Some applications of Fraïssé theory in functional analysis

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In 2005, Kechris, Pestov and Todorcevic [KPT05] published a paper linking in a very beautiful way three different domains : model theory, combinatorics and topological dynamics. More precisely, they showed that when a group G can be realized as a group of automorphisms $G = \mathrm{Aut}(\mathfrak{A})$ of a given structure $\mathfrak{A}$ which is homogeneous in some sense, then dynamical properties of G such as extreme amenability are closely related to Ramsey-type combinatorial properties of $\mathfrak{A}$. This fact, now called the KPT correspondence has lead to fruitful applications for the study of Polish groups. For instance, this method turned out useful to compute universal minimal flows of various groups. This memoir aims to review some ideas behind these correspondences and some applications.

Ultimately, the goal is to present applications in functional analysis, so we choose to present Fraïssé theory in the context of metric structures. Metric structures have been introduced as a framework to apply model-theoretic tools to non-discrete structures as classical model theory often fails at this level of generality [BBHU08]. We will see in Section 1 that metric structures can be used to treat any Polish group as automorphism groups of Fraïssé limits. Fraïssé limits have very nice properties as they are "determined by their finitely-generated substructures", which form a Fraïssé class. In Section 2, we will state two instances of KPT correspondence in this framework by giving for any Polish group G a combinatorial criterion for both amenability and extreme amenability, two properties of the actions of G on compact Hausdorff spaces. In Section 3, we will show how concentration of measure can lead to other approaches of the subject and how it can be seen as a strengthening of Ramsey properties. Finally, Section 4 reviews some examples, namely the very natural approximately ultrahomogeneous spaces that are the Urysohn space $\mathbb{U}$, the Gurarij space $\mathbb{G}$ and the Hilbert space ℓ_2 as well as the case of $L_p(\mathbf{R})$.

Main references

Here is a non-exhaustive list of references which are complementary to our introduction of the subject.

- To get a better understanding of Fraïssé theory, the simplest solution is to start with the classical theory (see e.g. [Hod93]).
- KPT correspondence in the classical case is already well presented in the original paper of Kechris, Pestov and Todorcevic [KPT05].
- The work of Melleray and Tsankov [MT14] and the survey from Vignati [Vig22] are very straightforward in their presentation of both Fraïssé theory and KPT correspondence in the metric setting and what follows is largely inspired by their work.
- Kaïchouh's PhD thesis [Kaï15] contains more details, clear proofs and a nice overview of related subjects.

- As we will see, Fraïssé theory has also connections with the concentration of measure phenomenon. Pestov's book [Pes06] is a good reference to get a deeper understanding of the interactions between these subjects.

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1 Fraïssé theory

Metric model theory

One motivation for introducing Fraïssé theory in analysis is to study Polish groups. Recall that a Polish group is a separable and completely metrizable topological group. It is well-known that every Polish group G can be seen as a closed subgroup of $\mathrm{Iso}(\mathbb{U})$) where $\mathbb{U}$ is the Urysohn space (see [Gao09], Theorem 2.5.2). Now, it turns out that the set of uniformly continuous function $X^n \to \mathbb{R}$ which are G-invariant contain all the information to know G. Formally, we have the following :

Theorem 1.1 ([Mel10], Theorem 6)**.** *Any Polish group is isomorphic to* $\mathrm{Aut}(\mathfrak{A})$ *for some* $\mathfrak{A}$ *which is an approximately ultrahomogeneous separable metric structure over a countable language*[1]*.*

Before explaining what is a metric structure and what means approximate ultrahomogeneity, let us make a historical remark. Theorem 1.1 is a generalization of a classical result of descriptive set theory stating that any closed subgroup of the permutation group of $\mathbf{N}$ can be seen as the automorphism group of a countable discrete structure (see for instance [Gao09] Section 2.4). This allows to reduce the study of *non-archimedean* Polish groups (i.e. groups with a countable basis at the identity consisting of open subgroups) to the study of certain automorphism groups. Now, using metric structures instead of discrete structures thus allows (hopefully) to study any Polish group via similar methods.

A metric structure $\mathfrak{M}$ is a complete metric space[2] (M, d), together with elements of M called constants, uniformly continuous functions $M^n \to M$ called n-ary functions, and uniformly continuous functions $M^n \to \mathbb{R}$ called n-ary predicates[3]. Note that M^n is endowed with the metric given by $d(x, y) := \max(d(x_1, y_1), ..., d(x_n, y_n))$ and that d can be considered as a 2-ary predicate. Given a structure $\mathfrak{M}$, we can bind to it a *signature* $\mathcal{L}$ which is a set of symbols : we have a symbol for each constant of $\mathfrak{M}$, a symbol (equipped with the arity and a modulus of uniform continuity) for each function of $\mathfrak{M}$ and a symbol (equipped with the arity and a modulus of uniform continuity) for each predicate symbol of $\mathfrak{M}$. In this case, we say that $\mathfrak{M}$ is an *$\mathcal{L}$-structure.* Note that from our definition, different choices of moduli of uniform continuity will lead to different signature $\mathcal{L}$ and thus $\mathfrak{M}$ can be an $\mathcal{L}$-structure for different signatures $\mathcal{L}$. In the following, we will only be interested in countable signatures. If $\mathfrak{M}$ is an $\mathcal{L}$-structure, we refer to its domain as M, to the constant corresponding to a constant symbol $c \in \mathcal{L}$ as $c^{\mathfrak{M}}$, the

[1] We can even require the language to be relational, but it is convenient to work with function symbols in some cases. For some proofs however, studying relational structures is enough since every structure "can be seen" as a structure in an appropriate relational language and dealing with function symbols can be notationally painful.

