Fourier transform and Langlands correspondence

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The Hecke eigensheaf is a core object in Langlands correspondence for function field: under Grothendieck's functions-sheaves dictionary, they are just the automorphic objects Langlands-corresponding to Galois representations. In this survey, we give a way of constructing Hecke eigensheaves for GL_n for function fields, first given by Frenkel–Gaitsgory–Vilonen in [FGV]. The method essentially comes from Laumon's previous work [Lau], in the spirit of Fourier transforms. Our next step is to review this construction in the setting of the geometric Langlands conjecture, which involves some notions from geometric representation theory. Finally, we breifly study the shadow of this construction in local Langlands correspondence, both in the ℓ-adic and p-adic coefficients.

The structure of this survey is as follows:

- Section 1 is a revision of the Langlands correspondence, especially on the notion of Hecke eigensheaves.
- Section 2 is a revision of Fourier transform.
- Section 3 is the details of the main construction.
- Section 4 is a collection of results and methods in geometric representation theory.
- Section 5 is a continuation of section 4, talking about geometric Langalnds correspondence and the occurrence of the previous construction in it.
- Section 6 deals with local Langlands correspondence.

1. The philosophy of Langlands correspondence

Before going into the main construction, we firstly introduce briefly about the Langlands correspondence for GL_n for function fields. Denote $\ell \neq p$ two different prime numbers. Let X be a proper smooth curve over the finite field $\mathbb{F}_p$ with function field F, with its Adeles $\mathbb{A}_F$. Denote $\{\sigma\}_n$ the set of rank n irreducible $\bar{\mathbb{Q}}_\ell$-local system on the generic point of X, i.e. $\bar{\mathbb{Q}}_\ell$-representations of Gal_F of dimension n. For such a representation, we can form its local L-factors $L(t, \sigma_x)$, which essentially comes from eigenvalues of Frobenius at x under this representation. For a precise definition, one can consult [LLaf]. On the other hand, denote $\{\pi\}_n$ the set of irreducible cuspidal automorphic representations of GL_n, i.e. irreducible cuspidal $\mathrm{GL}_n(\mathbb{A}_F)$-sub-representations of $\mathrm{Func}(\mathrm{GL}_n(F)\backslash \mathrm{GL}_n(\mathbb{A}_F), \bar{\mathbb{Q}}_\ell)$. For such a representation, we can also form its local L-factors $L(t, \pi_x)$, which essentially comes from eigenvalues of the Hecke actions at x. For a precise definition, one can consult [LLaf].

Theorem 1.1. *There is a bijection between $\{\pi\}_n$ and $\{\sigma\}_n$, such that their local L-factors agree, i.e. for almost all $x \in X$, $L(t, \pi_x) = L(t, \sigma_x)$.*

The bijection is called the Langlands correspondence, where $\{\pi\}_n$ is called the automorphic side, while $\{\sigma\}_n$ is called the Galois/spectral side. To get the correspondence it is a long history: for $n = 1$ it is abelian class field theory; for $n = 2$ it is firstly proven by Drinfeld [Dri]; for $n > 2$ it is firstly proven by Laurent Lafforgue [LLaf], by using the theory of shtukas. The notion of shtukas gives one direction of the correspondence, that from the automorphic side to the Galois side, and by some method, if we know one direction of this correspondence, then the other side is automatic. Later, Vincent Lafforgue, in [VLaf], uses the theory of shtukas to give the "automorphic to Galois" correspondence for general reductive group G, but in general this does not give the other direction of the correspondece as in the case of GL_n.

In this summary we concern about the other direction of the correspondence, i.e. the Galois side to the automorphic side, and that is what Laumon and Frenkel–Gaitsgory–Vilonen exactly did. More precisely, we construct a corresponding π for every *unramified* σ in theorem 1.1. Here such a rank n irreducible $\bar{\mathbb{Q}}_\ell$-local system is said to be unramified if it is a local system on X instead of only on the generic point of X.

The method really relies on the geometrization of the Langlands correspondence. Recall that Grothendieck's functions-sheaves dictionary describes a mechanism, passing from the derived category of $\bar{\mathbb{Q}}_\ell$-constructible shevaes on any algebraic stack Y over $\mathbb{F}_p$, to the set of $\mathrm{Func}(Y(\mathbb{F}_p), \bar{\mathbb{Q}}_\ell)$, by taking the trace of Frobenius. In particular, this applies to the stack $Y = \mathrm{Bun}_n$, which is the stack classifying rank n vector bundles on X. By Weil uniformization, $\mathrm{Bun}_n(\mathbb{F}_p) = \mathrm{GL}_n(F)\backslash\mathrm{GL}_n(\mathbb{A}_F)/\prod_x \mathrm{GL}_n(\mathcal{O}_x)$, hence the functions on $\mathrm{Bun}_n(\mathbb{F}_p)$ are exactly those *unramified* automorphic representations. We desire the corresponding π to an unramified σ is also unramified, and hope it to come from geometry, i.e. to come from some constructible sheaves under functions-sheaves dictionary. We call the desired constructible sheaves as the Hecke eigensheaf for σ. Now we can reformulate the main theorem of this survey geometrically as

Theorem 1.2. *For every geometrically irreducible $\bar{\mathbb{Q}}_\ell$-local system E of rank n on X, there exists a perverse sheaf $\mathrm{Aut}_E \in \mathrm{Perv}(\mathrm{Bun}_n, \bar{\mathbb{Q}}_\ell)$ on Bun_n such that it is a Hecke eigensheaf with eigenvalue E.*

We will develop a method for proving it in the following several sections. For now, let's explain some notions appearing in the main theorem.

