Character varieties and moduli space of L-parameters

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1 Introduction

The research is about some properties of character varieties: the base change theorem and the finiteness of morphisms and its applications in the local Langlands program. For an abstract group Γ and a group scheme G (over a base scheme S), the character variety associated to Γ and G is roughly the moduli space of all conjugate classes of semisimple representations $\Gamma \to G$. More precisely, it is the GIT (geometric invariant theory) quotient $\mathrm{Hom}(\Gamma, G) /\!\!/ G$, where G acts by conjugation on the space of all homomorphisms from Γ to G.

The study of character varieties has its own interest in representation theory and geometric invariant theory. Our motivation of the study of the character varieties comes from the local Langlands program, which will be introduced in section 2.

We will introduce and prove two results in the theory of character varieties: the base change theorem of character varieties and the finiteness of morphisms of character varieties. In section 3 we introduce some basic notions of character varieties and state the two results. In section 4 we sketch the proof of the base change theorem and in section 5 we sketch the proof of the finiteness theorem.

Finally we return to the local Langlands program. In section 6 and 7 we will provide some perspectives of the local Langlands programs from our results, though many things in this domain is still unclear.

I sincerely express my gratitude to Jean-François Dat, who supervised me to do these research.

2 Motivation

In this section we introduce our motivation of the study of character varieties from the local Langlands program.

The aim of the local Langlands program is to correspond the automorphic representations to the Galois representations. More precisely, let F be a finite extension of the p-adic number field $\mathbb{Q}_p$ and G be a (split) reductive group (for example, $G = \mathrm{GL}_n$) over F. The aim of the local Langlands program is to establish a correspondence (called the local Lagnlands correspondence):

$$\{\text{irreducible smooth representations of } G(F)\} \to \{\text{continuous representations of } W_F \text{ in } G^\vee\},$$

where W_F is the Weil group, a slight modification of the absolute Galois group $\mathrm{Gal}(\bar{F}/F)$ and $G^\vee$ is the Langlands dual group of G (for example, if $G \simeq \mathrm{GL}_n$ then

$G^\vee \simeq \mathrm{GL}_n$), which is also a reductive group. The representation of W_F in $G^\vee$ is called the "L-parameters" (if G is not split, the objects on the right hand side are not the representations but the "1-cocycles").

Some special cases of the local Langlands correspondence are constructed. For example, Henniart ([Hen00]) constructed the case for $G = \mathrm{GL}_n$ is a purely algebraic way. In recent years, Fargues and Scholze ([FS21]) constructed a geometric version of the local Langlands correspondence, which provides a ring homomorphism between the function rings on both sides:

$$\mathcal{O}(\mathrm{Hom}(W_F, G^\vee) /\!\!/ G^\vee) \to \mathcal{Z}(G(F)),$$

where the left hand side is the function ring of the moduli space of L-parameters and the right hand side is the Bernstein center of the category of $G(F)$-representations, which can be viewed as the function ring on the "space of irreducible representations". The ring homomorphism induces a map from irreducible representations of $G(F)$ to semisimple representations of W_F in $G^\vee$ and Fargues and Scholze proved that their correspondence is compatible to the Henniart's (up to a semisimplification).

In the Fargues–Scholze correspondence, we are interested in the moduli space of L-parameters. The moduli space of L-parameters is the character variety associated to W_F and $G^\vee$ (not precisely, there are issues about the continuity) and we can use the results on character varieties to deal with it. Moreover, the local Langlands correspondence may enable us to deduce some results in representations theory of p-adic groups.

Our first result is the base change theorem of character varieties. In the context of Langlands program, the base ring of the character variety corresponds to the coefficient ring of the representations of $G(F)$ and W_F (in the automorphic and Galois side, respectively). Thus the base change theorem may give us some relations of representations of different coefficient rings (the case we will mainly focus on is the base change from $\mathbb{Z}_l$ to $\mathbb{F}_l$ for a prime $l \neq p$).

The second result is the finiteness of morphisms between character varieties for G and its subgroup H. This may provides us some relations between representations on different groups. Using a theorem of Dat–Helm–Kurinczuk–Moss [DHKM23] about restricted Bernstein centers, we deduce a finiteness theorem between restricted Bernstein centers, which is a result in pure representation theory of p-adic groups.