[2] In metric model theory, it is convenient to also add the condition that $\mathfrak{M}$ is bounded to get the Compactness Theorem, a fundamental tool for logicians. For pure Fraïssé Theory however, this condition is not useful but is just bothering for the treatment of some spaces.

[3] In our applications, functions and predicates will actually be Lipschitz.

function corresponding to a function symbol $f \in \mathcal{L}$ as $f^{\mathfrak{M}}$ and the predicate corresponding to a predicate symbol $P \in \mathcal{L}$ as $P^{\mathfrak{M}}$. If $a \in M^n$, we denote by $\langle a \rangle$ the *substructure* generated by a i.e. the closure of $\{a_1, ..., a_n\} \cup \{c^{\mathfrak{M}} : c \text{ constant}\}$ by all the functions.

An *embedding* between two $\mathcal{L}$-structures $\mathfrak{A}$ and $\mathfrak{B}$ is an isometry $F : A \to B$ where for each constant symbol $c \in \mathcal{L}$, $F(c^{\mathfrak{A}}) = c^{\mathfrak{B}}$, for each function symbol $f \in \mathcal{L}$ of arity n and $a_1, ..., a_n \in A$, $F(f^{\mathfrak{A}}(a_1, ..., a_n)) = f^{\mathfrak{B}}(F(a_1), ..., F(a_n))$ and for each predicate symbol $P \in \mathcal{L}$ of arity n and $a_1, ..., a_n \in A$, $P^{\mathfrak{A}}(a_1, ..., a_n) = P^{\mathfrak{B}}(F(a_1), ..., F(a_n))$. If $A \subseteq B$ and the inclusion of $\mathfrak{A}$ in $\mathfrak{B}$ is an embedding, we write $\mathfrak{A} \subseteq \mathfrak{B}$. An embedding is an *isomorphism* if it is surjective. An isomorphism $\mathfrak{A} \to \mathfrak{A}$ is called an *automorphism* of $\mathfrak{A}$ and we denote by $\mathrm{Aut}(\mathfrak{A})$ the set of automorphisms of $\mathfrak{A}$, endowed with the pointwise convergence topology. If $\mathfrak{A}$ is separable and $\mathcal{L}$ is countable, then $\mathrm{Aut}(\mathfrak{A})$ is a Polish group and a consequence of Theorem 1.1 is that conversely, any Polish group is of this form. A *partial isomorphism* f between two tuples $a, b \in A^n$ is an isomorphism $\langle a \rangle \to \langle b \rangle$ such that $f(a_i) = b_i$ for every i. A structure $\mathfrak{A}$ is called *ultrahomogeneous* (AuH) if for every partial isomorphism $f : a \to b$ and every $\varepsilon > 0$, there exists $\hat{f} \in \mathrm{Aut}(\mathfrak{A})$ such that $d(\hat{f}(a_i), b_i) < \varepsilon$ for every i.

Fraïssé classes and limits

We fix a countable metric signature $\mathcal{L}$. By "structure", we will now mean $\mathcal{L}$-structure. As in the classical setting for Fraïssé theory, we are interested in classes of finitely-generated structures. A natural way to have such classes is by taking the age of any structure.

Definition 1.2. Let $\mathfrak{M}$ be a structure. The *age* of $\mathfrak{M}$ consists of the class of every finitely-generated structure $\mathfrak{A}$ such that there is an embedding $\mathfrak{A} \to \mathfrak{M}$.

Definition 1.3. Let $\mathcal{K}$ be a class of finitely-generated $\mathcal{L}$-structures closed under isomorphisms. We say that $\mathcal{K}$ has :

- the *Hereditary Property* (HP) if whenever $\mathfrak{A} \in \mathcal{K}$ and $\mathfrak{B} \to \mathfrak{A}$ is an embedding, then $\mathfrak{B} \in \mathcal{K}$.
- the *Joint Embedding Property* (JEP) if whenever $\mathfrak{A}, \mathfrak{B} \in \mathcal{K}$, there exist $\mathfrak{C} \in \mathcal{K}$ and embeddings $\mathfrak{A} \to \mathfrak{C}$ and $\mathfrak{B} \to \mathfrak{C}$.
- the *Near Amalgamation Property* (NAP) if for all $\mathfrak{A}, \mathfrak{B}, \mathfrak{C} \in \mathcal{K}$, all embeddings $\varphi_1 : \mathfrak{A} \to \mathfrak{B}$ and $\varphi_2 : \mathfrak{A} \to \mathfrak{C}$, every $\overline{a} \in \mathfrak{A}$ and $\varepsilon > 0$, we can find $\mathfrak{D} \in \mathcal{K}$ and embeddings $\psi_1 : \mathfrak{B} \to \mathfrak{D}$ and $\psi_2 : \mathfrak{C} \to \mathfrak{D}$ such that $d(\psi_1\varphi_1\overline{a}, \psi_2\varphi_2\overline{a}) < \varepsilon$.

