Introduction au domaine de recherche: Scaling relations in the near-critical planar random-cluster model

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Contents

1 Random-cluster model and phase transition, basic set-up

Definition of the model The random-cluster model (also called Fortuin-Kasteleyn percolation) is an example of a percolation model, introduced by Fortuin and Kasteleyn around 1970 [FK72] as a generalization of Bernoulli percolation. It was found to be related to many other models of statistical mechanics, including the Ising and Potts models, and to exhibit a very rich critical behavior.

For background, we direct the reader to the monograph [Gri06] and to the lecture notes [DC17] for most recent results.

Consider the square lattice $(\mathbb{Z}^2, \mathbb{E})$, that is the graph with vertex-set $\mathbb{Z}^2 = \{(n,m) : n,m \in \mathbb{Z}\}$ and edges between nearest neighbours. In a slight abuse of notation, we will write $\mathbb{Z}^2$ for the graph itself.

Write Λ_n for the subgraph of $\mathbb{Z}^2$ spanned by the vertex-set $\{-n,\dots,n\}^2$. For $1 \le r < R$, write $\mathrm{Ann}(r,R)$ for the annulus $\Lambda_R \setminus \Lambda_{r-1}$. We also write $\Lambda_n(x)$ and $\Lambda_n(e)$ for the boxes of size n recentred around x and the bottom left endpoint of the edge e, respectively.

Consider a finite subgraph $G = (V,E)$ of the square lattice (V denotes the vertex-set and E the edge-set) and let ∂G be the set of vertices in V incident to at most three edges in E.

A random-cluster configuration ω on G is an element of $\{0,1\}^E$. An edge e is said to be *open* (in ω) if $\omega_e = 1$, otherwise it is *closed*. We then consider the following probability measure on these configurations.

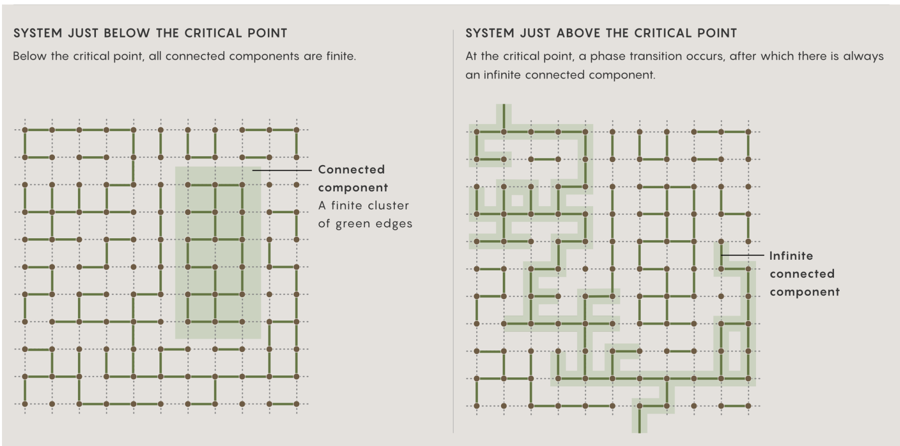

Figure 1: On the left, a typical configuration of subcritical random-cluster model; on the right, a typical configuration of supercritical random-cluster model. Green solid edges are opened, dotted edges are closed. Clusters are shows in light green. Picture by Merrill Sherman/Quanta Magazine.

Definition 1.1. The random-cluster measure on G with edge-weight p, cluster-weight $q > 0$ and boundary conditions ξ is given by

$$\phi^{\xi}_{G,p,q}[\omega] = \frac{1}{Z^{\xi}(G,p,q)}(\tfrac{p}{1-p})^{|\omega|}q^{k(\omega^{\xi})}, \tag{1.1}$$

where $|\omega| = \sum_{e\in E}\omega_e$, $k(\omega)$ is the number of connected components of the graph, ω^{ξ} is the graph obtained from ω by identifying wired vertices together, and $Z^{\xi}(G,p,q)$ is a normalising constant called the *partition function* chosen in such a way that $\phi^{\xi}_{G,p,q}$ is a probability measure.

Remark 1.1. When $q = 1$, the model becomes rather simple: each edge is opened with probability p independently. Such model is called the Bernoulli percolation, and has been studied for a long time (See e.g. [Gri06]). Moreover, the model can be defined for any G, but in this report we only focus on the case where G is a subgraph of $\mathbb{Z}^2$.