Remark 1.3. Grothendieck's functions-sheaves dictionary is far from being an equivalence: shifting by degree 2 of a constructible sheaf gives the same function as the original one. That makes us put the condition of being perverse for the Hecke eigensheaf being unique, somehow, which will also facilitate the construction.

The geometric version of Langlands correspondence even sounds more natural: after all we start from a geometric object σ, so the resulting object should also be a geometric object instead of a function. For example, the definition of a Hecke eigensheaf is naturally geometric.

Definition 1.4. Let $\mathrm{Hecke}_n^{\mathbf{i}}$ be the stack classifying $(V, V', x, \alpha\colon V \to V')$ such that $x \in X$, and α is a modification of vector bundles at x of length i. It serves as a correspondence

$$\mathrm{Bun}_n \xleftarrow{p} \mathrm{Hecke}_n^{\mathbf{i}} \xrightarrow{q} \mathrm{Bun}_n \times X.$$

Then $K \in D_c^b(\mathrm{Bun}_n)$ is called a Hecke eigensheaf with eigenvalue E if we have $q_! p^* K = K \boxtimes \wedge^i E$.

Later we will see a better description of Hecke eigensheaves in the formalism of geometric Langlands conjecture.

2. Fourier transform

Recall three classical objects which are in the same spirit:

(1) the most classical Fourier transform says that for a Schwarz function f, we can do the Fourier transform

$$\hat{f}(y) = \int_{\mathbb{R}} f(x) e^{-2\pi i x y} dx.$$

This gives a bijection of the set of Schwarz functions. In general, this theory works for any locally compact abelian groups;

(2) in algebraic geometry, there is a Fourier–Mukai transform, giving an equivalence between $D(A)$ and $D(A^\vee)$ for an abelian variety A and its dual $A^\vee = \mathrm{Pic}^0(A)$. Here $D(-)$ denotes the derived category of quasi-coherent sheaves;

(3) for an affine abelian scheme or formal scheme G, we can consider its Cartier dual $G^\vee$. The Cartier duality says an equivalence between $D(G)$ and $D(BG^\vee)$.

We want to treat them in a unified way in this section. We cannot avoid to use the language of ∞-stacks and derived stacks. Breifly, compared to stacks, derived stacks allow us to test on the larger category of simplicial commutative rings, and produce animas, a.k.a. ∞-groupoids, for each test object. A good reference for this is Lurie's thesis [Lur].

Then we can define what a group-like derived stack is, furthur what an abelian derived stack is: an abelian derived stack is a derived stack that it is a sheaf valued in $\mathbb{E}_\infty$-animas. As a special case, we can recover the notion of Picard stack à la Deligne in [Del], as those abelian derived stacks with only nontrivial homotopy groups in degree 0 and 1.

The advantage of using derived stacks is that we can form the mapping stack $\mathrm{Map}(X, Y)$ for two derived stacks X and Y, and it usually shares good properties.

Definition 2.1. For any abelian derived stack X, we define its 1-shifted character stack $\chi[1](X)$ as fitting in the Cartesian diagram

$$\begin{array}{ccc} \chi[1](X) & \longrightarrow & \mathrm{Map}(X, B\mathbb{G}_\mathrm{m}) \\ \downarrow & & \downarrow {\scriptstyle m_X^*} \\ \mathrm{Map}(X, B\mathbb{G}_\mathrm{m}) & \xrightarrow{\Delta_{m_{\mathbb{G}_\mathrm{m}},*} \circ (-)^2} & \mathrm{Map}(X \times X, B\mathbb{G}_\mathrm{m}) \end{array}$$

Here $m_X \colon X \times X \to X$ and $(-)^2 \colon \mathrm{Map}(X, Y) \to \mathrm{Map}(X^2, Y^2)$. In intuition, this is just the categorical way of considering group homomorphisms from X to $B\mathbb{G}_\mathrm{m}$.

One can verify $\chi[1](X)$ is also an abelian derived stack; we say X to be self-dual if $\chi[1](\chi[1](X)) \cong X$.

We also assume X to admit six functor formalisms. This can be in different settings: for example, we can use the six functor formalism of ind-coherent sheaves developped by Gaitsgory–Rozenblyum in [GR2]; or use the six functor formalism of solid sheaves developped by Clausen–Scholze in [CS19], [CS20] and [CS22].

For a self-dual abelian derived stack X, which admits six functor formalisms, denoted $D(X)$, we can do the Fourier transform abstractly as follows. From

$$p\colon X \times \chi[1](X) \longrightarrow B\mathbb{G}_{\mathrm{m}},$$

we define $\mathrm{Four}\colon D(X) \to D(\chi[1](X))$ as

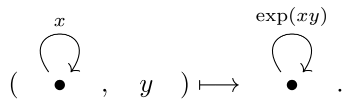
$$\mathrm{Four}(M) = \mathrm{pr}_{\chi[1](X),!}(\mathrm{pr}_X^* M \otimes p^*\mathcal{O}(1)).$$

As a folkfore, this functor Four is usually an equivalence.