The idea is to use the combinatorial properties to build, given a class $\mathcal{K}$, a separable structure $\mathfrak{M}$ such that :

(a) $\mathcal{K} = \mathrm{Age}(\mathfrak{M})$.

(b) $\mathfrak{M}$ is (AuH).

From (a), it is clear that $\mathcal{K}$ must have (HP) and (JEP). (NAP) on the other hand, is a necessary condition to get (b). Moreover, since we wish in the end to have a metric space M which is separable and complete, $\mathcal{K}$ must reflect these two properties in order to get (a). Let us formalize these ideas.

Definition 1.4. Let $\mathcal{K}$ be a class of finitely-generated structures. Define for each $n \geq 1$, the class $\tilde{\mathcal{K}}_n$ consisting of all pairs $(\mathfrak{A}, \overline{a})$ where $\mathfrak{A} \in \mathcal{K}$, $\overline{a}$ has length n and $\langle \overline{a} \rangle = \mathfrak{A}$. We then define a pseudometric d_n on $\tilde{\mathcal{K}}_n$ by setting[4] :

$$d_n((\mathfrak{A}, \overline{a}), (\mathfrak{B}, \overline{b})) := \inf \{d(\varphi\overline{a}, \psi\overline{a}) : \mathfrak{C} \in \mathcal{K}, \varphi : \mathfrak{A} \to \mathfrak{C}, \psi : \mathfrak{B} \to \mathfrak{C} \text{ embeddings}\}$$

[4]Note the similarity with the Gromov-Hausdorff distance.

The quotient $\tilde{\mathcal{K}}_n/(d_n = 0)$ will be denoted by $\mathcal{K}_n$ and the metric induced by d_n will also be denoted by d_n. If every $(\mathcal{K}_n, d_n)$ is a separable metric space, then we say that $\mathcal{K}$ has the *weak Polish property* (WPP). If moreover every $\mathcal{K}_n$ is complete, then we say that $\mathcal{K}$ has the *Polish property* (PP).

If one wants to have (a) for a separable structure $\mathfrak{M}$ then it must be that $\mathcal{K}$ has (PP). In practice, if $\mathcal{K}$ only has (WPP), one can consider a natural class $\hat{\mathcal{K}}$ which has (PP) and such that $\hat{\mathcal{K}}_n$ is the completion of $\mathcal{K}_n$ for each n. This completion does not change the combinatorial properties (HP), (JEP), (NAP) of the original class. We are now ready to state the Fraïssé correspondence in our framework.

Theorem 1.5 ([Ben15]). *Let $\mathcal{K}$ be a class of finitely-generated structures. The following are equivalent :*

(i) $\mathcal{K}$ has (HP), (JEP), (NAP) and (PP).

(ii) $\mathcal{K} = \mathrm{Age}(\mathfrak{M})$ for some separable (AuH) structure $\mathfrak{M}$.

If they hold, $\mathfrak{M}$ is unique up to isomorphism. $\mathcal{K}$ is then called a Fraïssé class *and $\mathfrak{M}$ its* Fraïssé limit.

Related model-theoretic notions

In this section, we add boundedness conditions to the signature $\mathcal{L}$ i.e. $\mathcal{L}$ contain bounds for each predicate symbol, d included. This allows to build ultraproducts of structures[5].

Definition 1.6. Let $\mathcal{U}$ be an ultrafilter on a set I, and $\mathfrak{M}_i$ be $\mathcal{L}$-structures. We define the ultraproduct $\mathfrak{N} = \prod_i \mathfrak{M}_i/\mathcal{U}$ on the ultralimit $\lim_{i\to\mathcal{U}} M_i$ by putting for each function symbol f and predicate symbol P :

$$f^{\mathfrak{N}}((x_i)_{i\in I}) = (f^{\mathfrak{M}_i}(x_i))_{i\in I}$$
$$P^{\mathfrak{N}}((x_i)_{i\in I}) = \lim_{i\to\mathcal{U}} P^{\mathfrak{M}_i}(x_i)$$

Let $\overline{x} = (x_1, ..., x_n)$ be constant symbols that are not in $\mathcal{L}$ and $\mathcal{L}(\overline{x})$ be the signature containing $\mathcal{L}$ and $x_1, ..., x_n$. An $\mathcal{L}(\overline{x})$-structure is thus determined by a pair $(\mathfrak{A}, \overline{a})$ where $\mathfrak{A}$ is an $\mathcal{L}$-structure and $\overline{a} \in \mathfrak{A}^{\overline{x}}$. A *first-order* formula $\varphi(\overline{x})$ is a map defined on the class of $\mathcal{L}(\overline{x})$-structures such that :