A configuration ω can be seen as a subgraph of G with vertex-set V and edge-set $\{e \in E : \omega_e = 1\}$. When speaking of connections in ω, we view ω as a graph. For sets of vertices A and B, we say that A is connected to B if there exists a path of edges of ω with endpoints that connect a vertex of A to a vertex of B. This event is denoted by $A\longleftrightarrow B$. We also speak of connections in a set of vertices C if the endpoints of the edges of the path are all in C.

A *cluster* is a connected component of ω. The *boundary conditions* ξ on G are given by a partition of ∂G. We say that two vertices of G are *wired together* if they belong to the same element of the partition ξ.

Two specific families of boundary conditions will be of special interest to us. On the one hand, the *free* boundary conditions, denoted 0, correspond to no wiring between boundary vertices. On the other hand, the *wired* boundary conditions, denoted 1, correspond to all boundary vertices being wired together.

For $q \geq 1$ and $i = 0, 1$, the family of measures $\phi^{i}_{G,p,q}$ converges weakly as G tends to $\mathbb{Z}^2$. The limiting measures on $\{0,1\}^{\mathbb{E}}$ are denoted by $\phi^{i}_{\mathbb{Z}^2,p,q}$ and are called *infinite-volume* random-cluster measures with free (wired) boundary conditions, and are proved to be

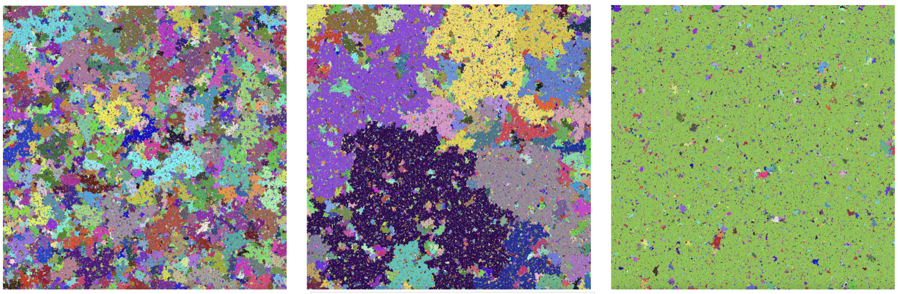

Figure 2: A typical configuration of Bernoulli percolation model for $p < p_c$, $p = p_c$, $p > p_c$. Clusters are shown in different colors. Picture by Raphaël Cerf.

invariant under translations and ergodic. Moreover, it is proved that $\phi^0_{\mathbb{Z}^2,p,q} = \phi^1_{\mathbb{Z}^2,p,q}$ for $p \neq p_c$.

Phase transition and near critical behavior As a percolation model, the random-cluster model undergoes a phase transition at a critical parameter $p_c = p_c(q)$ in the following sense.

Consider

$$\theta(p) := \phi^1_{\mathbb{Z}^2,p,q}[0 \text{ is in an infinite cluster}].$$

- When $p < p_c(q)$, there are no infinite cluster, so $\theta(p) = 0$. Moreover, all the clusters are typically small.
- When $p > p_c(q)$, there is a unique infinite cluster that occupies a positive portion of the whole space, so $\theta(p) > 0$. Moreover, excluding this large cluster, all other clusters are typically small, like the sub-critical case.
- When $p = p_c$ things become more subtle, and the behavior varies in the model. In the random-cluster models for $q \geq 1$, it was proved in [DCST17, DCGH$^+$21] (see also [RS20] when $q > 4$) that $\phi^0_{\mathbb{Z}^2,p_c,q} = \phi^1_{\mathbb{Z}^2,p_c,q}$, and the correlation length(defined below) at p_c is infinite if and only if $q \leq 4$.

For the random-cluster model on $\mathbb{Z}^2$, considerable progress has been made in the past ten years in the understanding of this phase transition. It was proved in [BDC12] (see also [DCM16, DCRT16]) that

$$p_c(q) = \frac{\sqrt{q}}{1+\sqrt{q}}.$$

It was also shown in these papers that the *correlation length*

$$\xi(p) := \lim_{n\to\infty} -n/\log[\pi_1(p,n) - \theta(p)] \in [0,\infty] \tag{1.2}$$

is finite as soon as $p \neq p_c$, where

$$\pi_1(p,n) := \phi^1_{\mathbb{Z}^2,p,q}[0 \longleftrightarrow \partial\Lambda_n] \tag{1.3}$$

(when $p = p_c$ we drop p_c from this notation).