Example 2.2. Let's compute 1-shifted character stack for $X = B\mathbb{G}_{\mathrm{a}}$. Firstly $\mathrm{Map}(B\mathbb{G}_{\mathrm{a}}, B\mathbb{G}_{\mathrm{m}})$ is the so-called *unipotent loop space* of $B\mathbb{G}_{\mathrm{m}}$, and is computed as $\widehat{\mathbb{G}_{\mathrm{a}}}/\mathbb{G}_{\mathrm{m}}$ as in [BN]. Therefore, we have a Cartesian diagram

$$\begin{array}{ccc} \chi[1](B\mathbb{G}_{\mathrm{a}}) & \longrightarrow & \widehat{\mathbb{G}_{\mathrm{a}}}/\mathbb{G}_{\mathrm{m}} \\ \downarrow & & \downarrow{\scriptstyle \Delta/\mathbf{id}} \\ \widehat{\mathbb{G}_{\mathrm{a}}}/\mathbb{G}_{\mathrm{m}} & \xrightarrow{\Delta/(-)^2} & (\widehat{\mathbb{G}_{\mathrm{a}}} \times \widehat{\mathbb{G}_{\mathrm{a}}})/\mathbb{G}_{\mathrm{m}} \end{array}$$

Hence $\chi[1](B\mathbb{G}_{\mathrm{a}}) = \widehat{\mathbb{G}_{\mathrm{a}}}$. In fact, we can write the pairing $B\mathbb{G}_{\mathrm{a}} \times \widehat{\mathbb{G}_{\mathrm{a}}} \to B\mathbb{G}_{\mathrm{m}}$ as



Moreover, $B\mathbb{G}_{\mathrm{a}}$ is self-dual, hence by the folkfore, we recover the Cartier duality $D(B\mathbb{G}_{\mathrm{a}}) \cong D(\widehat{\mathbb{G}_{\mathrm{a}}})$ as the Fourier transform. Similar things happen for $B\widehat{\mathbb{G}_{\mathrm{a}}}$ and $\mathbb{G}_{\mathrm{a}}$.

Inspired by this example, we can write the several constructions mentioned in the beginning as a unified way:

(1) take X to be the de Rham stack $\mathbb{G}_{\mathrm{a}}^{\mathrm{dR}}$ of $\mathbb{G}_{\mathrm{a}}$; we have an isomorphism $\mathbb{G}_{\mathrm{a}}^{\mathrm{dR}} \cong \mathbb{G}_{\mathrm{a}}/\widehat{\mathbb{G}_{\mathrm{a}}}$. Then by the example above, $\chi[1](\mathbb{G}_{\mathrm{a}}^{\mathrm{dR}}) = \mathbb{G}_{\mathrm{a}}^{\mathrm{dR}}$. Hence we have a Fourier transform $\mathrm{Four}\colon D(\mathbb{G}_{\mathrm{a}}^{\mathrm{dR}}) \to D(\mathbb{G}_{\mathrm{a}}^{\mathrm{dR}})$. Since $D(\mathbb{G}_{\mathrm{a}}^{\mathrm{dR}}) \cong \mathcal{D}(\mathbb{G}_{\mathrm{a}})$ recovers the theory of $\mathcal{D}$-modules on $\mathbb{G}_{\mathrm{a}}$; see [GR1], we recover the Fourier transform of $\mathcal{D}$-modules on $\mathbb{G}_{\mathrm{a}}$, which is a categorical version of the classical Fourier transform over $\mathbb{R}$;
(2) take X to be an abelian variety A. Then $\chi[1](A) = A^\vee$, so we recover the Fourier–Mukai transform;
(3) take X to be an affine abelian scheme or formal scheme G. Then $\chi[1](G) = BG^\vee$ and we recover Cartier duality. This is not an easy fact, while we have computed a special case $G = \widehat{\mathbb{G}_{\mathrm{a}}}$ above to convince the readers;

(4) take X to be a Banach–Colmez space. Then as we will see in future, the theory of Fourier transform depends heavily on whether the coeeficient is ℓ-adic or p-adic; we must treat it case by case. We will come back to the precise formulation when we need it.

However, this theory of Fourier transform only recovers stacky cohomology theories, that means those cohomology theories able to be described in the category of quasi-coherent sheaves on some stack. One example is the theory of de Rham cohomology and $\mathcal{D}$-modules of X, which is equivalent to the category of quasi-coherent sheaves on $\mathcal{X}^{\mathrm{dR}}$, hence is stacky. A non-example is the category of ℓ-adic constructible sheaves on a variety over $\mathbb{F}_p$, so we cannot develop Fourier transform by the method above; while the Fourier transform is quite important, which is crucial in the function field setting; so we discuss about it in a separate way.

We first treat the simplest case that $X = \mathbb{A}^1$ over $\mathbb{F}_p$. Then we have a simple answer: their is an Artin–Schreier sheaf $\mathcal{L}_\psi \in D^b_c(\mathbb{A}^1, \bar{\mathbb{Q}}_\ell)$ which corresponds to a character $\psi\colon \mathbb{F}_p \to \bar{\mathbb{Q}}_\ell$ under functions-sheaves dictionary. Then the Fourier transform is stated as:

Proposition 2.3. *Consider the diagram*

$$\begin{array}{ccccc} & & \mathbb{A}^1 \times \mathbb{A}^1 \xrightarrow{\ \Sigma\ } \mathbb{A}^1 & & \\ & \overset{p}{\swarrow} & & \overset{q}{\searrow} & \\ \mathbb{A}^1 & & & & \mathbb{A}^1 \end{array}$$

Define the Fourier–Deligne transform Four: $D^b_c(\mathbb{A}^1, \bar{\mathbb{Q}}_\ell) \to D^b_c(\mathbb{A}^1, \bar{\mathbb{Q}}_\ell)$ *as*

$$\mathrm{Four}(\mathcal{F}) = q_!(p^*\mathcal{F} \otimes \Sigma^*\mathcal{L}_\psi).$$

Then this transform is an equivalence of category, and preserves the perverse t-structure on each side.

