Introduction to Complex Cobordism

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0 Introduction

The main goal of this article is to introduce the complex cobordism theory and prove that it is the universal complex-oriented cohomology theory. The precise statement is given in Theorem 3.2 followed by a detailed proof of the theorem.

To begin with, we will discuss generalized cohomology theories in the first chapter. A generalized cohomology theory is similar to singular cohomology in many aspects, while its value at a point can be much more complicated and hence contains much more information than the singular cohomology. In Section 1.2, we will introduce the notion of spectrum which is highly related to the theory of generalized cohomology and bears its own interest. Since a comprehensive discussion of spectrum is rather complicated, we will be satisfied with only presenting the basics. Readers who are interested in this topic are welcome to consult Adams' book [1].

In Chapter Two, we will first define the MU spectrum, and then define the complex cobordism theory as the cohomology theory associated with it. In fact, this is just a special case of a bigger machinery. In general, one can consider a $(\mathfrak{B}, f)$-structure which imposes a special structure on normal bundles of manifolds. From such a structure, one can construct the associated spectrum, called Thom spectrum, and then define the associated cohomology theory. It turns out that the resulting cohomology theory has a concrete geometric interpretation which follows from the special structure imposed on normal bundles. In our case, the MU spectrum corresponds to the stable complex structure on normal bundles, and the complex cobordism ring of a manifold X classifies all complex-oriented morphisms to it up to cobordant relations. The proof of this geometric interpretation relies on the Thom–Pontryagin construction.

Among all the generalized cohomology theories, a significant portion are of special interest. They are the so-called complex-oriented cohomology theories and constitute the topics of the third chapter. One of the reasons they are particularly useful is that we can define Conner–Floyd Chern classes in these cohomology theories, which mimic the ordinary Chern classes in singular cohomology. We will prove that the complex cobordism defined in Chapter II is the universal complex-oriented cohomology theory. We will also see that Conner–Floyd Chern classes play an important role in the proof of this result.

1 Spectrum and generalized cohomology theory

In this section, we will first recall the definition of a generalized cohomology theory. Then, we will introduce spectrum and define the stable homotopy category. Finally, in the last section, we will show how to construct a generalized cohomology theory from a spectrum.

The effort to generalize singular cohomology was first made by Eilenberg and Steenrod in their paper [4] and book [3]. However, at that time, they still required the cohomology ring of a point to be centralized at degree zero, which is called "dimension axiom" by them. Later, mathematicians removed this axiom to include more cohomology theories into account. For more modern treatment, one can view Chapter II of Kono and Tamaki's book [6].

1.1 Generalized cohomology theory

In this section, we will briefly recall what a generalized cohomology theory is without writing down all the details. We will first introduce the definition of a relative cohomology theory, and then the reduced version. Finally, we will define multiplicative cohomology theories. For more detailed discussion, readers can consult Chapter II of Kono and Tamaki's book [6].

A CW-pair (X, A) consists of a CW-complex X and its subcomplex A. A morphism from a CW-pair (X, A) to another (Y, B) is a continuous map $f : X \to Y$ such that $f(A) \subseteq B$. A *relative cohomology theory* is a contravariant functor E^* from the category of CW-pairs to the category of graded abelian groups, together with a connection morphism $\delta : E^*(A) := E^*(A, \varnothing) \to E^{*+1}(X, A)$ for any CW-pair (X, A), and satisfies the properties similar to singular cohomology. To be more concrete, it should be homotopy invariant, and should admit the excision isomorphism. And for any CW-triple (X, A, B) with $B \subseteq A \subseteq X$, we have a long exact sequence

$$\cdots \to E^*(X, A) \to E^*(X, B) \to E^*(A, B) \xrightarrow{\delta} E^{*+1}(X, A) \to \cdots$$

Here, the first two maps are induced by inclusions, and the third one is given by the composition $E^*(A, B) \to E^*(A) \xrightarrow{\delta} E^{*+1}(X, A)$, and will be still denoted by δ.

Similar to singular cohomology, given a relative cohomology theory E^*, one can consider the reduced version $\widetilde{E}^*$ of it. It is defined by $\widetilde{E}^*(X) := E^*(X, x)$ for any pointed CW-complex X with the basepoint $x \in X$. Now, $\widetilde{E}^*$ becomes a contravariant functor from the category of pointed CW-complexes to the category of graded abelian groups. Moreover, it is equipped with a suspension isomorphism $\sigma : \widetilde{E}^{*+1}(\Sigma X) \xrightarrow{\sim} \widetilde{E}^*(X)$. It is also homotopy invariant. And for any CW-pair (X, A), we have an exact sequence

$$\widetilde{E}^*(X \cup_A CA) \to \widetilde{E}^*(X) \to \widetilde{E}^*(A),$$

where $X \cup_A CA$ is the mapping cone of the inclusion $A \hookrightarrow X$. This sequence can be extended by taking iterated mapping cones.

Although a generalized cohomology theory is similar to singular cohomology in many ways, there is a huge difference between them. The value of a generalized cohomology theory at a point is not required to be centralized at degree zero. That is to say, the graded group $E^*(pt)$, called the coefficient group of E^*, can be rather complicated and has elements of non-zero degrees. Although this makes generalized cohomology theories hard to compute in general, they often contain much more information than singular cohomology and hence are more useful in many cases.