Generally speaking, the correlation length characterizes the "size" of the random-cluster model at p. It is expected that when looking at the "window" of size $\xi(p)$ for $p \to p_c$, the behavior will be chosen and converge to a limit (i.e. the near-critical scaling limit mentioned in open question section).

Monotonic property and Spacial Markov property The following properties serve as the main tool of understanding the random-cluster model properly. They can be found in [Gri06], and we only recall them briefly below. Fix a subgraph $G = (V, E)$ of $\mathbb{Z}^2$.

Monotonic properties. An event A is called *increasing* if for any $\omega \leq \omega'$ (for the partial ordering on $\{0,1\}^E$), $\omega \in A$ implies that $\omega' \in A$. Fix $q \geq 1$, $1 \geq p' \geq p \geq 0$, and some boundary conditions $\xi' \geq \xi$, where $\xi' \geq \xi$ means that any wired vertices in ξ are also wired in ξ'. Then, for every increasing events A and B,

$$\phi^{\xi}_{G,p,q}[A \cap B] \geq \phi^{\xi}_{G,p,q}[A]\phi^{\xi}_{G,p,q}[B], \tag{FKG}$$

$$\phi^{\xi}_{G,p',q}[A] \geq \phi^{\xi}_{G,p,q}[A], \tag{p-MON}$$

$$\phi^{\xi'}_{G,p,q}[A] \geq \phi^{\xi}_{G,p,q}[A]. \tag{CBC}$$

The inequalities above will respectively be referred to as the *FKG inequality*, the *monotonicity in* p, and the *comparison between boundary conditions*. A basic application of (CBC) is that $\phi^1_{\mathbb{Z}^2,p,q}[A] \geq \phi^0_{\mathbb{Z}^2,p,q}[A]$ for any increasing event A. As a remark, these monotonic properties serve as the base point of the whole theory. For $q < 1$, monotonicity fails and most theories cannot be extended.

Spatial Markov property. For any configuration $\omega' \in \{0,1\}^E$ and any $F \subset E$,

$$\phi^{\xi}_{G,p,h}[\cdot_{|F} \,|\, \omega_e = \omega'_e, \forall e \notin F] \geq \phi^{\xi'}_{H,p,h}[\cdot], \tag{SMP}$$

where H denotes the graph induced by the edge-set F, and ξ' the boundary conditions on H defined as follows: x and y on ∂H are wired if they are connected in $(\omega')^{\xi}_{|E\setminus F}$.

2 Physical picture, renormalization

Critical exponents In physics, the near-critical behavior of the model is expected to be encoded by by certain observables, which naturally decay algebraically. Such observables are called *critical exponents*. Some of them, named α, β, γ, η, ν, ζ and ξ_1, are defined as follows (below $o(1)$ denotes a quantity tending to 0):

$$\begin{aligned}
f''(p) &= |p - p_c|^{-\alpha+o(1)} && \text{as } p \to p_c,\\
\theta(p) &= (p - p_c)^{\beta+o(1)} && \text{as } p \searrow p_c,\\
\chi(p) &= |p - p_c|^{-\gamma+o(1)} && \text{as } p \to p_c,\\
\phi_{p_c}[0 \longleftrightarrow x] &= |x|^{-\eta+o(1)} && \text{as } |x| \to \infty,\\
\pi_1(R) &= R^{-\xi_1+o(1)} && \text{as } R \to \infty,\\
\xi(p) &= |p - p_c|^{-\nu+o(1)} && \text{as } p \to p_c,\\
\phi_{p_c}[|\mathsf{C}| \geq n] &= n^{-\zeta+o(1)} && \text{as } n \to \infty,
\end{aligned}$$

where $|\mathsf{C}|$ is the number of vertices in the cluster C of the origin, and $f(p)$ and $\chi(p)$ are the thermodynamical quantities respectively called the *free-energy* and the *susceptibility* defined[1] by

$$f(p) := \lim_{n\to\infty} -\tfrac{1}{|\Lambda_n|}\log Z^0(\Lambda_n, p),$$
$$\chi(p) := \phi_p[|\mathsf{C}|\mathbb{1}_{|\mathsf{C}|<\infty}].$$

Let us mention that the first equation only applies when $f''(p)$ diverges as p approaches p_c, which is to say that the phase transition is of second order.