Then we can say more about general cases. First imagine that there is a fictive "stack" X^ℓ whose quasi-coherent sheaves gives ℓ-adic sheaves on X. Then the Fourier–Deligne transform for $\mathbb{A}^1$ is interpreted as $\chi[1](\mathbb{A}^{1,\ell}) = \mathbb{A}^{1,\ell}$, and under the pairing

$$\mathbb{A}^{1,\ell} \times \mathbb{A}^{1,\ell} \longrightarrow B\mathbb{G}_{\mathrm{m}}$$

the pullback of $\mathcal{O}(1)$ is given by $\Sigma^*\mathcal{L}_\psi$. Then for a general derived affine space V, we can do the following fictive computation:

$$\chi[1](V^\ell) \cong \mathrm{Hom}_{\mathbb{A}^{1,\ell}}(V^\ell, \chi[1](\mathbb{A}^{1,\ell})) \cong \mathrm{Hom}_{\mathbb{A}^{1,\ell}}(V^\ell, \mathbb{A}^{1,\ell}) \cong \mathrm{Hom}_{\mathbb{A}^1}(V, \mathbb{A}^1)^\ell \cong V^{\vee,\ell}.$$

This inspires the form of the Fourier–Deligne transform in general:

Proposition 2.4. *Consider the diagram*

$$\begin{array}{ccccc} & & V \times V^\vee \xrightarrow{\ \Sigma\ } \mathbb{A}^1 & & \\ & \overset{p}{\swarrow} & & \overset{q}{\searrow} & \\ V & & & & V^\vee \end{array}$$

Define the Fourier–Deligne transform $\mathrm{Four}\colon D_c^b(V,\bar{\mathbb{Q}}_\ell)\to D_c^b(V^\vee,\bar{\mathbb{Q}}_\ell)$ *as*

$$\mathrm{Four}(\mathcal{F})=q_!(p^*\mathcal{F}\otimes\Sigma^*\mathcal{L}_\psi).$$

Then it induces an equivalence of category and preserves the perverse t-structure.

Remark 2.5. One can easily generalize the whole picture to the relative case, i.e. we can consider derived vector bundles over X/derived relative abelian stack over X, and do the same thing to get a version of Fourier transform.

3. The main construction

Given a geometrically irreducible local system E of rank n on X, we want to construction a Hecke eigensheaf Aut_E on Bun_n. Denote Coh_n the algebraic stack parametrizing coherent sheaf of generic rank n on X. Denote Coh_n^d its degree d part, then we have a decomposition $\mathrm{Coh}_n=\bigsqcup_{d\geq 0}\mathrm{Coh}_n^d$.

When $n=0$, we have an open immersion $i\colon X^{(d)}=X^d/S_d\to\mathrm{Coh}_0^d$ by sending a divisor to its corresponding skyscraper sheaf. In general, we have a stratification of Coh_0^d with each stratum given by $X^{(d_1)}\times X^{(d_2)}\times\cdots\times X^{(d_n)}\to\mathrm{Coh}_0^d$ for some d_i such that $\sum id_i=d$.

Definition 3.1. Denote $r\colon X^d\to X^{(d)}$, and $\Delta\subset X^{(d)}$ the diagonal with complement $X^{(d)}-\Delta$ classifying pairwisely different points. The Laumon sheaf $\mathcal{L}_E$ is a sheaf on Coh_0 such that $\mathcal{L}_E|_{\mathrm{Coh}_0^d}$ is defined as $i_{!*}\left(r_*E^{\boxtimes d}\right)^{S_d}|_{X^{(d)}-\Delta}[d]$. Here $i_{!*}$ denotes the intermediate extension; so $\mathcal{L}_E$ is a perverse sheaf on Coh_0.

Remark 3.2. A clearer way to describe the Laumon sheaf is as follows. Denote $\widetilde{\mathrm{Coh}}_0^d$ the stack classifying a flag $(\mathcal{F}_d\twoheadrightarrow\mathcal{F}_{d-1}\twoheadrightarrow\cdots\twoheadrightarrow\mathcal{F}_1)$ such that $\mathcal{F}_i\in\mathrm{Coh}_0^i$. We have a resolution $\pi\colon\widetilde{\mathrm{Coh}}_0^d\to\mathrm{Coh}_0^d$ sending such a flag to $\mathcal{F}_d$, and a structure morphism $p\colon\widetilde{\mathrm{Coh}}_0^d\to X^d$ sending such a flag to $(\mathrm{supp}(\ker(\mathcal{F}_i\twoheadrightarrow\mathcal{F}_{i-1})))_i$. The resolution is an analogue of the Grothendieck–Springer resolution, and we have an isomorphism

$$\left(R\pi_*p^*E^{\boxtimes d}\right)^{S_d}\cong i_{!*}\left(r_*E^{\boxtimes d}\right)^{S_d}|_{X^{(d)}-\Delta}[d]=\mathcal{L}_E|_{\mathrm{Coh}_0^d}.$$

Denote Hecke_d^d the (extended) Hecke stack classifying two rank d bundles V,V' over X and a modification $\alpha\colon V\to V'$ such that $\mathrm{coker}(\alpha)\in\mathrm{Coh}_0^d$. Then there exists a structure morphism $\mathrm{Hecke}_d^d\to\mathrm{Coh}_0^d$ such that its fiber on the stratum $X^{(n_1)}\times\cdots\times X^{(n_k)}$ is Hecke_d^λ where $\lambda=(n_1+\cdots+n_k,\cdots,n_k,\cdots,0)$. The fusion property for Hecke stacks implies the fusion property for Coh_0^d. We get:

Proposition 3.3. *The pullback of $\mathcal{L}_E$ to $X^{(n_1)}\times\cdots\times X^{(n_k)}$ has fiber over $D_i=\sum n_{i,x}[x]$ of the form*

$$\bigotimes_{x\in X}(R^{(d_{1,x},\cdots,d_{k,x})}E_x)\left(-\sum_{i=1}^k n_{i,x}\frac{i(i-1)}{2}\right).$$

Here we can find the definition of $R^{(d_{1,x},\cdots,d_{k,x})}$ in [Lau, 3.3.8].