3 Main results

In this section we will give the statements and intuitions of the main results. We firstly precisely define the character variety. Let Γ be an abstract group and G be a group scheme over a base scheme S (the notion of group scheme is a generalization of the notion of algebraic group over an arbitrary base). We remark that all rings in this article are commutative.

Definition 3.1. *The character variety associated to Γ and G is the GIT quotient scheme*

$$\mathrm{Hom}(\Gamma, G) /\!\!/ G,$$

where:

- $\mathrm{Hom}(\Gamma, G)$ *is the scheme whose A-points for a ring A are the group homomorphisms* $\mathrm{Hom}(\Gamma, G(A))$ *(for example, if $\Gamma = F^N$ be the free group of N generators then* $\mathrm{Hom}(\Gamma, G) \simeq G^N$*).*

- *The action of G is the conjugation and the GIT quotient is defined as follows: if S is affine the quotient is the spectrum of invariant functions under the action of G*

$$\mathrm{Hom}(\Gamma, G) /\!\!/ G := \mathrm{Spec}(\mathcal{O}(\mathrm{Hom}(\Gamma, G))^G)$$

and for general S we define it by gluing.

The inclusion of function rings induces a morphism of schemes

$$\mathrm{Hom}(\Gamma, G) \to \mathrm{Hom}(\Gamma, G) /\!\!/ G.$$

The quotient map can be roughly described in geometry as the following: for a representation $\Gamma \to G$, we send it to the conjugate class of its semisimplification.

3.1 The base change result

Let G be a reductive group scheme over a base scheme S and $\Gamma = F^N$ be a free group. Then for a morphism $T \to S$ between schemes, the base change $G_T := G \times_S T$ is a group scheme over T and we have the following.

Theorem 3.2 (Base change)**.**

$$G^N /\!\!/ G \times_S T \simeq G_T^N /\!\!/ G_T$$

To illustrate this, we give an example where $G = \mathrm{GL}_n$, $S = \mathrm{Spec}(A)$ and $T = \mathrm{Spec}(B)$ are affine and $N = 1$. By Chevalley's theorem, we have

$$\mathrm{GL}_{n,A} /\!\!/ \mathrm{GL}_{n,A} \simeq \mathbb{G}_{m,A}^n / S_n \simeq \mathrm{Spec}(A[\sum x_i, \sum x_i x_j, \cdots, x_1 \cdots x_n][(x_1 \cdots x_n)^{-1}])$$

where $\mathbb{G}_m$ is the multiplication group. In the isomorphism, the symmetric polynomial of degree k corresponds to the coefficient of t^k of the function $M \mapsto \det(1+tM)$ over $\mathrm{GL}_{n,A}$, which is invariant under the conjugation. The case for $G_T \simeq \mathrm{GL}_{n,B}$ is similar. Thus the theorem holds in this case as both sides are of the same form.

The theorem in the case $S = \mathrm{Spec}(\mathbb{Z}_l)$ and $T = \mathrm{Spec}(\mathbb{F}_l)$ is implicitly mentioned and proved in the article of Fargues and Scholze ([FS21]), using the modular representation theory developed by Cline ([Cli77]), Koppinen ([Kop84]) and Mathieu ([Mat90]). Using some descent steps, we can reduce the case for general S and T to this special case.

3.2 The finiteness result

Let S be noetherian group scheme, G be a reductive group scheme over S and H be a closed subgroup scheme of G such that H is also reductive. Let Γ be a finitely generated group. Then we have the following.

Theorem 3.3 (Finiteness result)**.** *The morphism*

$$\mathrm{Hom}(\Gamma, H) /\!\!/ H \to \mathrm{Hom}(\Gamma, G) /\!\!/ G$$

is finite.

The notion "finite" is defined as follows.

Definition 3.4. *A morphism of schemes $f\colon S' \to S$ is called finite, if for any open affine subscheme $U = \operatorname{Spec}(A) \subset S$, $f^{-1}(U)$ is affine and of the form $\operatorname{Spec}(A')$, where A' is of finite type as an A-module.*

Roughly speaking, the notion "finite" geometrically means closed and that the inverse image of a point contains only finite points. For example, a quotient map $S \to S/\Gamma$ is finite, if Γ is a finite group.

Thus for algebraic groups $H \subset G$ over a field K, the theorem implies the following.