- It is invariant under isomorphisms i.e. if $(\mathfrak{A}, \overline{a}) \simeq (\mathfrak{B}, \overline{b})$ then $\varphi^{\mathfrak{A}}(\overline{a}) = \varphi^{\mathfrak{B}}(\overline{b})$ where $\varphi^{\mathfrak{A}}(\overline{a})$ is the value of φ on the structure determined by $(\mathfrak{A}, \overline{a})$.
- For every family of $\mathcal{L}(\overline{x})$-structures $(\mathfrak{A}_i, \overline{a}_i)_{i\in I}$ and any ultrafilter $\mathcal{U}$ on I we have :

$$\varphi^{\prod_i \mathfrak{A}_i/\mathcal{U}}((\overline{a}_i)_{i\in I}) = \lim_{i\to\mathcal{U}} \varphi^{\mathfrak{A}_i}(\overline{a}_i)$$

If $\overline{x}$ is the empty tuple, φ is called a *first-order sentence*. A set T of conditions of the form $[\varphi \leq r]$ where φ is a first-order sentence and $r \in \mathbb{R}$ is called a *theory* and if $\mathfrak{M}$ a metric structure, we write $\mathfrak{M} \models T$ when $\varphi^{\mathfrak{M}} \leq r$ for every condition $[\varphi \leq r] \in T$. If $\mathfrak{M}$ is a metric structure, then its *theory* $\mathrm{Th}(\mathfrak{M})$ is the maximal theory T such that $\mathfrak{M} \models T$. If T, S are theories, we write $T \models S$ if for every $\mathfrak{M} \models T$, we have $\mathfrak{M} \models S$.

[5]The boundedness assumption is here to get to the same framework as in [BBHU08]. Standard tricks can be used to also deal with unbounded structures [Ben08]

A syntactic and less abstract definition of first-order formulas is one of the starting point of metric model theory (see [Han23]) and can justify a bit more the name of "first-order formula" for someone who knows what first-order means for logicians but who is not aware of Keisler-Shelah's Theorem. One goal of metric model theory is then to understand the link between properties of $\mathfrak{M}$ and properties of $\mathrm{Th}(\mathfrak{M})$. We say that a separable structure $\mathfrak{M}$ is *ω-categorical* if whenever $\mathfrak{N}$ is another separable structure such that $\varphi^{\mathfrak{M}} = \varphi^{\mathfrak{N}}$ for every $\mathcal{L}$-sentence φ, then $\mathfrak{M} \simeq \mathfrak{N}$. By Ryll-Nardzewski's Theorem for continuous logic, this is equivalent to the action $\mathrm{Aut}(\mathfrak{M}) \curvearrowright \mathfrak{M}$ being approximately oligomorphic.

The spirit of the (AuH) property is that these structures are determined by their finite chunks. More precisely, two isometric tuples in an (AuH) structure $\mathfrak{M}$ cannot be distinguished by first-order formulas. We say that a theory T has *quantifier elimination* if for every $\mathfrak{M}, \mathfrak{N} \models T$, every $\overline{a} \in \mathfrak{M}$ and $\overline{b} \in \mathfrak{N}$ and every isomorphism $f : \overline{a} \to \overline{b}$, we have $\varphi^{\mathfrak{M}}(\overline{a}) = \varphi^{\mathfrak{N}}(\overline{b})$ for any first-order formula $\varphi(\overline{x})$. The terminology "quantifier elimination" has a syntactical meaning because in a theory with quantifier elimination, first-order formulas can be approximated by quantifier-free ones.

Theorem 1.7. *Let $\mathfrak{M}$ be an ω-categorical separable structure. Then $\mathfrak{M}$ is approximately ultra-homogeneous if and only its theory has quantifier elimination.*

A first-order formula $\varphi(\overline{x})$ is a *Σ_1-formula* in a theory T if for every models $\mathfrak{M}, \mathfrak{N} \models T$ with $\mathfrak{M} \subseteq \mathfrak{N}$ and every $\overline{a} \in \mathfrak{M}^{\overline{x}}$, we have $\varphi^{\mathfrak{N}}(\overline{a}) \leq \varphi^{\mathfrak{M}}(\overline{a})$. If T is a theory, $\mathfrak{M} \models T$ is *existentially closed* if whenever $\mathfrak{M} \subseteq \mathfrak{N} \models T$, we have for every Σ_1-formula $\varphi(\overline{x})$ and every $\overline{a} \in \mathfrak{M}^{\overline{x}}$ we have $\varphi^{\mathfrak{M}}(\overline{a}) = \varphi^{\mathfrak{N}}(\overline{a})$. For some theories, the class of existentially closed models can be axiomatized by some theory T^*. In this case, we say that T^* is the *model companion* of T. If it exists, T^* enjoys nice model-theoretic properties and in particular, it has a weaker form of quantifier elimination, namely *model-completeness*. Sometimes however, T^* has full quantifier elimination and in this case we say that T^* is the *model completion* of T. We will see some examples of theories for which the model companion is the theory of some ω-categorical Fraïssé limit. In these cases, it will be clear that the model companion is in fact a model completion because its theory has quantifier elimination.