In next step, we lift the Laumon sheaf $\mathcal{L}_E$ to a perverse sheaf Aut'_E over Bun'_n. Here Bun'_n classifies a vector bundle V over X of rank n and an embedding $\Omega_X^{n-1} \hookrightarrow V$.

Remark 3.4. Denote Mir_n the mirabolic subgroup of GL_n, i.e.

$$\mathrm{Mir}_n = \begin{pmatrix} & & & * \\ & * & & \vdots \\ & & & * \\ 0 & \cdots & 0 & 1 \end{pmatrix}$$

Then $\mathrm{Bun}'_n(\mathbb{F}_p) = \mathrm{Mir}_n(F)\backslash \mathrm{GL}_n(\mathbb{A}_F)/\prod_{x\in X} \mathrm{GL}_n(\mathcal{O}_x)$. Therefore, constructible sheaves on Bun'_n corresponds to *mirabolic* automorphic forms under functions-sheaves dictionary.

In fact, the construction is automatically defined over a bigger stack Coh'_n, in which Bun'_n serves as an open substack; here Coh'_n parametrizes those $(\mathcal{M}, s)$ with $\mathcal{M} \in \mathrm{Coh}_n$ and $s\colon \Omega_X^{n-1} \hookrightarrow \mathcal{M}$.

Definition 3.5. Let

- $\mathcal{E}_k$ be the stack classifying $(\mathcal{M}_k, s_k), \mathcal{M}_k \in \mathrm{Coh}_k, s_k \in R\,\mathrm{Hom}(\Omega_X^{k-1}, \mathcal{M}_k)$,
- $\mathcal{E}_k^\vee$ be the stack classifying $(\mathcal{M}_k, R\,\mathrm{Hom}(\mathcal{M}_k, \Omega_X^k)[1]), \mathcal{M}_k \in \mathrm{Coh}_k$.

They are dual vector bundles (in the derived sense) over Coh_k by Serre duality. Notice $j_k\colon \mathrm{Coh}'_k \hookrightarrow \mathcal{E}_k$ serves as a substack of $\mathcal{E}_k$, and Coh'_k also admits a natural morphism $\mathcal{E}_{k-1}^\vee$. Hence we have a diagram

$$\begin{array}{ccccccccc} \mathcal{E}_k^\vee & & \mathcal{E}_k & \longleftarrow & \mathrm{Coh}'_k & \longrightarrow & \mathcal{E}_{k-1}^\vee & & \mathcal{E}_{k-1} \\ & \searrow \quad \swarrow & & & & & & \searrow \quad \swarrow & \\ & \mathrm{Coh}_k & & & & & & \mathrm{Coh}_{k-1} & \end{array}$$

Now we construct the desired sheaf inductively.

(1) denote $\mathcal{F}_{E,0} = \mathcal{L}_E$;
(2) define $\mathcal{F}_{E,1}$ as the pullback (up to degree shifting) of $\mathcal{L}_E$ on Coh'_1 along the structure morphism $\mathrm{Coh}'_1 \to \mathrm{Coh}^0$, which is a perverse sheaf;
(3) if we have already defined $\mathcal{F}_{E,k}$ over Coh'_k, define $\mathcal{F}_{E,k+1}$ inductively as the pullback to Coh'_{k+1} of the (derived) Fourier transform of $j_{k!}(\mathcal{F}_{E,k})$, i.e.

$$\mathcal{F}_{E,k+1} = \mathrm{Four}(j_{k!}(\mathcal{F}_{E,k}))|_{\mathrm{Coh}'_{k+1}}.$$

Finally, we get the desired sheaf $\mathcal{F}_{E,n}$ on Coh'_n. It is perverse because of the cleanness property below, and that Fourier transform preserves perversity.

Theorem 3.6 (Cleanness)**.** *The natural morphism $j_{k!}\mathcal{F}_{E,k} \to j_{k*}\mathcal{F}_{E,k}$ is an isomorphism. As a result, $j_{k!}\mathcal{F}_{E,k}$ equals to its intermediate extension $j_{k!*}\mathcal{F}_{E,k}$, hence perverse. In particular, $\mathcal{F}_{E,n}$ is an irreducible perverse sheaf over Coh'_n.*

Remark 3.7. There is a slight difference between our construction and the one in [FGV]. This is because at that time only the underived Fourier transform is developped, so we must restrict ourselves to some good substack of Coh_k where the underived duality behaves well. We briefly recall the setting there. Use the convention of degree as $\deg(\det(\mathcal{M})) - k(k-1)(g-1)$ (so that $\bigoplus_{i=0}^{k-1} \Omega^i$ has degree 0). Fix a line bundle $\mathcal{L}$ such that for any $\mathcal{M} \in \mathrm{Bun}_{k\leq n}$, $\mathrm{Hom}(\mathcal{M}, \mathcal{L}) = 0$ implies $\deg(\mathcal{M}) > nk(2g-2)$ and $\mathrm{Ext}^1(\Omega^{k-1}, \mathcal{M}) = 0$. Let $c_{g,n}$ be an integer such that for any $\mathcal{M} \in \mathrm{Bun}_n^{\geq c_{g,n}}$ satisfying $\mathrm{Hom}(\mathcal{M}, \mathcal{L}) \neq 0$, then $\mathcal{M}$ is very unstable. Let $\mathcal{C}_k^d$ be the substack of Coh_k^d such that $\mathrm{Hom}(\mathcal{M}, \mathcal{L}) = 0$. Set $\mathcal{C}_k = \cup_{d \geq c_{g,n}} \mathcal{C}_k^d$. Then we replace every Coh_k above by $\mathcal{C}_k$, and do Fourier transform, and finally get the usual construction of the sheaf $\mathcal{F}_{E,n}$.