These exponents are quantities of central interest in physics and have been the object of many studies. A beautiful prediction is that these exponents should depend on each other via *scaling relations*. We list some of them here:

$$\eta = 2\xi_1, \qquad (\mathbf{R1})$$
$$\zeta = \xi_1/(2-\xi_1), \qquad (\mathbf{R2})$$
$$\beta = \nu\xi_1, \qquad (\mathbf{R3})$$
$$\gamma = (2-2\xi_1)\nu, \qquad (\mathbf{R4})$$
$$\alpha = 2-2\nu. \qquad (\mathbf{R5})$$

An important feature of the relations above is that they are independent of q: the exponents vary from model to model, but not the formulae. Relations **R1–5** were proved for Bernoulli percolation (i.e. cluster-weight $q = 1$) in a milestone paper by Harry Kesten [Kes87] without any of them being computed, or indeed even be shown to exist (see also [Nol07, GPS18, DCMT21]). Let us mention that similar relations should hold in all dimensions that are below the so-called *upper-critical dimension* (with certain values of 2 replaced by the dimension d).

Renormalization group formalism The arrival of the renormalization group formalism (see [Fis98] for a historical exposition) led to a new era of understanding statistical physics models (albeit mostly non-rigorous).

A naive idea of renormalization is to consider the "block-spin" configuration of the original one (replacing a block of neighboring sites with one site having a spin equal to the dominant spin in the block, see figure 3) This new model can be *approximately* viewed as the original one with a change of scales and parameters (i.e. p in our situation). If you emit the approximation, we just find a way to compare observables for different p, which characterize the critical exponents in a non-precise way. (See [Car96, 3.1]) for a more detailed review of block-spin renormalization)

Conformal invariance at criticality The critical point arises as the fixed point of the renormalization transformations. Specifically, under simple rescaling the model at the critical temperature should converge to a scaling limit, a "continuous" version of the originally discrete model, corresponding to a quantum field theory. This leads to the idea of universality: models on different regular lattices or even more general planar graphs belong to the same renormalization space, with a unique critical point, and so at criticality, the scaling limit should always be the same (be independent of the lattice while the critical temperature depends on it).

Being unique, the scaling limit at the critical point must satisfy translation, rotation, and scale invariance, leading to the conformal invariance of the critical field.

Understand the conformal invariance of the critical model has always been a huge problem in mathematical physics, see [Smi10, CDCH$^+$14] for some approach for the Ising

[1] The definition of $f(p)$ for $q = 1$ is slightly different and is given by $f(p) := \phi_p[1/|\mathsf{C}|]$.

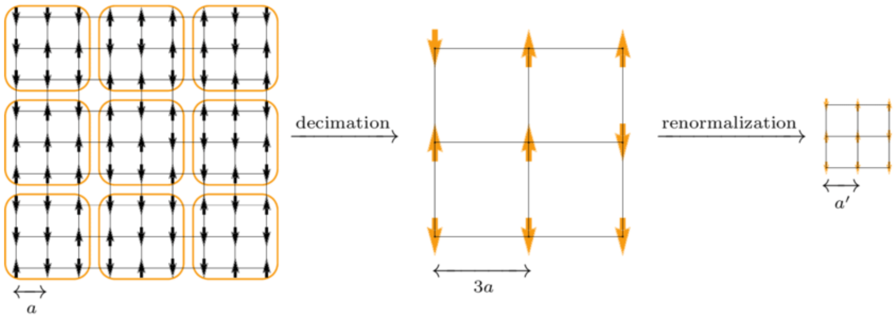

Figure 3: A picture of block-spin renormalization of the Ising model. Picture by Charlie Duclut.

model. In case of random-cluster model, a very recent progress is the rotation invariance [DCKK+20].

3 Mathematical approach to 2D: Russo-Seymour-Welsh theory

As we explained in the physical picture, the critical random-cluster model is expected to be scaling invariant, which is the starting point of the mathematical treatment of models at criticality. Mathematically, we will need to use crossing estimates in rectangles and more generally in quads, as well as estimates on certain universal and non-universal critical exponents. Such crossing estimates initially emerged in the study of Bernoulli percolation in the late 1970s under the coined name of Russo-Seymour-Welsh (RSW for short) theory. [Rus78, SW78]

Consider the "crossing event" that there exists a crossing in a box $[0, \rho n] \times [0, n]$ (denoted by $\mathcal{C}([0, \rho n] \times [0, n])$). It is predicted that when $p > p_c$ (resp. $p < p_c$), the probability of this event will converge to 1 (resp. 0) when n converge to infinity; and when $p = p_c$, it will converge to some constant related to ρ.

