t$ is a connected smooth projective variety. $\mathcal{X}$ can be base changed to a smooth projective variety X over $K = \mathbb{C}((t))$ via the morphism Spec $\mathbb{C}((t)) \to$ Spec $\mathbb{C}[t, t^{-1}]$. We use X^{an} to denote the analytification of X.

2. We say π is a polarized degeneration family if there is a line bundle $\mathcal{L}$ on $\mathcal{X}$ such that $\mathcal{L}|_{\mathcal{X}_t}$ is ample for any $t \in \mathbb{C}^*$.

3. We say that π is a polarized degeneration family of CY manifolds if there exists a holomorphic volume form Ω on $\mathcal{X}$. For each $t \in \mathbb{C}^*$, the normalized Calabi-Yau measure on $\mathcal{X}_t$ is defined to be the probability measures
$$d\mu_t := \frac{\Omega_t \wedge \overline{\Omega}_t}{\int_{X_t} \Omega_t \wedge \overline{\Omega}_t}$$

4. We say that π is a polarized maximally degeneration family of CY n-manifolds if the essential skeleton $\mathrm{Sk}(X)$ is of dimension n. Here the essential skeleton is a topological space which can be realized a sub-simplicial complex in the dual complex of any SNC model of X, and the homeomorphism type of the essential skeleton does not depends on the SNC model. For the precise definition, refer to for example [NX16].

Example 2.16. The Fermat family

$$\mathcal{X}_t = \{Z_0 Z_1 \cdots Z_{n+1} + t \sum_{i=0}^{n+1} Z_i^{n+2} = 0\} \subset \mathbb{CP}^{n+1} \times \mathbb{C}^*, \quad t \in \mathbb{C}^*, 0 < |t| << 1$$

is an example of maximally degeneration family of polarized CY n-manifolds. The special fibre $\mathcal{X}_0 = \{Z_0 Z_1 \cdots Z_{n+1} = 0\}$ has $n+1$ irreducible components.

Conjecture 2.17 (Strong SYZ Conjecture)**.** *Let $\mathcal{X}$ be a maximally degeneration family of polarized CY manifolds over a punctured disc $\mathbb{D}^*$. Then there exists $\epsilon > 0$ such that for any $|t| \leq \epsilon$, $\mathcal{X}_t$ admits a special Lagrangian torus fibration.*

However, [Joy00] suggests that the study of strong SYZ conjecture in the above version faces some essential difficulties and might be too optimistic for families of CY 3-folds. While Gross and Siebert has studied "soft" versions of SYZ fibration for the purpose of the studies of mirror symmetry ([Gro12]), it is also interesting to consider the following weak version SYZ conjecture from the perspective of complex geometry.

Conjecture 2.18 (Weak Metric SYZ Conjecture)**.** *Let $\mathcal{X}$ be a maximally degeneration family of polarized CY manifolds over a punctured disc $\mathbb{D}^*$. The for each $t \in \mathbb{D}^*$, there exists a subset U_t in $\mathcal{X}_t$, such that*

1. *There exists a special Lagrangian torus fibration on U_t.*
2. $$\lim_{t \to 0} \mu_t(U_t) = 1$$ *where μ_t is the normalized volume measure on X_t defined in 2.15.*

Definition 2.19. Let ϕ be a psh metric on (X, L). We say

1. ϕ is weakly regular if there is a SNC model $\mathcal{X}$ of X and a function ψ on the dual complex $\Delta_{\mathcal{X}}$, such that $$\phi = \psi \circ r_{\mathcal{X}}$$ where $r_{\mathcal{X}}$ is the retraction map $X^{an} \to \Delta_{\mathcal{X}}$.
2. We say ϕ is $\mathbb{R}$-regular if it is weakly regular and ψ is an $\mathbb{R}$-linear function on the simplicial set $\Delta_{\mathcal{X}}$.

Theorem 2.20. *[Li20] Let $\mathcal{X}$ be a maximally degeneration family of polarized CY manifolds. We use X^{an} be the Berkovich space over $\mathbb{C}((t))$, and ω to denote the Lebesgue measure on the essential skeleton $\mathrm{Sk}(X)$. Let ϕ be the solution to the NA MA equation*

$$MA(\phi) = \omega$$

If ϕ is weakly regular, then the weak metric SYZ conjecture holds for the family $\mathcal{X}$.

Question 2.21. Let X be a smooth projective variety over $k((t))$ where k is an algebraically closed field of characteristic zero. Let X^{an} be the analytification of X, $\mathrm{Sk}(X)$ be the essential skeleton, ω be the Lebesgue measure on $\mathrm{Sk}(X)$. Then is the solution to the equation

$$MA(\phi) = \omega$$

always weakly regular?

Example 2.22.
1. ([Thu05]) The weakly regularity properties holds for smooth projective curves over $k((t))$.
2. ([Liu11]) The maximally degenerating family of Abelian varieties satisfies the weakly regular properties in the above question.
3. ([HJMM22] [PS22]) The Fermat family in Example 2.16 satisfies the weakly regular properties in the above question. In their paper, they answered the above question positively for a more general class of CY hypersurfaces in certain toric varieties.

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