Finally, if the image of the functor E^* actually lies in the subcategory of graded rings, then we say E^* is a multiplicative cohomology theory. This means that for any CW-pair (X, A), the graded abelian group $E^*(X, A)$ is equipped with a graded product that is compatible with the pull-back morphisms, and we will call it the cup product of E^*.

1.2 Spectrum

Apart from the abstract definition of generalized cohomology theories given above, we will introduce a more concrete way using spectra. Since the rigorous description of the theory of spectrum is rather complicated, we are satisfied to give a somewhat "naive" version here, which is, nevertheless, sufficient for our discussion. More comprehensive discussion about spectrum can be found in Adams' book [1].

Definition 1.1. *A spectrum E is a sequence of pointed CW-complexes E_n indexed by $n \in \mathbb{Z}_{\geqslant 0}$ with cellular structure maps $\alpha_n : \Sigma E_n \to E_{n+1}$.*

A natural way to construct a spectrum is the suspension spectrum $\Sigma^\infty X$ of a pointed CW-complex X. It is defined by $(\Sigma^\infty X)_n = \Sigma^n X$, and the structure maps are just the identity maps

$\Sigma(\Sigma^n X) = \Sigma^{n+1}X$. In particular, when X is a point, the n-th component of the spectrum $\Sigma^\infty X$ is just the n-sphere. So we will call the resulting spectrum the sphere spectrum and denote it by $\mathbb{S}$.

For any interger k, we can shift a spectrum E by k and denote the resulting spectrum by $\Sigma^k E$. It is defined by $(\Sigma^k E)_n = E_{n+k}$ with the structure maps induced from those of E. In particular, we can shift a suspension spectrum $\Sigma^\infty X$ by k and denote it by $\Sigma^{\infty+k}X$.

The definition of morphisms between spectra is more complicated. Since each spectrum is a sequence of pointed CW-complexes and what we are really interested in is what happens when the index tends to infinity, we should allow a morphism to be defined on a given cell of E_n only after possibly increasing n by the structure maps. To this end, we quote Adams' slogan "cells now - maps later" from his book [1]. The precise definition of morphisms between spectra can be found in Chapter 2, Part III of the same book of Adams and will be omitted here.

Spectra are similar to pointed CW-complexes in many aspects. For example, one can define the homotopy between two spectrum morphisms. Following Adams' notation, we will use $[E, F]_r$ to define the homotopy classes of morphisms of degree r from spectrum E to F. Here, a morphism of degree r means that it decreases the indexes of the components of E by r. Or naively, it maps E_n to F_{n-r} for every $n \in \mathbb{Z}_{\geqslant 0}$. And we have natural ismorphism $[E, F]_r \cong [\Sigma^r E, F]_0$. When $r = 0$, we will often omit the subcript and write only $[E, F]$. The category consists of spectra and homotopy classes of morphisms between them is called the stable homotopy category.

Now, for any interger r, we can define the r-th homotopy group of a spectrum E by $\pi_r(E) := [\mathbb{S}, E]_r$. In particular, every spectrum E admits a canonical morphism $\mathbb{S} \to E$ which is induced by the inclusion of the basepoint of E_0. This morphism will be called the unit morphism of E and will be denoted by η. The element it represents in $\pi_0(E) = [\mathbb{S}, E]$ will be called the unit element of $\pi_*(E)$, and denoted by 1.

Another similarity between spectra and pointed CW-copmlexes is that we can also define the smash product of two spectra. However, it is only well-defined in the stable homotopy category, *i.e.*, it can only be defined up to homotopy. Let E and F be two spectra, we will use $E \otimes F$ to denote their smash product. One should always view it as an object in the stable homotopy category, or a spectrum up to homotopy. It is easy to see that the sphere spectrum is the unit object under this product, *i.e.*, we have $\mathbb{S} \otimes E \cong E \otimes \mathbb{S} \cong E$ in the stable homotopy category for any spectrum E. The smash products of suspension spectra are easy to describe, and we have $\Sigma^\infty X \otimes \Sigma^\infty Y \cong \Sigma^\infty(X \wedge Y)$ for any pointed CW-complexes X and Y.

1.3 From spectrum to generalized chomology theory

Given a spectrum E, we can define a reduced cohomology theory $\widetilde{E}^*$. The cohomology group of any pointed CW-complex X is given by $\widetilde{E}^*(X) := [\Sigma^\infty X, E]_{-*}$. And the suspension isomorphism is given by

$$\sigma : \widetilde{E}^{*+1}(\Sigma X) = [\Sigma^\infty(\Sigma X), E]_{-*-1} \cong [\Sigma^{\infty-*}X, E] \cong [\Sigma^\infty X, E]_{-*} = \widetilde{E}^{-*}(X).$$

We will denote the corresponding relative cohomology theory by E^*.

To make E^* into a multiplicative cohomology theory, we introduce the notion of a ring spectrum.