As a remark, estimate on such crossing probabilities at the self-dual point $p = \frac{\sqrt{q}}{1+\sqrt{q}}$ serves as the main idea to determine p_c [BDC12].

Theorem 3.1 (Crossing estimates)**.** For $\rho, \epsilon > 0$, there exist $c' > 0$ and $c = c(\rho, \epsilon) > 0$ such that for $i = 0, 1$,

$$c \leq \phi^i_{\mathbb{Z}^2, p_c}[\mathcal{C}([0, \rho n] \times [0, n])] \leq 1 - c. \tag{RSW-crit}$$

Go to the near-critical regime In the near-critical regime, we don't expect that the field is scaling invariant, but we expect a local limit at the window of the correlation length $\xi(p)$. However, $\xi(p)$ itself is not easy to characterize, and that's why the *Characteristic length* is introduced, base on the RSW theory.

The Characteristic length Now, fix a constant $\delta > 0$ small enough (independent of $q \in [1, 4]$ and can easily be estimated).

Definition 3.1 (Characteristic length)**.** For each $q \geq 1$ and $p \in (0, 1)$, let

$$L(p) = L(p, q) := \inf\{R \geq 1 : \phi_p[\mathcal{C}(\Lambda_R)] \notin [\delta, 1 - \delta]\} \in [1, \infty]. \tag{3.1}$$

Note that $L(p) < +\infty$ for every $p \neq p_c$ by [BDC12]; by duality, $L(p_c) = +\infty$ as long as $\delta < 1/2$, which we will always assume. The interest of the characteristic length lies in its connection with the *scaling window*, i.e. the regime of parameters (R, p) for which one expects typical properties of the random-cluster model in Λ_R with parameters p to be similar to the critical ones. In physics, the statement that the system looks critical is usually related to another length-scale, namely the correlation length $\xi(p)$ defined in (1.2). The correlation length encodes the rate of exponential decay of the probability of being connected to distance n but not to infinity as n tends to infinity; it is not a priori directly related to $L(p)$. Nevertheless, the following result reunites the two notions of correlation and characteristic lengths, thus affirming that the characteristic length is simply the correlation length in disguise.

Theorem 3.2 (Equivalence correlation/characteristic lengths)**.** Fix $1 \leq q \leq 4$, we have that for $p \in (0, 1)$,

$$L(p) \asymp \xi(p). \tag{3.2}$$

The proof is based on a coarse-grained procedure.

RSW in near-critical The RSW theory simply extends to the near-critical case, with the scale $n < L(p)$, which serves as the main technical tool.

Theorem 3.3 (Crossing estimates below the characteristic length)**.** For $\rho, \epsilon > 0$, there exist $c' > 0$ and $c = c(\rho, \epsilon) > 0$ such that for every p, every $1 \leq n \leq L(p)$, every graph G containing $[-\epsilon n, (\rho + \epsilon)n] \times [-\epsilon n, (1 + \epsilon)n]$ and every boundary conditions ξ,

$$c \leq \phi^{\xi}_{G,p}[\mathcal{C}([0, \rho n] \times [0, n])] \leq 1 - c. \tag{RSW}$$

Moreover, if Ann_n denotes the event that there exists an open circuit surrounding Λ_n in $\mathrm{Ann}(n, 2n)$,

$$\phi^{0}_{\mathrm{Ann}(2n,n),p}[\mathrm{Ann}_n] \geq c' > 0. \tag{RSW'}$$

4 A mathematical proof of scaling relations: intuition

In this section, we will give an intuitive proof of the scaling relations (**R3**), i.e. $\beta = \nu\xi_1$. The statement is given as follows. ([DCM22, (1.25)]).

Theorem 4.1. Fix $1 < q \leq 4$. For $p > p_c$,

$$\theta(p) \asymp \pi_1(\xi(p)). \tag{4.1}$$

Here we use the notation $\asymp$, $\gtrsim$, $\lesssim$ as follows. For two families $(f_i)_{i\in I}$ and $(g_i)_{i\in I}$, introduce $f \asymp g$ (resp. $f \lesssim g$ and $f \gtrsim g$) to refer to the existence of constants $c, C \in (0, \infty)$ such that for every $i \in I$, $cg_i \leq f_i \leq Cg_i$ (resp. $f_i \leq Cg_i$ and $f_i \geq cg_i$). In most cases, the family I will be obvious from context and omitted. In the special case where I contains (implicitly or explicitly) the edge-parameter $p \in (0, 1)$, we in fact further require that p is not close to 0 or 1 (which is justified for every application that we have in mind since we are interested in properties for p close to p_c).