Theorem 3.5. *Given a group homomorphism $\rho\colon \Gamma \to G(K)$, there is at most finitely many conjugate classes (in $H(K)$) of homomorphisms $\sigma\colon \Gamma \to H(K)$ such that the image of σ in $G(K)$ is conjugate with ρ.*

We give an example of the result. Let $H \subset G$ be $\mathrm{GL}_k \subset \mathrm{GL}_n$ and $\Gamma = \mathbb{Z}$. By Chevalley's theorem, we have the following commutative diagram:

$$\begin{array}{ccc} H /\!\!/ H & \xrightarrow{\sim} & \mathbb{G}_m^k/S_k \\ \downarrow & & \downarrow \\ G /\!\!/ G & \xrightarrow{\sim} & \mathbb{G}_m^n/S_n \end{array}$$

Thus $H /\!\!/ H \to G /\!\!/ G$ is the composition of a closed immersion and a quotient by a finite group and thus it is finite.

The special case of the theorem for the base S being the spectrum a field $\operatorname{Spec}(K)$ and $\Gamma = F^N$ being free is proved by Vinberg ([Vin96], for K of characteristic 0) and Martin ([Mar03], for general K), using a different idea of proof. Dat formulated the conjecture for general base S in a lecture note ([Dat22]) and Cotner ([Cot24]) proved the theorem for general G, H and Γ, using the Bruhat–Tits building theory. We will follow Cotner's proof in this memoir.

Although the Weil group W_F is not finitely generated, these results hold for $\Gamma = W_F$ and base $S = \operatorname{Spec}(\mathbb{Z}_l)$ (for good l), we will mention the idea of proof in this case.

4 The base change theorem

In this section we sketch the idea of proof of the base change theorem 6.5.

Let G be a reductive group scheme over a base scheme S. Then for a morphism $T \to S$ between schemes, the base change $G_T := G \times_S T$ is a reductive group scheme over T and we have the following.

Theorem 4.1 (Base change)**.**

$$G^N /\!\!/ G \times_S T \simeq G_T^N /\!\!/ G_T$$

The first step is the reduction to the case $S = \operatorname{Spec} \mathbb{Z}_l$. The arguments in this steps are formal.

- The result is local, thus it suffices to prove the theorem for S and T being affine. Denote $S = \operatorname{Spec}(A)$ and $T = \operatorname{Spec}(B)$ and rephrase the theorem in terms of function rings. The theorem is equivalent to

$$\mathcal{O}(G^N)^G \otimes_A B \simeq \mathcal{O}(G_B^N)^G.$$

- The result holds for flat morphisms $T \to S$, as the invariant $(-)^G$ is given by H^0 of the bar complex (the H^i of the bar complex is called the group cohomology $H^i(G, V)$)
$$V^G = H^0(G, V) = H^0(V \to V \otimes \mathcal{O}(G) \to V \otimes \mathcal{O}(G) \otimes \mathcal{O}(G) \to \cdots)$$
and cohomology commutes with tensoring a flat module. Thus we can use the flat descent: if there is a flat morphism $S' \to S$ such that S' is flat over S and the result for $G_{S'}$ and $T' = S' \times_S T$ is correct, then the result for G and $T \to S$ is correct.

- There exists a finite étale morphism $S' \to S$ such that the group scheme $G_{S'}$ is split ([DG70]). We may reduce to the case that G is split by flat descent.

- By [DG70] all split reductive group scheme is the base change of a split reductive group over $\mathbb{Z}$. We may reduce to the case $S = \mathrm{Spec}(\mathbb{Z})$ and then $S = \mathrm{Spec}(\mathbb{Z}_l)$ by localization.

- By universal coefficient theorem, it suffices to prove the first group cohomology $H^1(G, \mathcal{O}(G^N))$ vanishes for split G over $\mathbb{Z}_l$.

The second step is to calculate the group cohomology. This step uses the modular representation theory.

Over a field k of positive characteristic ($\mathbb{F}_l$ here), the category of representations is not semisimple and the higher group cohomology may not be 0. However, there is a family of representations, called "induced" (from the Borel subgroup B) or "costandard" representations
$$\nabla(\lambda) := H^0(G/B, \mathbb{F}_{l,w_0\lambda}),$$
(∇ can also be defined over $\mathbb{Z}_l$) where λ is a dominant weight (for example, if $G = \mathrm{GL}_2$, $G/B \simeq \mathbb{P}^1$ and $\nabla(n) = H^0(\mathbb{P}^1, \mathcal{O}(n)) \simeq \mathrm{Sym}^n V$ is the symmetric power of the standard representation).