Definition 1.2. *A spectrum E is a ring spectrum if there exists a spectrum morphism $\mu : E \otimes E \to E$ such that it is associative and unitary with respect to the unit morphism η of E.*

The morphism μ allows us to define a product structure on the reduced cohomology theory $\widetilde{E}$, making it into a multiplicative cohomology theory. It is given as follows: let X and Y be two pointed CW-complexes, then we have

$$\begin{aligned}\widetilde{E}^n(X) \otimes_{\pi_*(E)} \widetilde{E}^m(Y) &= [\Sigma^\infty X, E]_{-n} \otimes_{\pi_*(E)} [\Sigma^\infty Y, E]_{-m} \\ &\to [\Sigma^\infty X \otimes \Sigma^\infty Y, E \otimes E]_{-n-m} \\ &\to [\Sigma^\infty (X \wedge Y), E]_{-n-m} = \widetilde{E}^{n+m}(X \wedge Y).\end{aligned}$$

Here, the first morphism is given by taking the smash product of spectrum maps, and the second is induced by $\mu : E \otimes E \to E$. If we take Y equal to X and pull back the image by the diagonal morphism $\Delta : X \to X \wedge X$, then we get the cup product of $\widetilde{E}^*(X)$:

$$\widetilde{E}^n(X) \otimes_{\pi_*(E)} \widetilde{E}^m(X) \to \widetilde{E}^{n+m}(X \wedge X) \xrightarrow{\Delta^*} \widetilde{E}^{n+m}(X).$$

Therefore, the associated cohomology of a ring spectrum is always multiplicative, and we will only consider cohomology theories and spectra of this kind in the following.

Remark 1.1. *We have shown how to construct a cohomology theory from a spectrum. In fact, any cohomology theory can be constructed in this way. This is due to Brown's representation theorem. See Brown's original paper [2] for detailed discussion.*

It is easy to see that a spectrum morphism $\varphi : E \to F$ induces a morphism of the corresponding cohomology theories. For any pointed CW-complex X, the morphism $\widetilde{E}^*(X) \to \widetilde{F}^*(X)$ is defined by

$$\varphi_*(X) : \widetilde{E}^*(X) = [\Sigma^\infty X, E]_{-*} \to [\Sigma^\infty X, F]_{-*} = \widetilde{F}^*(X),$$

where the morphism in the middle is given by the composition with φ.

2 Complex cobordism theory

In the first section, we will first define the so-called MU spectrum, and define the complex cobordism as the cohomology theory associated to it. Although complex cobordism is defined for any topological space, when restriced to manifolds, it admits a rather concrete geometric interpretation. This will be the content of the second section.

2.1 The MU spectrum

For any positive interger n, we may consider the classigying space $BU(n)$ of the uniraty group $U(n)$ and the associated universal complex vector bundle ζ_n of rank n over it. When $n = 0$, we take $BU(n)$ to be a point, and ζ_0 be the zero vector bundle over it, *i.e.*, its fiber is just a point.

Definition 2.1. *For every $n \geqslant 0$, we define the spectrum $MU(n)$ to be $MU(n) := \Sigma^{\infty-2n} Th(\zeta_n)$. Here, $Th(\zeta)$ denotes the Thom space of a vector bundle ζ, which is defined by the quotient of the disc bundle $D(\zeta)$ of ζ by the sphere bundle $S(\zeta)$ of ζ.*

Remark 2.1. *When $n = 0$, we see that $MU(0)$ is isomorphic to the sphere spectrum $\mathbb{S}$.*

Now we define morphisms between these spectra.

If we consider the rank $n+1$ vector bundle $\zeta_n \oplus \mathbb{C}$ over $BU(n)$, where $\mathbb{C}$ denotes the trivial complex vector bundle of rank one, then there exists a unique (up to homotopy) classifying map from $BU(n)$ to $BU(n+1)$ making the following diagram a pullback diagram

$$\begin{tikzcd}
\zeta_n \oplus \mathbb{C} \arrow[r] \arrow[d] & \zeta_{n+1} \arrow[d] \\
BU(n) \arrow[r] & BU(n+1).
\end{tikzcd}$$

Thus, it induces a morphism of Thom spaces $Th(\zeta_n \oplus \mathbb{C}) \to Th(\zeta_{n+1})$. One can verify without difficulty that $Th(\zeta_n \oplus \mathbb{C}) \cong \Sigma^2 Th(\zeta_n)$, therefore we get a morphism $\Sigma^2 Th(\zeta_n) \to Th(\zeta_{n+1})$ and hence a morphism between spectra $\alpha_n : MU(n) \to MU(n+1)$.

Now, we take all $MU(n)$ and the morphisms constructed above into account, we can define the spectrum MU as the colimit of them.

Definition 2.2. *The sepctrum MU is defined by $MU = \varinjlim MU(n)$.*

Remark 2.2. *By the definition of colimit, for every n there exists a natural morphism of spectra $\beta_n : MU(n) \to MU$. In particular, when $n = 0$, we have $MU(0) \cong \underline{\mathbb{S}}$, and the morphism β_0 is just the unit morphism of MU.*

Moreover, the spectrum MU is also a ring specturm. Let m and n be two non-negative intergers, then we have a natural morphism $BU(n) \times BU(m) \to BU(n+m)$ induced by taking the direct sum of vector bundles. After taking Thom spaces, it induces $Th(\zeta_n) \wedge Th(\zeta_m) \cong Th(\zeta_n \times \zeta_m) \to Th(\zeta_{n+m})$, and hence a spectrum morphism $\gamma_{n,m} : MU(n) \otimes MU(m) \to MU(n+m)$. After taking colimit, we get a specturm morphism $\gamma : MU \otimes MU \to MU$. One can check that this morphism is compatible with the unit morphism $\beta_0 : \underline{\mathbb{S}} \cong MU(0) \to MU(n)$, and hence makes MU into a ring spectrum

Recall that we have seen in Section 1.3 how to construct a cohomology theory from a spectrum.