Let us first explain why we have (4.1) induce (**R3**). In fact, simply by definition,

$$|p - p_c|^{\beta+o(1)} = \theta(p) \asymp \xi(p)^{-\xi_1+o(1)} = |p - p_c|^{\nu\xi_1+o(1)}.$$

Remark 4.1. We should notice that there is a cheat in these computations, i.e. we don't prove such critical exponents really exist. So this is a conditional proof.

We first claim that

$$\theta(p) \asymp \pi_1(p, L(p)). \tag{4.2}$$

Proof of the claim. It is clear that $\theta(p) \leq \pi_1(p, L(p))$ and we only need to prove the inverse.

Now, by definition of $L(p)$, and simply apply (FKG) we have $\phi_p[\mathrm{Ann}_{L(p)}] \gtrsim 1$.

Moreover, we want to show that

$$\phi_p[\Lambda_{L(p)} \longleftrightarrow \infty] \gtrsim 1.$$

The proof basically follows the idea of Peierls' argument. By choosing the constant δ in the definition of $L(p)$ properly small, the probability that two boxes of size $L(p)$ and distance at least $L(p)$ are linking together by a dual path is dominated by a small constant $c(\delta)$.

Suppose that the event $(\Lambda_{L(p)} \longleftrightarrow \infty)^c$ happens, there must be a circuit that disconnect $\Lambda_{L(p)}$ to ∞. If we separate space into blocks of size $L(p)$, the existence of such circuit means a series of such blocks are connected in the dual. For a fix list of n blocks, such event happens with probability at most $c(\delta)^n$. Finally, by summing over all possible choice of blocks (at most $Cn^2 8^n$) and all n, pick δ sufficiently small we get a uniform upper bound smaller than 1.

Finally, by (FKG) we have

$$\theta(p) \geq \phi_p[0 \longleftrightarrow \partial\Lambda_{2L(p)}, \mathrm{Ann}_{L(p)}, \Lambda_{L(p)} \longleftrightarrow \infty] \gtrsim \pi_1(p, 2L(p)) \gtrsim \pi_1(p, L(p)).$$

□

Problem comes to show that $\pi_1(p, L(p)) \asymp \pi_1(L(p))$. Generally speaking, we don't have any good idea to directly compare an event under the measure ϕ_p and ϕ_{p_c}, except that to calculate the difference between them. So we want to show that

$$\log \pi_1(p_c, L) - \log \pi_1(p, L) \asymp 1,$$

where $L = L(p)$.

Now, notice that for any function $X : \{0,1\}^{\mathbb{Z}^2} \to \mathbb{R}$, simply by calculation

$$\frac{d}{dp}\phi^1_{\mathbb{Z}^2,p,q}[X] = \frac{1}{p(1-p)} \sum_{e \in \mathbb{Z}^2} \mathrm{Cov}(X, \omega_e). \tag{DF}$$

And hence

$$\frac{d}{dp} \log \pi_1(p, L) \asymp \frac{\sum_{e \in \mathbb{Z}^2} \mathrm{Cov}(A_L, \omega_e)}{\pi_1(p, L)} = \phi^1_{p,q}[\omega_e = 1 | A_L] - \phi^1_{p,q}[\omega_e = 1], \tag{4.3}$$

Where A_L is the event that $0 \longleftrightarrow \Lambda_L$.

the only thing we can do is to check the difference between these two and write it in derivatives about p. That is, we want to show that for $L < L(p)$, $\log(\pi_1(p_c, L) - \log(\pi_1(p, L)) < \infty$. Using the fact that

$$\log(\pi_1(p_c, L) - \log(\pi_1(p, L)) \asymp \int_p^{p_c} \sum_{e \in \mathbb{Z}^2} \phi^1_{p',q}[\omega_e = 1 | A_L] - \phi^1_{p',q}[\omega_e = 1] \; dp'.$$

So the remaining task is to give a bound on

$$\phi^1_{p',q}[\omega_e = 1|A_L] - \phi^1_{p',q}[\omega_e = 1]. \tag{4.4}$$

Since both A_L and $\omega_e = 1$ are increasing events, by (FKG), the value above is non-negative. However, the usual RSW theory only enables us to compare between events, while here we are considering the difference between measures.