Denote Aut'_E as the restriction of $\mathcal{F}_{E,n}$ to Bun'_n. In order to get a sheaf Aut_E on Bun_n, we need to do descent of Aut'_E from Bun'_n to Bun_n. We breifly summarize the process.

We first do the descent in a smaller collection of component, saying that on the part $\bigcup_{d \gg 0} \mathrm{Bun}_n^d$ of degree sufficiently large.

Since Aut'_E is an irreducible perverse sheaf, it is an intermediate extension of some local system on an subopen of $\mathrm{Bun}_n'^d$.

Lemma 3.8 ([FGV, lemma 6.4])**.** *Let Y be a smooth scheme or stack and let $\mathcal{K}$ be an irreducible perverse sheaf on Y. If the Euler characteristics of the stalks of $\mathcal{K}$ are the same at all $\overline{\mathbb{F}_p}$–points of Y, then $\mathcal{K}$ is a local system. If these Euler characteristics are not identically equal to 0, then $\mathcal{K} \neq 0$.*

Then one can compute the Euler characteristics of this perverse sheaf, and show that they are constant along the fibers of $p\colon \mathrm{Bun}_n'^d \to \mathrm{Bun}_n^d$. Therefore, we can assume $\mathcal{F}_{E,n}$ is an intermediate extension of a local system $\mathcal{K}$ on a subopen of the form $p^{-1}(U)$ for some U. Then to do the descent, it suffices to do the descent $\mathcal{K}$ from $p^{-1}(U)$ to U, because the intermediate extension of such a descent will give the desired descent of Aut'_E.

We come back to study the map $p\colon \mathrm{Bun}_n'^d \to \mathrm{Bun}_n^d$. When d is sufficiently large, each fiber of this map is of the form $H^0(X, \mathcal{L}) - 0$ of some line bundle $\mathcal{L}$, hence $H^0(X, \mathcal{L}) - 0/\mathbb{G}_\mathrm{m}$ is a projective space. The local system $\mathcal{K}$ is $\mathbb{G}_\mathrm{m}$-equivariant, as Fourier transform preserves $\mathbb{G}_\mathrm{m}$-action; hence $\mathcal{K}$ descents to $\mathrm{Bun}_n'^d/\mathbb{G}_\mathrm{m}$. However, $\mathrm{Bun}_n'^d/\mathbb{G}_\mathrm{m}$ over Bun_n^d has fibers as projective spaces, which is simply connected; so $\mathcal{K}$ must descent to a local system on U.

Furthurmore, we can extend the descent of Aut'_E to other connected components, instead of the part of $d \gg 0$. This is because if we want a sheaf to be Hecke eigenvalued, then it is uniquely determined by finitely many connected components, as Hecke operators shift degrees. This finishes the process of the descent, and we call the resulting sheaf as Aut_E on Bun_n.

Finally we need to prove the Hecke eigen-property of Aut_E, but let's say it to be tautological: because the Laumon sheaf $\mathcal{L}_E$ satisfies the Hecke-Laumon property with respect to E; see [FGV, section 8], so the construction of Aut_E insures itself to be a Hecke eigensheaf with eigenvalue E. As before, the better viewpoint is via the geometric Langlands conjecture.

4. Methods from geometric representation theory

To fully understand the technical construction of Hecke eigensheaves, we need tools from geometric representation theory to unpack, integrate, and repack the whole construction. Let's recall them one by one. We work over a field k of characteristic 0 from now on.

We firstly discuss about Schur–Weyl duality. Classically, there is an "unexpected" connection between irreducible representations of GL_n and irreducible representations of S_d: after all they are all parametrized by combinatorial objects called Young diagrams. The exact relation is given as:

Theorem 4.1. *Let Std_n be the standard GL_n-representation. Consider the $\mathrm{GL}_n \times S_d$-representation $\mathrm{Std}_n^{\otimes d}$. We have a decomposition of $\mathrm{GL}_n \times S_d$-representation*

$$\mathrm{Std}_n^{\otimes d} = \bigoplus_{D \in \mathrm{Young}_d^{\leq n}} \rho_n^D \boxtimes \pi_d^D.$$

Here $\mathrm{Young}_d^{\leq n}$ collects Young diagrams with d boxes at most n rows, and ρ_n^D, π_d^D are the corresponding irreducible representations.

We want to categorify this. Let X be a projective smooth curve over k. Denote $\mathrm{Ran} := \varinjlim_{|I|<\infty} X^I$ the Ran space of X, which one can imagine as a geometric version of Adele. If an object C is defined for every point $x \in X$, then we can extend its definition to the Ran space, and denote it as C_{Ran}.

Then we can form the category $(\mathrm{Rep}\,\mathrm{GL}_n)_{\mathrm{Ran}}$, classifying the collection of finitely many points of X and a GL_n-representation for each such a point. This is the Ran version of the category of GL_n-representation. On the other hand, the Ran version of the category of S_d-representation is $X^{(d)}$. However, a better version is Coh_0^d because it contains more information, for example, the fusion property.