Definition 2.3. *The complex cobordism MU^* is the cohomology theory associated with the spectrum MU.*

Remark 2.3. *Since MU is a ring spectrum, MU^* is a multiplicative cohomology theory.*

2.2 Geometric interpretation

It turns out that the complex cobordism, despite its abstract definition, admits a rather concrete geometric interpretation for spaces X that are nice enough. In this section, we will only consider spaces which are closed smooth submanifolds of some Euclidean spaces, and refer to them as manifolds. Before giving the geometric interpretation, we will need the notion of a complex orientation of a smooth morphism between manifolds.

Definition 2.4. *Let $f : X \to Y$ be a smooth morphism between manifolds. Then we define its virtual normal bundle to be $v_f := f^*TY - TX \in KO(X)$, where $KO(X)$ denotes the real K-theory of X. A complex orientation of f is defined to be a stable complex structure on the virtual normal bundle v_f.*

Remark 2.4. *A stable complex structure on a vector bundle E is a complex structure on the vector bundle $E \oplus \mathbb{R}^n$ for some n, where $\mathbb{R}$ denotes the trivial bundle. And we identify the complex structure on $E \oplus \mathbb{R}^n$ with the induced complex structure on $E \oplus \mathbb{R}^{n+2} \cong E \oplus \mathbb{R}^n \oplus \mathbb{C}$, where $\mathbb{R}^2 \cong \mathbb{C}$ comes from a fixed isomorphism as $\mathbb{R}$-vector spaces.*

Next, we will define when two proper complex-oriented morphisms are cobordant.

Definition 2.5. *Let X_i, $i = 0, 1$, be two manifolds with the same dimension n. And let $f_i : X_i \to Y$ be two proper complex-oriented morphisms. They are cobordant if and only if there exists a manifold Z of dimension $n + 1$ and a proper complex-oriented morphism $c : Z \to Y \times \mathbb{R}$ such that c is transversal to $Y \times \{i\}$, $i \in \{0, 1\}$, and the pull-back of c to these submanifolds coincide with the morphisms f_i with the same complex orientations.*

Now for any manifold X and interger n, we use $\Omega_U^n(X)$ to denote the set of homotopy classes of all proper complex-oriented morphisms to X of dimension $-n$, where the dimension of a morphism from Y to X is defined to be $dim(Y) - dim(X)$. And we use $\Omega_U^*(X)$ to denote the disjoint union of all $\Omega_U^n(X)$ for intergers n.

It turns out that $\Omega_U^*(X)$ admits a graded ring structure. Let $f_i : Y_i \to X$, $i = 0, 1$, be two proper complex-oriented morphisms. The product of them is defined by pulling back the product morphism $f_0 \times f_1 : Y_0 \times Y_1 \to X \times X$ along the diagonal map $\Delta : X \to X \times X$. Here, we may assume that $f_0 \times f_1$ and Δ are transversal by Thom's transversality theorem. And when Y_0 and Y_1 have the same dimension, we can define the sum of them to be the disjoint union $f_0 \sqcup f_1 : Y_0 \sqcup Y_1 \to X$. One can verify that these two operations make $\Omega_U^*(X)$ into a graded ring. Moreover, it is isomorphic to the complex cobordism ring $MU^*(X)$ by the following theorem.

Theorem 2.1. *Let X be a manifold, and n be any interger. Then we have $MU^*(X) \cong \Omega_U^*(X)$ as graded rings.*

The proof of this theorem is rather complicated and will be omitted here. In fact, this theorem can be generalized to any so called $(\mathfrak{B}, f)$-structure, which are used to impose spectial structures on normal bundles of manifolds. For example, the $(\mathfrak{B}, f)$-structure we used here is the BU-structure, and it imposes a stable complex structure on the normal bundle. From any $(\mathfrak{B}, f)$-structure, one can associate a spectrum and hence a cohomology theory (the complex cobordism MU^* in our case). And it turns out that the resulting cohomology ring of any manifold X admits a similar geometric interpretation which concerns the morphisms with the virtual normal bundle admitting such a special structure up to cobordant relation. The heart of the proof of this result relies on the so-called Thom–Pontryagin construction. Detailed discussion on this topic can be found in Chapter I of Kochman's book [5].

Remark 2.5. *As a special case of Theorem 2.1, if we take X to be a point, then we get a geometric interpretation for the coefficient ring $MU^*(pt) = \pi_*(MU)$. In this case, a complex orientation of a morphism from a manifold M to a point is equivalent to a stable complex structure on the tangent bundle of M. Meanwhile, the cobordant relation between two morphisms from M_i, $i \in \{0, 1\}$, to a point becomes the bordant relation between manifolds M_i. If we define a weakly complex manifold to be a manifold with a stable complex structure on its tangent bundle, then $\pi_*(MU)$ is the ring of bordant classes of closed weakly complex manifolds, and is called the complex cobordism ring. In 1960, this ring was first computed by Milnor as $\pi_*(MU) \cong \mathbb{Z}[y_1, y_2, \cdots]$ with $y_i \in \pi_{2i}(MU)$ in his paper [7].*

3 Complex-oriented cohomology theory

In this chapter, we will define when is a generalized cohomology theory complex-oriented. And we will prove that complex cobordism is the universal complex-oriented cohomology theory.