2 KPT correspondence

Extreme amenability

Fix G a Hausdorff topological group. Recall that a *G-flow* is a continuous action $G \times X \to X$ on a compact Hausdorff space X.

A nice property of topological group is their amenability. G is said to be *amenable* if every G-flow admits an invariant probability measure (on the Borel algebra). For instance, every compact group is amenable due to the existence of the Haar measure. G is *extremely amenable* if every G-flow admits a fixed point. This notion is interesting for "big" groups because by a theorem of Veech, the only extremely amenable locally compact group is the trivial one (see [Aus88], Chapter 8).

A G-flow is *minimal* if the action $G \curvearrowright X$ cannot be restricted to some nonempty closed subset $Y \subsetneq X$. This is equivalent to all the orbits of X being dense.

Theorem 2.1. *G has a universal minimal flow X, i.e. a minimal flow such that for every other minimal flow X', there exists a continuous map $f : X \to X'$ such that for every $(g, x) \in G \times X$, $f(gx) = gf(x)$. The universal minimal flow of G is unique up to isomorphism and denoted by $M(G)$.*

One can easily see that $M(G)$ is a single point if and only if the group is extremely amenable. For non-extremely amenable groups, computing $M(G)$ can serve as a consolation prize.

The KPT correspondence for metric structures

Here, we fix a Fraïssé class $\mathcal{K}$, $\mathfrak{M}$ is its Fraïssé limit and G the automorphism group of $\mathfrak{M}$. G is endowed with the topology of pointwise convergence.

If $\mathfrak{A}$ and $\mathfrak{B}$ are two structures, let us denote by $\mathrm{Emb}(\mathfrak{A},\mathfrak{B})$ the set of embeddings $\varphi : \mathfrak{A} \to \mathfrak{B}$. If $\overline{a}$ is a finite tuple which generates $\mathfrak{A}$, then we define a distance on $\mathrm{Emb}(\mathfrak{A},\mathfrak{B})$ by

$$d_{\overline{a}}(\varphi,\psi) := d^{\mathfrak{B}}(\varphi\overline{a},\psi\overline{a}) \ .$$

When we write $\mathrm{Emb}(\langle\overline{a}\rangle,\mathfrak{B})$, this will mean $\mathrm{Emb}(\mathfrak{A},\mathfrak{B})$ endowed with this metric. A *coloring* of $\mathrm{Emb}(\langle\overline{a}\rangle,\mathfrak{B})$ is then a 1-Lipschitz map $c : \mathrm{Emb}(\langle\overline{a}\rangle,\mathfrak{B}) \to [0,1]$.

We say that a Fraïssé class $\mathcal{K}$ has the *approximate Ramsey property* (ARP) if for all $\langle\overline{a}\rangle = \mathfrak{A}, \mathfrak{B} \in \mathcal{K}$, all nonempty finite $F \subseteq \mathrm{Emb}(\mathfrak{A},\mathfrak{B})$ and every $\varepsilon > 0$, there exists $\mathfrak{C} \in \mathcal{K}$ such that for every coloring c of $\mathrm{Emb}(\langle\overline{a}\rangle,\mathfrak{C})$, there exist $\beta \in \mathrm{Emb}(\mathfrak{B},\mathfrak{C})$ and $r \in \mathbb{R}$ such that :

$$|c(\beta\circ\alpha) - r| < \varepsilon \text{ for any } \alpha \in F$$

Note that in the case of a relational signature, $\mathrm{Emb}(\mathfrak{A},\mathfrak{B})$ is finite so the definition is a bit simpler. Put it mildly in this case, the approximate Ramsey property of $\mathcal{K}$ means that for every $\mathfrak{A} \subseteq \mathfrak{B} \in \mathcal{K}$, there exists a big enough $\mathfrak{C}$ such that any coloring c of the copies of $\mathfrak{A}$ in $\mathfrak{C}$ includes a copy of $\mathfrak{B}$ in $\mathfrak{C}$ on which c is almost monochromatic.

The (ARP) is by definition a property of $\mathcal{K}$ but it has reformulations in which $\mathfrak{M}$ appears more clearly. In a class with the (ARP), the superstructure $\mathfrak{C}$ we build is itself contained in the Fraïssé limit $\mathfrak{M}$. Conversely, using the axiom of choice, bad colorings for every $\mathfrak{C} \in \mathcal{K}$ can be used to obtain a bad coloring of $\mathrm{Emb}(\langle\overline{a}\rangle,\mathfrak{M})$. With these ideas we obtain the following characterization of the (ARP) in which the Fraïssé limit is more clearly involved.

Lemma 2.2 ([MT14] Proposition 3.4)**.** *$\mathcal{K}$ has the (ARP) if and only if for all $\langle\overline{a}\rangle = \mathfrak{A}, \mathfrak{B} \in \mathcal{K}$, all nonempty finite $F \subseteq \mathrm{Emb}(\mathfrak{A},\mathfrak{B})$, every $\varepsilon > 0$ and every coloring c of $\mathrm{Emb}(\langle\overline{a}\rangle,\mathfrak{M})$, there exist $\beta \in \mathrm{Emb}(\mathfrak{B},\mathfrak{M})$ and $r \in \mathbb{R}$ such that :*

$$|c(\beta\circ\alpha) - r| < \varepsilon \textit{ for any } \alpha \in F$$

Let us now state the KPT correspondence for metric structures :

Theorem 2.3 ([MT14] Theorem 3.10)**.** *The following are equivalent :*

(i) G is extremely amenable.