Let $R := d(e, \Lambda_L)$. It is expected (and not difficult to show) that when $R > L(p')$, (4.4) decays exponentially in R. However, when $L < R < L(p')$, it is expected to have a polynominal decay in R, so the exact exponent become crucial. In order to treat this, a crucial observable, i.e. the *mixing rate* comes out.

The mixing rate In [DCM22], the mixing rate is introduced in order to characterize the covariance between events.

Definition 4.1 (Mixing rate)**.** For $1 < q \leq 4$, $1 \leq r < R$, $p \in (0, 1)$ and e an edge incident to the origin, write

$$\Delta_p(R) := \phi^1_{\Lambda_R,p}[\omega_e] - \phi^0_{\Lambda_R,p}[\omega_e], \tag{4.5}$$

$$\Delta_p(r, R) := \phi^1_{\Lambda_R,p}[\mathcal{C}(\Lambda_r)] - \phi^0_{\Lambda_R,p}[\mathcal{C}(\Lambda_r)]. \tag{4.6}$$

The quantity $\Delta_p(R)$ is called the *mixing rate.*

We list here some properties of the mixing rate without proof.

Theorem 4.2 (Properties of the mixing rate)**.** Fix $1 < q \leq 4$.

(i) (Quasi-multiplicativity) For every $p \in (0, 1)$ and $1 \leq r \leq R \leq L(p)$,

$$\Delta_p(r)\Delta_p(r, R) \asymp \Delta_p(R). \tag{4.7}$$

(ii) (Stability below the characteristic length) For every $p \in (0, 1)$ and $1 \leq R \leq L(p)$,

$$\Delta_p(R) \asymp \Delta_{p_c}(R). \tag{4.8}$$

(iii) (Mixing interpretation) For every $1 \leq 2r \leq R \leq L(p)$,

$$\Delta_p(r, R) \asymp \max\Big\{\Big|\frac{\phi_p[A \cap B]}{\phi_p[A]\phi_p[B]} - 1\Big| \;:\; A \in \mathcal{F}(\Lambda_r) \text{ and } B \in \mathcal{F}(\mathbb{Z}^2 \setminus \Lambda_R)\Big\}, \tag{4.9}$$

where $\mathcal{F}(S)$ is the σ-algebra generated by the edges with both endpoints in S.

(iv) (Covariance characterization) For every $p \in (0, 1)$, $1 \leq r \leq R \leq L(p)$, and two edges e, f with $d(e, f) = R$,

$$\mathrm{Cov}_p(\omega_e, \omega_f) \asymp \Delta_p(R)^2. \tag{4.10}$$

Back to our problem, property (iii) tell us that when $L < R < L(p')$,

$$\phi^1_{p',q}[\omega_e = 1|A_L] - \phi^1_{p',q}[\omega_e = 1] \lesssim \Delta_{p'}(R).$$

This is not the best bound we have, but at least good to treat since (ii) deduce p case into p_c case. In fact, the right-hand-side can be improved to $\Delta_{p'}(R)\Delta_{p'}(L, R)$, which is expected to be optimal. Finally, when $R < L$, a more careful analysis should be applied, but the idea is the same.

To conclude this section, we mention that the mixing rate has also a critical exponent, namely $\Delta_{p_c}(R) = R^{-\iota+o(1)}$. It is predicted that $\iota = \frac{3\kappa}{8} - 1$ where $\kappa(q) := 4\pi/\arccos(-\frac{\sqrt{q}}{2})$. Although this has not been proved, some good estimations of ι are given in [DCM22], and scaling relations related to ι are shown.

5 Open questions

We only mention 2 big open problems in this field.

It is expected that the field of the near-critical window (of scale $L(p)$) will converge when $p \to p_c$. For Bernoulli percolation, such scaling limit has been studied in [GPS18]. For the critical random cluster model with $1 \leq q \leq 4$, a special value is when $q = 2$ (related to the Ising model), such that extra observables can be constructed and almost everything is known on the square lattice, including the conformal invariance of the model and its interfaces [Smi10, CDCH+14]. It is therefore natural to discuss the question of the construction of the near-critical scaling limit in this context, especially since one expects subtle differences with the corresponding result for Bernoulli percolation.

Question 1. Construct the near-critical scaling limit of the model, i.e. the limits of random-cluster models on $\frac{1}{R}\mathbb{Z}^2$ at p such that R is of order of $\lambda L(p)$, where λ is a fixed strictly positive parameter. One may start by studying the case of $q = 2$.