3.1 Complex-orientation

Definition 3.1. *A multiplicative cohomology theory* E^* *is said to be complex orientable if there exists an element* $t_E \in \widetilde{E}^2(\mathbb{CP}^\infty)$ *such that* $i^*(t_E) \in \widetilde{E}^2(\mathbb{CP}^1) \cong \widetilde{E}^2(\mathbb{S}^2) \cong \pi_0(E)$ *is the unit. Here,* $i : \mathbb{CP}^1 \hookrightarrow \mathbb{CP}^\infty$ *is the natural inclusion. The element* t_E *is called a complex orientation of the cohomology theory* E^**, and* E^* *together with* t_E *is called a complex-oriented cohomology theory.*

Remark 3.1. *A cohomology theory may admit different complex orientations.*

As an example, one can show that singular cohomology is complex-oriented. In fact, we can choose the complex orientation $t \in H^2(\mathbb{CP}^\infty, \mathbb{Z})$ to be the generator of $H^*(\mathbb{CP}^\infty, \mathbb{Z}) \cong \mathbb{Z}[[t]]$ as a ring of power series over $\mathbb{Z}$. It is indeed a complex orientation since the restriction of t to $\mathbb{CP}^1$ is the generator of $H^2(\mathbb{CP}^1, \mathbb{Z}) \cong \mathbb{Z}[t]/(t^2) \cong \mathbb{Z}$.

Another example is the complex cobordism we defined before. In fact, since $U(1) \cong \mathbb{S}^1$, the sphere bundle of ζ_1 can be identified with the universal $U(1)$-principle bundle over $BU(1)$, and hence is contractible. Therefore, the natural inclusion $\mathbb{CP}^\infty \cong BU(1) \hookrightarrow Th(\zeta_1)$ is a homotopy equivalence. And we have

$$\widetilde{MU}^2(\mathbb{CP}^\infty) = [\Sigma^{\infty-2}\mathbb{CP}^\infty, MU] \cong [\Sigma^{\infty-2}Th(\zeta_1), MU] = [MU(1), MU].$$

Therefore, the morphism $\beta_1 : MU(1) \to MU$ that appears in the definition of MU corresponds to an element $t_{MU} \in \widetilde{MU}^2(\mathbb{CP}^\infty)$. In the following, we will prove that t_{MU} gives a complex orientation of MU.

In fact, unwrapping the definition, t_{MU} being a complex orientation means that the morphism

$$\underline{\mathbb{S}} \cong \Sigma^{\infty-2}\mathbb{CP}^1 \to \Sigma^{\infty-2}\mathbb{CP}^\infty \to \Sigma^{\infty-2}Th(\zeta_1) = MU(1) \to MU$$

is the unit morphism of MU. Meanwhile, recall that the unit morphism of MU is given by

$$\beta_0 : \underline{\mathbb{S}} \cong MU(0) \to MU(1) \to MU.$$

Since the map $MU(0) \to MU(1)$ is induced by the bundle map $\zeta_0 \oplus \mathbb{C} \to \zeta_1$, it is equivalent to consider the inclusion of a fiber F into ζ_1. And the unit morphism is given by

$$\underline{\mathbb{S}} \cong \Sigma^{\infty-2}Th(F) \to \Sigma^{\infty-2}Th(\zeta_1) = MU(1) \to MU.$$

Therefore, t_{MU} is indeed a complex orientation of MU by the following lemma.

Lemma 3.1. *The following two elements in* $\pi_2(\mathbb{CP}^\infty)$ *are the same.*

- *The first one is induced by the inclusion* $\mathbb{S}^2 \cong \mathbb{CP}^1 \hookrightarrow \mathbb{CP}^\infty$*.*
- *The second is given as follows. Let* ζ_1 *be the tautological bundle over* $BU(1) \cong \mathbb{CP}^\infty$*, then the inclusion of a fiber* F *of* ζ_1 *induces a morphism* $\mathbb{S}^2 \cong Th(F) \to Th(\zeta_1)$ *and hence defines an element of* $\pi_2(Th(\zeta_1)) \cong \pi_2(\mathbb{CP}^\infty)$*. Here, we recall that the inclusion* $BU(1) \hookrightarrow Th(\zeta_1)$ *is a homotopy equivalence.*

Proof. We first prove that $\pi_2(\mathbb{CP}^\infty) \cong \mathbb{Z}$. First, we consider $\mathbb{CP}^n$. If we view $\mathbb{CP}^n$ as the set of lines in $\mathbb{C}^{n+1}$ and identify $\mathbb{S}^{2n+1}$ with the unit sphere of $\mathbb{C}^{n+1}$, then we get a fibration $\mathbb{S}^1 \hookrightarrow \mathbb{S}^{2n+1} \to \mathbb{CP}^n$. It induces a long exact sequence of homotopy groups

$$\cdots \to \pi_2(\mathbb{S}^{2n+1}) \to \pi_2(\mathbb{CP}^n) \to \pi_1(\mathbb{S}^1) \to \pi_1(\mathbb{S}^{2n+1}) \to \cdots$$

When $n \geqslant 1$, both $\pi_2(\mathbb{S}^{2n+1})$ and $\pi_1(\mathbb{S}^{2n+1})$ are trivial, and hence we get an isomorphism between $\pi_2(\mathbb{CP}^n)$ and $\pi_1(\mathbb{S}^1) \cong \mathbb{Z}$. After taking colimits, we have $\pi_2(\mathbb{CP}^\infty) \cong \mathbb{Z}$. Therefore, the Hurewicz homomorphism $\pi_2(\mathbb{CP}^\infty) \to H_2(\mathbb{CP}^\infty, \mathbb{Z}) \cong \mathbb{Z}$ is an isomorphism. And to prove that two elements in $\pi_2(\mathbb{CP}^\infty)$ are equal is equivalent to prove that they are equal in $H_2(\mathbb{CP}^\infty, \mathbb{Z})$.