(ii) $\mathcal{K}$ has the (ARP).

We do not sketch a proof here, but the proof relies on the fact that extreme amenability is equivalent to some kind of oscillation stability (see [MT14] for a complete proof).

The KPT correspondence can be used to prove that some groups are extremely amenable (or if not, to compute their minimal flows) as in the original paper of Kechris, Pestov and Todorcevic. They for instance used this method to prove that $\mathrm{Aut}(\mathbb{Q},<)$ and the group of isometries of the Urysohn space [KPT05] are extremely amenable. The other way of the KPT correspondence is also useful, especially when dealing with metric structures. We will see later that using concentration of measure is sometime a fruitful method to prove extreme amenability, so probabilistic and geometric intuitions can be used to get combinatorial informations on finitely generated structures.

KPT correspondence for amenability

In her PhD thesis, Kaïchouch [Kaï15] proved a KPT-like criterion for amenability. $\mathcal{K}$ has the *metric convex Ramsey property* if for every $\varepsilon > 0$, every $\langle \overline{a} \rangle = \mathfrak{A}, \mathfrak{B} \in \mathcal{K}$ and every nonempty finite $F \subseteq \mathrm{Emb}(\mathfrak{A}, \mathfrak{B})$, there is $\mathfrak{C} \in \mathcal{K}$ such that for every coloring $c : \mathrm{Emb}(\langle \overline{a} \rangle) \to [0, 1]$, there exists $r \in [0, 1]$ and a finitely supported probability measure ν on $\mathrm{Emb}(\mathfrak{B}, \mathfrak{C})$ for which :

$$\left| \int c(\nu \circ \alpha) d\nu - r \right| < \varepsilon \text{ for every } \alpha \in F$$

The difference with the (ARP) is that we authorize to deal with convex combinations of copies of $\mathfrak{B}$ to get almost monochromatic sets. On the dynamical side, this weaker property naturally does not allow to get fixed points in G-flows but invariant probability measures instead. Formally, we get this KPT-like correspondence :

Theorem 2.4 ([Kaï15] Theorem 9.22)**.** *The following are equivalent :*

(i) G is amenable.

(ii) $\mathcal{K}$ has the metric convex Ramsey property.

3 Concentration of measure as a Ramsey property

Another useful method to prove extreme amenability is to use the concentration of measure phenomenon. We give a few definitions here, but Pestov's book [Pes06] is a good reference to understand how concentration of measure can be used to study dynamics of topological groups. A topological group G is *Lévy* if there is an increasing sequence $(G_n)_{n \in \mathbf{N}}$ of compact subgroups such that their union is dense in G and such that the family (μ_n) of the corresponding normalized Haar measures concentrate in G i.e. for every sequence $(A_n)_{n \in \mathbf{N}}$ of Borel sets satisfying $\liminf_n \mu_n(A_n) > 0$ and every neighbourhood V of the identity of G, one has $\lim_n \mu_n(VA_n) = 1$. Recall that the *normalized Haar measure* of a compact group H is the unique regular Borel probability measure μ such that for every Borel set $A \subseteq G$ and every $g \in A$, $\mu(gA) = \mu(A) = \mu(Ag)$.

It can be proved that every Lévy group is extremely amenable[6] [GP07]. The aim of this section is to make clearer the link between concentration of measure and Ramsey properties by proving that if G is Lévy and is the automorphism group of the limit of a Fraïssé class $\mathcal{K}$ then $\mathcal{K}$ has the (ARP), and is thus amenable by the KPT correspondence.

As earlier, we take a Fraïssé class $\mathcal{K}$, its Fraïssé limit $\mathfrak{M}$ and the automorphism group of $\mathfrak{M}$ is denoted by G. Recall that G is endowed with the pointwise convergence topology. Suppose that G is Lévy. Write G as the closure of $\bigcup_n G_n$ where the G_n's are compact groups with normalized Haar measure μ_n, (μ_n) concentrating in G.

In a concentrated space, uniformly continuous functions are "nearly constant nearly everywhere". The following useful lemma is a formulation of this fact in our particular case.

Lemma 3.1 ([GP07] Lemma 2.4)**.** *For every uniformly continuous $f : G \to [0, 1]$, there exists a sequence of constants (r_n) such that :*

$$\lim_{n \to \infty} \mu_n\{|f - r_n| < \varepsilon\} = 1$$

Theorem 3.2. *If G is a Lévy group then $\mathfrak{M}$ has the approximate Ramsey property.*

[6] We can prove this statement for some weakenings of our definition of Lévy group, but we stick here to that simpler form for the sake of clarity.