Then we have a functor $(\mathrm{Rep}\,\mathrm{GL}_n)_{\mathrm{Ran}} \to \mathcal{D}(\mathrm{Coh}_0)$, sending an irreducible representation ρ_n^D of GL_n with trivial variation along Ran space, to the corresponding perverse sheaf on Coh_0^d determined by π_d^D. The construction relies on geometric Satake equivalence, but let's say firstly some important observations:

(1) under this functor, the object in $(\mathrm{Rep}\,\mathrm{GL}_n)_{\mathrm{Ran}}$ determined by the local system E is sent to $\mathcal{L}_E$;
(2) any image under this functor has support contained in strata $X^{(d_1)} \times X^{(d_2)} \times \cdots \times X^{(d_m)}$ with $m \leq n$;
(3) combining two, we get a geometric intuition of proposition 3.3.

Now we back to the construction of this functor, which uses (derived) geometric Satake equivalence as follows.

We treat directly the derived version of geometric Satake, as the derived phenomenon itself serves an important role in geometric Langlands program. For an arbitrary reductive group G over k, with Langlands dual group $\check{G}$, denote the affine

Grassmannian as $\mathrm{Gr}_G = LG/L^+G$, where $LG = G(k(\!(t)\!))$ and $L^+G = G(k[\![t]\!])$. then the derived geometric Satake equivalence states:

Theorem 4.2. *There is an equivalence of symmetric monoidal ∞-categories*

$$\underline{\mathrm{IndCoh}}_{\mathrm{Nilp}}(0 \times_{\check{\mathfrak{g}}} 0/\check{G}) \cong \mathcal{D}(L^+G\backslash \mathrm{Gr}_G).$$

Remark 4.3. We give several remark:

(1) the symmetric monoidal structures come from convolutions;
(2) if we work on the heart of each side, then there is no derived phenomenon appearing, and the resulting is the (underived) geometric Satake equivalence
$$\underline{\mathrm{QCoh}}(\mathrm{B}\check{G}) \cong \mathrm{Perv}(L^+G\backslash \mathrm{Gr}_G).$$
(3) the nilpotent cone condition is explained by Koszul duality;
(4) the derived geometric Satake equivalence is the local behavior at $x \in X$ of geometric Langlands conjecture in the next section;
(5) The category of ind-coherent sheaves is really involved: take $G = \mathbb{G}_{\mathrm{m}}$, then there is a difference between $\underline{\mathrm{IndCoh}}(0 \times_{\mathbb{A}^1} 0)$ and $\underline{\mathrm{QCoh}}(0 \times_{\mathbb{A}^1} 0)$, and the geometric Satake only holds for ind-coherent sheaves by Koszul daulity.

Then the Schur–Weyl duality can be strictly constructed via derived geometric Satake. The Ran version of derived geometric Satake gives

$$\underline{\mathrm{IndCoh}}_{\mathrm{Nilp}}(0 \times_{\check{\mathfrak{g}}} 0/\check{G})_{\mathrm{Ran}} \cong \mathcal{D}(\mathrm{Hecke}_{\mathrm{GL}_n,\mathrm{Ran}})$$

where $\mathrm{Hecke}_{\mathrm{GL}_n,\mathrm{Ran}}$ is the Ran version of Hecke modification. It admits a natural morphism $q\colon \mathrm{Hecke}_{\mathrm{GL}_n,\mathrm{Ran}} \longrightarrow \mathrm{Coh}_0$ defined by modification. Then we define the desired functor as

$$(\mathrm{Rep}\,\mathrm{GL}_n)_{\mathrm{Ran}} \longrightarrow \underline{\mathrm{IndCoh}}_{\mathrm{Nilp}}(0 \times_{\check{\mathfrak{g}}} 0/\check{G})_{\mathrm{Ran}} \cong \mathcal{D}(\mathrm{Hecke}_{\mathrm{GL}_n,\mathrm{Ran}}) \xrightarrow{q_*} \mathcal{D}(\mathrm{Coh}_0).$$

This recovers geometrically the Schur–Weyl duality. We denote this functor as $\mathrm{SW}\colon (\mathrm{Rep}\,\mathrm{GL}_n)_{\mathrm{Ran}} \to \mathcal{D}(\mathrm{Coh}_0)$.

5. Occurrence in the geometric Langlands conjecture

Now we go into the world of (de Rham) geometric Langlands conjecture, and see the connection with Laumon sheaf. The geometric Langlands conjecture is firstly studied by Beilinson and Drinfeld, and then developped by Gaitsgory and others. Recently it is well formulated and proven in five articles [GLC1], [GLC2], [GLC3], [GLC4] and [GLC5]. Fix the setting that k is an algebraic closed field of characteristic 0, and X is a curve over k. The advantage of working over such a field k is that we can use the theory of $\mathcal{D}$-modules, which is a stacky theory; so the Fourier transform is better developped. Fix the standard convention of of $\mathcal{D}$-modules; for arbitrary $\mathcal{D}$-modules, we have $f^!$ and f_*; for holonomic ones, we have additionally f^* and $f_!$.

The statement of the geometric Langlands conjecture is as follows.