Cline proved in [Cli77] that:

Theorem 4.2. *For integer $i \geq 1$, the group cohomology*
$$H^i(G_{\mathbb{F}_l}, \nabla(\lambda)) = 0.$$

Thus the representations which have a filtration whose subquotients are costandard representations (these representation are called "having good filtrations") also have vanishing higher cohomology and it suffices to prove that $\mathcal{O}(G^N)$ has a good filtration.

Koppinen proved in [Kop84] that:

Theorem 4.3. *As a $G \times G$-representation (via left and right action) $\mathcal{O}(G_{\mathbb{F}_l})$ has a filtration whose subquotients are $\nabla(\lambda) \boxtimes \nabla(-w_0\lambda)$ (in fact, all dominant weights λ appear once).*

We proved that the filtration can be lifted to $\mathcal{O}(G_{\mathbb{Z}_l})$.

Thus $\mathcal{O}(G^N)$ over $\mathbb{Z}_l$ has a filtration as a G^{2N}-representation whose subquotients are of the from $\nabla(\lambda_1) \boxtimes \cdots \boxtimes \nabla(\lambda_{2N})$ and it suffices to prove that the subquotients have vanishing higher cohomology. Their cohomology are finitely generated and thus it suffices to prove that their cohomology are 0 after modulo l by Nakayama's lemma. Then we conclude by the fact of Mathieu in [Mat90].

Theorem 4.4. *Over $\mathbb{F}_l$, the tensor product of costandard modules has a good filtration. Thus it has vanishing higher cohomology.*

5 The finiteness theorem

In this section we sketch the proof of the finiteness theorem of Cotner ([Cot24]).

Let S be noetherian group scheme, G be a reductive group scheme over S and H be a closed subgroup scheme of G such that H is also reductive over S. Let Γ be a finitely generated group. Then we have the following.

Theorem 5.1 (Finiteness result)**.** *The morphism*

$$\mathrm{Hom}(\Gamma, H) /\!\!/ H \to \mathrm{Hom}(\Gamma, G) /\!\!/ G$$

is finite.

We sketch the proof now.

The morphism
$$\mathrm{Hom}(\Gamma, H) /\!\!/ H \to \mathrm{Hom}(\Gamma, G) /\!\!/ G$$
is affine and of finite type ([Alp10]). By the valuative criterion of finite morphisms ([Sta23], Tag 01WM), it suffices to prove that for any discrete valuation ring A with fraction field K and a commutative diagram

$$\begin{CD} \mathrm{Spec}(K) @>>> \mathrm{Hom}(\Gamma, H) /\!\!/ H \\ @VVV @VVV \\ \mathrm{Spec}(A) @>>> \mathrm{Hom}(\Gamma, G) /\!\!/ G, \end{CD}$$

there exists a local extension $A \subset A'$ with fraction field K' and a morphism $\mathrm{Spec}(A') \to \mathrm{Hom}(\Gamma, H) /\!\!/ H$ such that the diagram

$$\begin{tikzcd} \mathrm{Spec}(K') \arrow[r] \arrow[d] & \mathrm{Spec}(K) \arrow[r] \arrow[d] & \mathrm{Hom}(\Gamma, H) /\!\!/ H \arrow[d] \\ \mathrm{Spec}(A') \arrow[r] \arrow[urr, dashed] & \mathrm{Spec}(A) \arrow[r] & \mathrm{Hom}(\Gamma, G) /\!\!/ G \end{tikzcd}$$

By [Alp10], we can replace A by its local extension such that the morphism

$$\mathrm{Spec}(A) \to \mathrm{Hom}(\Gamma, G) /\!\!/ G$$

comes from a group homomorphism $\Gamma \to G(A)$ and

$$\mathrm{Spec}(K) \to \mathrm{Hom}(\Gamma, H) /\!\!/ H$$

comes from a group homomorphism $\Gamma \to H(K)$, such that they are conjugate in $G(K)$. Thus the image of Γ in $H(K)$ is bounded. By Bruhat–Tits theory ([KP23]), we can replace A by its local extension such that $\mathrm{im}(\Gamma \to H(K))$ is conjugate to a subgroup of $H(A)$. This process provides a group homomorphism $\Gamma \to H(A)$ and the corresponding morphism
$$\mathrm{Spec}(A) \to \mathrm{Hom}(\Gamma, H) /\!\!/ H$$
is what we need.