Proof. Let $\langle \bar{a} \rangle = \mathfrak{A} \subseteq \mathfrak{B} \subseteq \mathfrak{M}$, $\alpha_1, ..., \alpha_d \in \mathrm{Emb}(\langle \bar{a} \rangle, \mathfrak{B})$ and c a coloring of $\mathrm{Emb}(\langle \bar{a} \rangle, \mathfrak{M})$. By approximate ultrahomogeneity of $\mathfrak{M}$ and density of $\bigcup G_n$, we can find a natural number N and $g_1, ..., g_d \in G_N$ such that $d^{\mathfrak{M}}(g_i \bar{a}, \alpha_i \bar{a}) < \varepsilon$ for every $i = 1, ..., d$.

Define the mapping $f : G \to [0, 1]$ by $f(g) := c(g \restriction \mathfrak{A})$. By Lemma 3.1, we have for some sequence of constants (r_n) :

$$\lim_{n \to \infty} \mu_n \{|f - r_n| < \varepsilon\} = 1$$

Hence, there exists $N' \geq N$ such that $\mu_{N'}\{|f - r_{N'}| \geq \varepsilon\} < 1/d$. We thus have :

$$\begin{aligned} \mu_{N'} \left[\{|f - r_{N'}| \geq \varepsilon\} g_1^{-1} \cup ... \cup \{|f - r_{N'}| \geq \varepsilon\} g_d^{-1}\right] &\leq \sum_i \mu_{N'} \left[\{|f - r_{N'}| \geq \varepsilon\} g_i^{-1}\right] \\ &\leq d\mu_{N'}\{|f - r_{N'}| \geq \varepsilon\} < d\frac{1}{d} = 1 \end{aligned}$$

So there exists $g \in G_{N'}$ such that $|c(gg_i \restriction \mathfrak{A}) - r_{N'}| < \varepsilon$ for every i. We can now deduce, since c is Lipschitz, that for $1 \leq i \leq d$:

$$\begin{aligned} |c(g \restriction \mathfrak{B} \circ \alpha_i) - r_{N'}| &\leq |c(g\alpha_i) - c(gg_i \restriction \mathfrak{A})| + |c(gg_i \restriction \mathfrak{A}) - r_{N'}| \\ &< d_{\bar{a}}(g\alpha_i, gg_i \restriction \mathfrak{A}) + \varepsilon \\ &\leq d^{\mathfrak{M}}(g\alpha_i \bar{a}, gg_i \bar{a}) + \varepsilon = d^{\mathfrak{M}}(\alpha_i \bar{a}, g_i \bar{a}) + \varepsilon \leq 2\varepsilon \end{aligned}$$

□

When we look closely to the previous proof then we can intuitively say that the Lévy property implies a stronger form of approximate Ramsey property. Indeed, we use a probabilistic argument to find a $g \in G$ for which $g \restriction \mathfrak{B}$ is almost monochromatic but the proof somehow shows that the g's having this property are generic in some sense.

4 Applications

The Urysohn space and the Rado Graph

One fundamental (and continuous) example of Fraïssé class is the class $\mathcal{K}$ of all finite metric spaces. Here, the signature $\mathcal{L}$ is empty. To see that $\mathcal{K}$ has (NAP), one can use the "free amalgamation". Let $\mathfrak{A}, \mathfrak{B}, \mathfrak{C} \in \mathcal{K}$, $\varphi_1 : \mathfrak{A} \to \mathfrak{B}$ and $\varphi_2 : \mathfrak{A} \to \mathfrak{C}$. One defines a pseudometric d on $B \sqcup C$ by putting $d(b, b') = d^{\mathfrak{B}}(b, b')$ if $b, b' \in \mathfrak{B}$, $d(c, c') = d^{\mathfrak{B}}(c, c')$ if $c, c' \in \mathfrak{C}$ and $d(b, c) = \min_{a \in \mathfrak{A}} d^{\mathfrak{B}}(b, \varphi_1 a) + d^{\mathfrak{C}}(c, \varphi_2 a)$ if $b \in \mathfrak{B}$ and $c \in \mathfrak{C}$. Define then $\mathfrak{D}$ as the metric space induced by d and $\psi_1 : \mathfrak{B} \to \mathfrak{D}$, $\psi_2 : \mathfrak{C} \to \mathfrak{D}$ the embeddings respectively induced by $B \subseteq B \sqcup C$ and $C \subseteq B \sqcup C$. Then it is straightforward to check that $d(\psi_1 \varphi_1 a, \psi_2 \varphi_2 a) = 0$ for every $a \in \mathfrak{A}$. From there, one easily gets that $\mathcal{K}$ is a Fraïssé class. The Fraïssé limit of $\mathcal{K}$ is called the *Urysohn space*.

Let us now highlight a nice comparison with a famous classical structure. The Rado graph RG is a countable graph and is also called the random graph because it is obtained almost surely by putting independently and with probability $p \in (0, 1)$ a vertex between n and m for every $n \neq m \in \mathbf{N}$. Another construction of RG is done by taking the Fraïssé limit of the class of finite graphs and in particular RG is ultrahomogeneous and contains every finite graph. One can see a metric space as a weighted graph. Thus, the construction of the Urysohn space as a Fraïssé limit is very similar to the one of RG, so much so that Vershik proved that the Urysohn space is the random Metric Space in some sense [Ver04] (see also [Usv08] for a more general treatment of this phenomenon). RG features nice model-theoretic properties and it is not surprising to see that the Urysohn space is a nice structure too. As RG, the Urysohn space is ω-categorical. Finally, it is not hard to see that the theory of RG is the model completion of the theory of graphs ; analogously, the theory of the Urysohn space is the model completion of the theory of metric spaces.