Now the first elements obviously corresponds to the generator of the group $H_2(\mathbb{CP}^\infty, \mathbb{Z})$. So to prove that the second elements also corresponds to the generator, we only need to prove that the induced morphism $H_2(Th(F), \mathbb{Z}) \to H_2(Th(\zeta_1), \mathbb{Z})$ is an isomorphism. This is true since it corresponds to the Thom isomorphism $H^2(Th(\zeta_1), \mathbb{Z}) \xrightarrow{\sim} H^2(F, F\backslash\{0\}, \mathbb{Z})$ under Poincaré duality.
□

3.2 Conner–Floyd Chern class and the computation of certain cohomology rings

One of the reasons why complex oriented cohomology theories are interesting is because the existence of a complex orientation allows us to mimic the definition of Chern classes in the context of our new cohomology. These classes are called Conner–Floyd Chern classes since they first appeared in Conner and Floyd's book [8] in 1966. Now we begin our definition of these classes by computing certain cohomology rings.

Proposition 3.1. *Let E^* be a complex-oriented cohomology theory with the complex orientation t_E. Then we have*

$$E^*(\mathbb{CP}^\infty) \cong \pi_*(E)[[t_E]].$$

Now, let E^* be a complex-oriented cohomology theory. For every non-negative interger n, we consider the rank n vector bundle $\zeta_1^n := \oplus_{i=1}^n p_i^* \zeta_1$ over $BU(1)^n$ where $p_i : BU(1)^n \to BU(1)$ is the projection onto the i-th component. Then it induces a classifying morphism $\gamma_n : BU(1)^n \to BU(n)$, and hence a morphism of cohomology rings

$$\gamma_n^* : E^*(BU(n)) \to E^*(BU(1))^{\otimes n} \cong \pi_*(E)[[x_1, \cdots, x_n]].$$

Here, the isomorphism comes from Proposition 3.1. Moreover, since permuting the components of $BU(1)^n$ does not change the morphism γ_n, the image of γ_n^* is stable under the permutation of x_i. Hence, if we define c_k to be the k-th elementary symmetric polynomial in x_i, then we have

$$\gamma_n^* : E^*(BU(n)) \to \pi_*(E)[[c_1, \cdots, c_n]] \subseteq E^*(BU(1))^{\otimes n}.$$

Moreover, we can prove that this is in fact an isomorphism.

Proposition 3.2. *Let E^* be a complex-oriented cohomology theory with the complex orientation t_E. Then for every non-negative interger n, we have*

$$E^*(BU(n)) \cong \pi_*(E)[[c_1, \cdots, c_n]],$$

where $c_k \in E^{2k}(BU(n))$ is defined as above. In particular, when $n = 1$, we have $c_1 = t_E \in E^2(BU(1)) \cong E^2(\mathbb{CP}^\infty)$.

Under this isomorphism, the morphism $E^*(BU(n)) \to E^*(BU(n-1))$ induced by the natural morphism $BU(n-1) \to BU(n)$ is given by sending c_i to c_i for $1 \leqslant i \leqslant n-1$ and sending c_n to 0.

The proofs of Proposition 3.1 and Proposition 3.2 rely on the Atiyah–Hirzebruch spectral sequence (AHSS), which allows us to compute generalized cohomology using singular cohomology with the coefficient being the coefficient ring of the generalized cohomology.

Theorem 3.1 (AHSS)**.** *Let E^* be a generalized cohomology theory. Then for any finite CW-complex X, we have a spectral sequence whose second page is given by $E_2^{p,q} = H^q(X, E^p(pt))$ and converges to $E^{p+q}(X)$.*

This spectral sequence is an important tool when dealing with generalized cohomology theories. One can find its proof and how to apply it to prove Propositions 3.1 and 3.2 in Kochman's book [5].

Following from Proposition 3.2, we can define Conner–Floyd Chern classes as the pullbacks of those classes c_i.

Definition 3.2. *Let E^* be a complex-oriented cohomology theoet, and let ζ be a complex vector bundle of rank n over a paracompact manifold X. Then there exists a classifying map φ from X to $BU(n)$ induced by ζ. We define $c_i^E(\zeta) := \varphi^* c_i \in E^{2i}(X)$ for $1 \leqslant i \leqslant n$. They are called Conner–Floyd Chern classes of ζ in the cohomology theory E^*.*

Remark 3.2. *When we take E^* to be the singular cohomology, then the Conner–Floyd Chern classes defined above are just the usual Chern classes. Therefore, Conner–FLoyd Chern classes are the generzlization of Chern classes in generalized cohomology theories.*

Now, we are ready to compute the cohomology ring $E^*(Th(\zeta_n))$. For every n, we use $EU(n) \to BU(n)$ to denoted the universal $U(n)$-bundle. Then we have $\zeta_n \cong EU(n) \times_{U(n)} \mathbb{C}^n$, and its sphere bundle is given by $S(\zeta_n) \cong EU(n) \times_{U(n)} \mathbb{S}^{2n-1}$. Meanwhile, we have an $U(n)$-equivariant isomorphism $\mathbb{S}^{2n-1} \cong U(n)/U(n-1)$. Therefore, we have

$$S(\zeta_n) \cong EU(n) \times_{U(n)} U(n)/U(n-1) \cong EU(n)/U(n-1).$$

Since $EU(n)$ is contractible, this space is homotopic to $BU(n-1)$.