6 Application in the theory of L-parameters

In this section we explain how to apply these theorems in the theory of L-parameters. We firstly give the precise definition of the Weil group. Intuitively, it is the discretization of the Frobenius in the absolute Galois group.

Definition 6.1. *Let F be a finite extension of the $\mathbb{Q}_p$. Let F^{nr} be the maximal unramified extension of F. Then the Galois group $\mathrm{Gal}(F^{nr}/F)$ is isomorphic to the completion of $\mathbb{Z}$, $\hat{\mathbb{Z}}$ over all primes. $1 \in \hat{\mathbb{Z}}$ corresponds to the Frobenius automorphism. We take the subgroup $\mathbb{Z} \in \mathrm{Gal}(F^{nr}/F)$ to be the subgroup generated by the Frobenius. The Weil group W_F is defined to be the pullback*

$$\begin{array}{ccc} W_F & \longrightarrow & \mathbb{Z} \\ \downarrow & & \downarrow \\ \mathrm{Gal}(\bar{F}/F) & \longrightarrow & \hat{\mathbb{Z}} \end{array}$$

We firstly explain the base change result. This is implied by a resolution theorem proved by Fargues and Scholze ([FS21]).

Theorem 6.2. *Let G be a split reductive group scheme over $\mathbb{Z}_l$ for a prime $l \neq p$, such that l does not divide the order of the torsion group of $\pi_1(G)$. In the infinite category of ind-perfect G-representation complexes, there is an isomorphism*

$$\mathrm{colim}_{F^N \to W_F} \mathcal{O}(\mathrm{Hom}(F^N, G)) \to \mathcal{O}(\mathrm{Hom}(W_F, G))$$

where the index in the colimit runs through all homomorphisms from a free group F^N to W_F. In particular, by taking G-invariants, we have that

$$\mathrm{colim}_{F^N \to W_F} \mathcal{O}(\mathrm{Hom}(F^N, G))^G \to \mathcal{O}(\mathrm{Hom}(W_F, G))^G$$

is an isomorphism.

The theorem shows that the character variety for W_F can be represented in terms of the character variety for F^N. Thus we deduce the base change theorem for $\mathrm{Hom}(W_F, G) /\!\!/ G$.

Theorem 6.3. *Let G over $\mathbb{Z}_l$ and l as above, then we have that*

$$\mathrm{Hom}(W_F, G) /\!\!/ G \times \mathbb{F}_l \simeq \mathrm{Hom}(W_F, G_{\mathbb{F}_l}) /\!\!/ G_{\mathbb{F}_l}$$

The result may have some consequences in comparison between the representations of W_F on G over $\mathbb{Z}_l$ and $\mathbb{F}_l$. But we do not know the precise statements.

Next we discuss the finiteness result.

Theorem 6.4. *Let G be a split reductive group scheme over $\mathbb{Z}_l$ for a prime $l \neq p$ and H be its closed subgroup scheme such that H is reductive over S, then the morphism*

$$\mathrm{Hom}(W_F, H) /\!\!/ H \to \mathrm{Hom}(W_F, G) /\!\!/ G$$

is finite.

We remark that here Hom is the moduli space of continuous homomorphisms.

We follow the treatments in Dat–Helm–Kurinczuk–Moss ([DHKM20]) to prove the theorem 6.4. The idea is to discretize W_F to become a finitely generated group. The kernel of $W_F \to \mathbb{Z}$ is the inertia subgroup I_F and it can be decomposed as

$$1 \to P_F \to I_F \to T_F \to 1$$

where T_F is the tame inertia and P_F is the wild inertia. The tame inertia is isomorphic to the group $\prod_{q \neq p} \mathbb{Z}_q$, the completion of $\mathbb{Z}$ over all primes other than p (which corresponds to the extension $F^{nr}(\pi^{1/n})/F^{nr}$, n runs through all numbers coprime with p). We take the subgroup $\mathbb{Z}[\frac{1}{p}]$ of T_F and take W_F^0 to be the extension of P_F, $\mathbb{Z}[\frac{1}{p}]$ and $\mathbb{Z}$. We notice that the extension of $\mathbb{Z}[\frac{1}{p}]$ and $\mathbb{Z}$ here is the group generated by two elements