It is a classical fact that the isometry group of $\mathbb{U}$ is Lévy and thus, extremely amenable. As pointed out by Tsankov and Melleray, the fact that $\mathrm{Iso}(\mathbb{U})$ is Lévy can be seen by using combinatorial properties of the class of finite metric spaces (see [MT14] Theorem 4.6).

The Hilbert space

The fact that the infinite-dimensional separable Hilbert space is the Fraïssé limit of the finite-dimensional Hilbert spaces is quite easy to see, the amalgamation being performed using direct sums. Gromov and Milman [GM83] proved directly that the isometry group of ℓ_2 is Lévy and thus, extremely amenable.

The Gurarij space

Instead of finite metric spaces or Hilbert spaces, it is natural to wonder what happens if we consider normed vector spaces. By a very similar procedure of free amalgamation, we obtain the universal (or random) Banach space named the Gurarij space $\mathbb{G}$. $\mathbb{G}$ is ω-categorical and is the model completion of the theory of Banach spaces.

The case of the Gurarij space is particularly interesting because the first known proof of the extreme amenability of $\mathrm{Aut}(\mathbb{G})$ was provided by Bartošová, López-Abad, Lupini and Mbombo and used a KPT correspondence [BLLM21]. The fact that the class of all finite dimensional Banach spaces has the (ARP) is not so easy to see, but they manage to reduce this problem to subclasses, either $\{\ell_\infty^n\}_{n\in\mathbf{N}}$ or the class of polyhedral spaces. It is not known if $\mathrm{Aut}(\mathbb{G})$ is a Lévy group.

L_p spaces

In this section, we fix $1 \leq p < \infty$, $p \neq 2$. Let us consider L_p spaces as metric structures, in the signature of Banach spaces. We will see below that some model-theoretic properties of L_p differ between $p = 4, 6, ...$ and $p \neq 2, 4, 6, ...$ but before that, let us explain some general facts. First of all, Krivine and Henson proved that the class of L_p spaces is axiomatizable which is not so easy to see as a syntactical property. Moreover, atomicity is in some sense definable in L_p spaces and as a consequence, the separable space $L_p(0,1)$ is ω-categorical. ω-categoricity implies some sort of homogeneity but it is not a sufficient condition to get approximate ultrahomogeneity. In the the general case, the theory of $L_p(0,1)$ does not have quantifier elimination (although it is model complete) so $L_p(0,1)$ cannot be (AuH). The best we can do is the following :

Theorem 4.1 ([FLMT20], Example 2.8)**.** *$L_p(0,1)$ is approximately ultrahomogeneous if and only if p is not even.*

On one hand, ultrahomogeneity of L_p is a rephrasing of Plothkhin and Rudin's equimeasurability Theorem that we state now.

Theorem 4.2 (Equimeasurability Theorem, [Rud76] Theorem I)**.** *Let $1 \leq p < +\infty$ not even. If $f = (f_1, ..., f_n) \in L_p(0,1)$ and $g = (g_1, ..., g_n) \in L_p(0,1)$ are such that :*

$$\int_0^1 |1 + a_1 f_1(\omega) + ... + a_n f_n(\omega)|^p d\omega = \int_0^1 |1 + a_1 g_1(\omega) + ... + a_n g_n(\omega)|^p d\omega$$

for every scalar $a_1, ..., a_n$, then f and g are equimeasurable, i.e. for every Borel set A, $\{\omega : f(\omega) \in A\}$ and $\{\omega : g(\omega) \in A\}$ have the same measure.

In model-theoretic terms, this theorem establishes that two tuples which have the same quantifier-free type (over the parameter 1, to be precise) are jointly equidistributed. With a bit of work, we can conclude that if two tuples f and g have the same quantifier-free type, then one can be sent to the other up to ε by an isometry which is exactly the property of approximate ultrahomogeneity.

Theorem 4.1 is essentially due to Lusky [Lus78] but the terminology is more modern. Ferenczi et al. [FLMT20] proved a stronger result, stating that $L_p(0,1)$ is *Fraïssé* according to their terminology. To this end, they naturally used a Fraïssé-like correspondence combined with an improvement of the Equimeasurability Theorem. It turns out that a separable Banach space is Fraïssé if it is approximately ultrahomogeneous and ω-categorical [FR23] so the use of the Fraïssé correspondence can be avoided. It is not known if there exist approximately ultrahomogeneous Banach spaces that are not ω-categorical.

On the side of Ramsey theory, Giordano and Pestov proved that the group of isometries of $L_p(0,1)$ is Lévy, and thus extremely amenable ([GP07] Theorem 6.6). This is done thanks to the Banach-Lamperti's Theorem describing the isometries as induced by transformations of the underlying probability space. This allows to write the group of isometries as a semi-direct product which is Lévy.

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