Recently, rotational invariance of the critical random-cluster model was obtained in [DCKK+20]. This rotational invariance is expected to carry over to the near-critical regime, leading to the following question.

Question 2. Prove that the near-critical scaling limit of the model is invariant under rotations.

References

[BDC12] Vincent Beffara and Hugo Duminil-Copin. The self-dual point of the two-dimensional random-cluster model is critical for q $\geq$ 1. *Probability Theory and Related Fields*, 153(3-4):511–542, 2012.

[Car96] John Cardy. *Scaling and renormalization in statistical physics*, volume 5. Cambridge university press, 1996.

[CDCH+14] Dmitry Chelkak, Hugo Duminil-Copin, Clément Hongler, Antti Kemppainen, and Stanislav Smirnov. Convergence of ising interfaces to schramm's sle curves. *Comptes Rendus. Mathématique*, 352(2):157–161, 2014.

[DC17] Hugo Duminil-Copin. Lectures on the ising and potts models on the hypercubic lattice. In *PIMS-CRM Summer School in Probability*, pages 35–161. Springer, 2017.

[DCGH+21] Hugo Duminil-Copin, Maxime Gagnebin, Matan Harel, Ioan Manolescu, and Vincent Tassion. Discontinuity of the phase transition for the planar random-cluster and potts models with q¿ 4. In *Annales Scientifiques de l'Ecole Normale Supérieure*, volume 54, pages 1363–1413. Société Mathématique de France, 2021.

[DCKK+20] Hugo Duminil-Copin, Karol Kajetan Kozlowski, Dmitry Krachun, Ioan Manolescu, and Mendes Oulamara. Rotational invariance in critical planar lattice models. *arXiv preprint arXiv:2012.11672*, 2020.

[DCM16] Hugo Duminil-Copin and Ioan Manolescu. The phase transitions of the planar random-cluster and potts models with q$\geq$ 1 are sharp. *Probability Theory and Related Fields*, 164(3-4):865–892, 2016.

[DCM22] Hugo Duminil-Copin and Ioan Manolescu. Planar random-cluster model: scaling relations. In *Forum of Mathematics, Pi*, volume 10, page e23. Cambridge University Press, 2022.

[DCMT21] Hugo Duminil-Copin, Ioan Manolescu, and Vincent Tassion. Near critical scaling relations for planar bernoulli percolation without differential inequalities. *arXiv preprint arXiv:2111.14414*, 2021.

[DCRT16] Hugo Duminil-Copin, Aran Raoufi, and Vincent Tassion. A new computation of the critical point for the planar random-cluster model with q ≥ 1. *Annales de l'Institut Henri Poincaré, Probabilités et Statistiques*, 54, 04 2016.

[DCST17] Hugo Duminil-Copin, Vladas Sidoravicius, and Vincent Tassion. Continuity of the phase transition for planar random-cluster and potts models with 1 ≤ q ≤ 4 1≤ q≤ 4. *Communications in Mathematical Physics*, 349:47–107, 2017.

[Fis98] Michael E Fisher. Renormalization group theory: Its basis and formulation in statistical physics. *Reviews of Modern Physics*, 70(2):653, 1998.

[FK72] Cornelius Marius Fortuin and Piet W Kasteleyn. On the random-cluster model: I. introduction and relation to other models. *Physica*, 57(4):536–564, 1972.

[GPS18] Christophe Garban, Gábor Pete, and Oded Schramm. The scaling limits of near-critical and dynamical percolation. *Journal of the European Mathematical Society*, 20(5):1195–1268, 2018.

[Gri06] Geoffrey Grimmett. *The random-cluster model*, volume 333. Springer, 2006.

[Kes87] Harry Kesten. Scaling relations for 2 d-percolation. *Communications in Mathematical Physics*, 109:109–156, 1987.

[Nol07] Pierre Nolin. Near-critical percolation in two dimensions. *Electronic Journal of Probability*, 13, 12 2007.

[RS20] Gourab Ray and Yinon Spinka. A short proof of the discontinuity of phase transition in the planar random-cluster model with $q > 4$. *Communications in Mathematical Physics*, 378, 08 2020.

[Smi10] Stanislav Smirnov. Conformal invariance in random cluster models. i. holmorphic fermions in the ising model. *Annals of Mathematics*, 172:1435–1467, 09 2010.