Recall that the Thom space $Th(\zeta_n)$ is defined to be the disc bundle $D(Th(\zeta_n))$ quotient by the sphere bundle $S(Th(\zeta_n))$. Therefore, we have the following long exact sequence

$$\cdots \to \widetilde{E}^*(Th(\zeta_n)) \to E^*(D(Th(\zeta_n))) \to E^*(S(Th(\zeta_n))) \to \widetilde{E}^{*+1}(Th(\zeta_n)) \to \cdots$$

Now, since $D(Th(\zeta_n))$ is homotopic to $BU(n)$ and $S(Th(\zeta_n))$ is homotopic to $BU(n-1)$, the long exact sequence can be rewritten as

$$\cdots \to \widetilde{E}^*(Th(\zeta_n)) \to E^*(BU(n)) \to E^*(BU(n-1)) \to \widetilde{E}^{*+1}(Th(\zeta_n)) \to \cdots$$

Moreover, it is not difficult to check that the second arrow is induced by the natural morphism $BU(n-1) \to BU(n)$. So it is surjective by Proposition 3.2, and hence the long exact sequence breaks down to short exact sequences

$$0 \to \widetilde{E}^*(Th(\zeta_n)) \to E^*(BU(n)) \to E^*(BU(n-1)) \to 0.$$

By using the same proposition again, we can compute the cohomology ring of $Th(\zeta_n)$ as follows.

Proposition 3.3. *The inclusion of the zero section* $BU(n) \hookrightarrow Th(\zeta_n)$ *identifies* $\widetilde{E}^*(Th(\zeta_n))$ *as the subring of* $E^*(BU(n))$ *generated by* $c_n \in E^{2n}(BU(n))$*, i.e., we have*

$$\widetilde{E}^*(Th(\zeta_n)) \cong c_n \cdot \pi_*(E)[[c_1, \cdots, c_n]] \subseteq \pi_*(E)[[c_1, \cdots, c_n]] \cong E^*(BU(n)).$$

Remark 3.3. *This proposition also implies that Thom isomorphism still holds in the cohomology theory* E^**. In fact, the top Conner–Floyd Chern class* $c_n \in \widetilde{E}^{2n}(Th(\zeta_n)) \cong E^{2n}(\zeta_n, \zeta_n \backslash BU(n))$ *serves as the universal Thom class of rank* n *bundles, and the Thom isomorphism is given by multiplication with* c_n *under the identifications given above*

$$c_n \cup \cdot : E^*(BU(n)) \cong \pi_*(E)[[c_1, \cdots, c_n]] \xrightarrow{\sim} c_n \cdot \pi_*(E)[[c_1, \cdots, c_n]] \cong \widetilde{E}^{*+2n}(Th(\zeta_n)).$$

3.3 Complex cobordism as the universal complex-oriented cohomology theory

In this section, we will prove that MU^* with the complex orientation t_{MU} is the universal complex-oriented cohomology theory. To be more specific, we have the following theorem.

Theorem 3.2. *Let* E^* *be a complex-oriented cohomology theory and* t_E *be its complex orientation. Then there exists an unique morphism (up to homotopy)* $\varphi : MU \to E$ *of spectra such that the induced morphism* $\varphi_*(\mathbb{CP}^\infty) : \widetilde{MU}^2(\mathbb{CP}^\infty) \to \widetilde{E}^2(\mathbb{CP}^\infty)$ *maps* t_{MU} *to* t_E*.*

Proof. For any non-negative interger n, we have

$$\widetilde{E}^{2n}(Th(\zeta_n)) \cong [\Sigma^{\infty-2n} Th(\zeta_n), E] = [MU(n), E].$$

Therefore, the class $c_n \in \widetilde{E}^{2n}(Th(\zeta_n))$ corresponds to a spectrum morphism $\varphi_n : MU(n) \to E$. Here, we make the convention that $c_0 = 1$ to be the unit element of $\pi_*(E)$. So for $n = 0$, we have $\varphi_0 : MU(0) \cong \mathbb{S} \to E$ as the unit morphism of E. To get a morphism from $MU = \varinjlim MU(n)$ to E, we need to prove that these morphisms φ_n are compatible for different n, *i.e.*, we have $\varphi_n = \varphi_{n+1} \circ \alpha_n$. Here, morphisms $\alpha_n : MU(n) \to MU(n+1)$ are used in the definition of the spectrum MU.

Recall that MU is a ring spectrum and we have constructed a spectrum morphism $\gamma_{n,m} : MU(n) \otimes MU(m) \to MU(n+m)$ for any $n, m \in \mathbb{Z}_{\geqslant 0}$. We claim that the following diagram commutes

$$\begin{tikzcd}
MU(n) \otimes MU(m) \arrow[r, "\gamma_{n,m}"] \arrow[d, "\varphi_n \otimes \varphi_m"'] & MU(n+m) \arrow[d, "\varphi_{n+m}"] \\
E \otimes E \arrow[r, "\mu"'] & E.
\end{tikzcd}$$

Here, the horizontal map $\mu : E \otimes E \to E$ is the structure map of E as a ring spectrum.