$$\langle s, t \mid sts^{-1} = t^p \rangle.$$

In [DHKM20], it is shown that

$$\mathrm{Hom}(W_F, G) /\!\!/ G \simeq \mathrm{Hom}(W_F^0, G) /\!\!/ G.$$

For the wild inertia P_F, it is a pro-p-group. Thus a continuous morphism from P_F to a $\mathbb{Z}_l$-algebra factors through a finite quotient of P_F. Thus

$$\mathrm{Hom}(W_F^0, G) \simeq \bigcup_{P \subset P_F \text{ open}} \mathrm{Hom}(W_F^0/P, G)$$

and $\mathrm{Hom}(W_F^0/P, G)$ is an open and closed subset of $\mathrm{Hom}(W_F^0, G)$. Thus it suffices to prove that the result holds for all W_F^0/P. We conclude by noticing W_F^0/P is a finitely generated group.

We can also deduce from the theorem a finiteness result for Bernstein centers using the local Langlands program. In [DHKM23], it is proved that for an open subgroup $G^0 \subset G(F)$, the correspondence

$$\mathcal{O}(\mathrm{Hom}(W_F, G^\vee) /\!\!/ G^\vee) \to \mathcal{Z}(G(F)/G^0)$$

(the latter is the "restricted" Bernstein center) is finite. Then for $H \subset G$ as above (satisfying the existence of $H^\vee \to G^\vee$) and $H^0 = H(F) \cup G^0$, we have a commutative diagram

$$\begin{array}{ccc} \mathcal{O}(\mathrm{Hom}(W_F, G^\vee) /\!\!/ G^\vee) & \longrightarrow & \mathcal{Z}(G(F)/G^0) \\ \downarrow & & \downarrow \\ \mathcal{O}(\mathrm{Hom}(W_F, H^\vee) /\!\!/ H^\vee) & \longrightarrow & \mathcal{Z}(H(F)/H^0) \end{array}$$

such that the horizontal morphisms and the left vertical arrow is finite. Thus the right vertical arrow between the restricted Bernstein centers is finite and we deduce a theorem in purely representation theory.

Theorem 6.5. *For G, H, G^0, H^0 as above, the morphism between Bernstein centers*

$$\mathcal{Z}(G(F)/G^0) \to \mathcal{Z}(H(F)/H^0)$$

is finite.

7 Further perspectives

In this section, let G be a split reductive group over $\mathbb{Z}_l$. The issue of the GIT quotient $\mathrm{Hom}(\Gamma, G) /\!\!/ G$ is that it does not reflect all information of the action of G on the representation space $\mathrm{Hom}(\Gamma, G)$. As mentioned, in the process of GIT quotient we take the semisimplification of a representation $\Gamma \to G$ and thus the construction of Fargues–Scholze ([FS21])

$$\mathcal{O}(\mathrm{Hom}(W_F, G^\vee) /\!\!/ G^\vee) \to \mathcal{Z}(G(F)),$$

only provides a "semisimplified" local Langlands correspondence, which means that the Galois representation corresponding to an automorphic representation under the Fargues–Scholze correspondence is the semisimplification of the "desired" Galois representation.

Moreover we can not read the stablizer of a representation under the conjugate action under G. To get the full information, it is necessary to consider the quotient stack (where a stack can be viewed as a space whose points have automorphism groups, here the automorphism of a point is its stablizer under G)

$$\mathrm{Hom}(\Gamma, G)/G.$$

Fargues–Scholze ([FS21]) conjectured that there is an equivalence of categories (called the categorical local Langlands correspondence)

$$D^{b,qc}_{\mathrm{Coh}}(\mathrm{Hom}(W_F, G^\vee)/G^\vee_{\bar{\mathbb{Q}}_l}) \simeq D_{lis}(\mathrm{Bun}_G, \bar{\mathbb{Q}}_l)$$

from the derived category of coherent sheaves on the quotient stack to the derived category of lisse sheaves on Bun_G (the moduli space of G-bundles).

The base change result still holds for quotient stacks $\mathrm{Hom}(\Gamma, G)/G$. However, the finiteness does not hold even for $H = 1$, $G = \mathbb{G}_m$ and $\Gamma = \mathbb{Z}$ as $1 \in G/G$ has the automorphism group $\mathbb{G}_m$, which is not finite. The finiteness of the morphism between character varieties may reflect some facts in the theory of quotient stack and the categorical local Langlands correspondence but it is not known yet.

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