To prove this claim, we first translate it into the language of cohomology theory. Since the morphism φ_k corresponds to the class $c_k \in \widetilde{E}^{2k}(Th(\zeta_k))$ for any $k \in \mathbb{Z}_{\geqslant 0}$, the commutativity of the diagram is equivalent to $c_{n+m} \in \widetilde{E}^{2n+2m}(Th(\zeta_{n+m}))$ maps to $c_n \times c_m \in \widetilde{E}^{2n+2m}(Th(\zeta_n) \wedge Th(\zeta_m))$ under the morphism induced by $Th(\zeta_n) \wedge Th(\zeta_m) \to Th(\zeta_{n+m})$. Now, from the commutative diagram

$$\begin{tikzcd}
BU(1)^{n+m} \arrow[r] \arrow[d, equal] & BU(n+m) \arrow[r, hook] & Th(\zeta_{n+m}) \\
BU(1)^n \times BU(1)^m \arrow[r] & BU(n) \times BU(m) \arrow[r, hook] \arrow[u] & Th(\zeta_n) \wedge Th(\zeta_m), \arrow[u]
\end{tikzcd}$$

we get another commutative diagram of cohomology rings

$$\begin{tikzcd}
E^*(BU(1)^{n+m}) \arrow[d, equal] & E^*(BU(n+m)) \arrow[l, hook'] \arrow[d] & \widetilde{E}^*(Th(\zeta_{n+m})) \arrow[l, hook'] \arrow[d] \\
E^*(BU(1)^n \times BU(1)^m) & E^*(BU(n) \times BU(m)) \arrow[l, hook'] & \widetilde{E}^*(Th(\zeta_n) \wedge Th(\zeta_m)). \arrow[l, hook']
\end{tikzcd}$$

Now, by Propositions 3.1, 3.2 and 3.3, we see that all the horizontal maps in the diagram above are injective. Suppose that the formal power series generators of $E^*(BU(1)^{n+m})$ are $x_1, \cdots, x_{n+m}$, then the images of $c_{n+m} \in \widetilde{E}^*(Th(\zeta_{n+m}))$ and $c_n \times c_m \in \widetilde{E}^*(Th(\zeta_n) \wedge Th(\zeta_m))$ are both equal to $x_1 \cdots x_{n+m}$ in $E^*(BU(1)^{n+m}) = E^*(BU(1)^n \times BU(1)^m)$ under the horizontal maps, and hence c_{n+m} maps to $c_n \times c_m$ under the right vertical map.

Now, we can view $BU(0)$ as a point of $BU(1)$ and identity $\zeta_0 \oplus \mathbb{C}$ as the fiber of ζ_1 over that point, then we have the following commutative diagram

$$\begin{tikzcd}
\zeta \oplus \mathbb{C} \cong \zeta_n \times \zeta_0 \arrow[r] \arrow[d] & \zeta_n \times \zeta_1 \arrow[r] \arrow[d] & \zeta_{n+1} \arrow[d] \\
BU(n) \cong BU(n) \times BU(0) \arrow[r] & BU(n) \times BU(1) \arrow[r] & BU(n+1).
\end{tikzcd}$$

Therefore, the morphism $\alpha_n : MU(n) \to MU(n+1)$, which is induced by the bundle morphism $\zeta_n \oplus \mathbb{C} \cong \zeta_n \times \zeta_0 \to \zeta_{n+1}$, can be identified with the upper row of the following diagram

$$\begin{tikzcd}
MU(n) \cong MU(n) \otimes \underline{\mathbb{S}} \arrow[r, "\sim"] \arrow[d, "\varphi_n \cong \varphi_n \otimes id"'] & MU(n) \otimes MU(0) \arrow[r, "id \otimes \alpha_0"] \arrow[d, "\varphi_n \otimes \varphi_0"] & MU(n) \otimes MU(1) \arrow[r, "\gamma_{n,1}"] \arrow[d, "\varphi_n \otimes \varphi_1"] & MU(n+1) \arrow[d, "\varphi_{n+1}"] \\
E \cong E \otimes \underline{\mathbb{S}} \arrow[r, "id \otimes \eta"'] & E \otimes E \arrow[r, equal] & E \otimes E \arrow[r, "\mu"'] & E,
\end{tikzcd}$$

where $\eta : \underline{\mathbb{S}} \to E$ is the unit morphism of E as a ring spectrum.

By the complatibility of η and μ, the lower row of this diagram is just the identity map of E, so the compatibility of φ_n and φ_{n+1} is equivalent to the commutativity of this diagram. Now, the left square trivially commutes since φ_0 is, by definition, equal to the unit morphism η of E. The right square also commutes by our claim proved above. Therefore, only the commutativity of the middle square remains to be proved, which is equivalent to $\varphi_1 \circ \alpha_0 = \varphi_0$. Now we again translate this equality using the language of cohomology theory. Since φ_1 corresponds to the complex orientation t_E, $\varphi_0 : \underline{\mathbb{S}} \cong MU(0) \to E$ corresponds to the unit element 1 of $\pi_0(E)$, and α_0 is induced by the inclusion of a fiber F into ζ_1, the equation $\varphi_1 \circ \alpha_0 = \varphi_0$ is equivalent to $i^*(t_E) = 1$ in $\widetilde{E}^2(\mathbb{S}^2)$, where $i : \mathbb{S}^2 \cong Th(F) \to Th(\zeta_1)$ is the morphism induced by fiber inclusion. This equality holds by Lemma 3.1 and the definition of a complex orientation.

The uniqueness of the morphism φ is clear from the construction above. $\square$

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