Theorem 5.1. *There exists a functor $\mathbb{L}_G$ which is an equivalence:*

$$\mathbb{L}_G\colon \mathcal{D}_{\frac{1}{2}}(\mathrm{Bun}_G) \xrightarrow{\mathbb{L}_G,\cong} \underline{\mathrm{IndCoh}}_{\mathrm{Nilp}}(\mathrm{LS}_{\check{G}}).$$

We denote its tempered quotient as

$$\mathbb{L}_G^{\mathrm{temp}}\colon \mathcal{D}_{\frac{1}{2}}(\mathrm{Bun}_G)_{\mathrm{temp}} \xrightarrow{\cong} \underline{\mathrm{QCoh}}(\mathrm{LS}_{\check{G}}).$$

On the left side, we use twisted $\mathcal{D}$-modules in critical level instead of original ones, as even in the local story the twisting appears naturally; nevertheless, there is a noncanonical identification $\mathcal{D}_{\frac{1}{2}}(\mathrm{Bun}_G) \cong \mathcal{D}(\mathrm{Bun}_G)$. On the right side, we use ind-coherent sheaves as the right object, because of being singular for $\mathrm{LS}_{\check{G}}$. This singularity makes difference between ind-coherent sheaves and quasi-coherent sheaves. The subscript "Nilp" refers to the nilpotent cone and the category $\underline{\mathrm{IndCoh}}_{\mathrm{Nilp}}(-)$ is explained in [AG]; it is a shadow of the Koszul duality phenomenon.

We don't discuss the proof of this conjecture. However, the proof relies on some tools that we will use below.

(1) Derived geometric Satake equivalence discussed in the last section.
(2) Spectral action à la Drinfeld–Gaitsgory.

By the Ran version of geometric Satake equivalence, $\mathrm{Rep}\,\check{G}_{\mathrm{Ran}}$ acts on $\mathcal{D}_{\frac{1}{2}}(\mathrm{Bun}_G)$. On the other hand, the pullback along the morphism $(X^I)^{\mathrm{dR}} \times \mathrm{LS}_{\check{G}} \to B\check{G}^I$ gives a map $\mathrm{Rep}\,\check{G}^I \to \underline{\mathrm{QCoh}}(\mathrm{LS}_{\check{G}}) \otimes \mathcal{D}(X^I)$. By self-duality of $\mathcal{D}(X)$, we get $\mathrm{Rep}\,\check{G}^I \otimes \mathcal{D}(X^I) \to \underline{\mathrm{QCoh}}(\mathrm{LS}_{\check{G}})$. Collecting all finite sets I, we get a map $\mathrm{Rep}\,\check{G}_{\mathrm{Ran}} \to \underline{\mathrm{QCoh}}(\mathrm{LS}_{\check{G}})$.

Remark 5.2. This map serves as a role of factorization homology in the sense of

$$\text{“}\bigotimes_X\text{”}\,\mathrm{Rep}\,\check{G} \longrightarrow \int_X \mathrm{Rep}\,\check{G}_x = \underline{\mathrm{QCoh}}(\mathrm{LS}_{\check{G}}).$$

Theorem 5.3 (Vanishing conjecture à la Drinfeld–Gaitsgory)**.** *The action of* $\mathrm{Rep}\,\check{G}_{\mathrm{Ran}}$ *on* $\mathcal{D}_{\frac{1}{2}}(\mathrm{Bun}_G)$ *factors through* $\underline{\mathrm{QCoh}}(\mathrm{LS}_{\check{G}})$*.*

By using this theorem, we can also redefine the notion of a Hecke eigensheaf:

Definition 5.4. Let $\sigma \in \mathrm{LS}_{\check{G}}$, and we denote the pullback along σ also as σ which gives a functor $\sigma\colon \underline{\mathrm{QCoh}}(\mathrm{LS}_{\check{G}}) \to D(k)$. A Hecke eigensheaf with eigenvalue σ is an object in the category $\mathcal{D}_{\frac{1}{2}}(\mathrm{Bun}_G) \otimes_{\underline{\mathrm{QCoh}}(\mathrm{LS}_{\check{G}}),\sigma} D(k)$.

By unpacking the definition, one finds this is exactly the definition in section 1.

(3) Whittaker coefficients and its enhancement.

Let 2ρ be the character that is the sum of all simple roots and $2\check{\rho}$ the corresponding cocharacter. Choose a square root $\Omega_X^{\frac{1}{2}}$ of Ω_X. Then $2\check{\rho}(\Omega_X^{\frac{1}{2}})$

is a T-bundle on X. Denote $\mathrm{Bun}_N^{\rho(\Omega_X)} = \mathrm{Bun}_G \times_{\mathrm{Bun}_T} 2\check{\rho}\{(\Omega_X^{\frac{1}{2}})\}$. It admits a morphism p to Bun_G. On the other hand, denote ψ the composition of:

$$\psi\colon \mathrm{Bun}_N^{\rho(\Omega_X)} \longrightarrow \prod_{\text{simple roots}} \mathrm{Bun}_{\mathbb{G}_\mathrm{a}}^{\Omega_X} \longrightarrow \prod H^1(X, \Omega_X) \cong \prod \mathbb{A}^1 \xrightarrow{\sum} \mathbb{A}^1.$$

In summary, we have a diagram

$$\mathrm{Bun}_G \xleftarrow{p} \mathrm{Bun}_N^{\rho(\Omega_X)} \xrightarrow{\psi} \mathbb{A}^1.$$

Let exp be the exponential $\mathcal{D}$-module on $\mathbb{A}^1$. We define

$$\mathrm{coeff}\colon D_{\frac{1}{2}}(\mathrm{Bun}_G) \longrightarrow \underline{\mathrm{Vect}},\ \mathcal{F} \longmapsto C^\bullet_{\mathrm{dR}}(\mathrm{Bun}_N^{\rho(\Omega_X)}, p^! \mathcal{F} \otimes \psi^* \exp).$$

Since exp is holonomic, this functor is well-defined.

Remark 5.5. Whittaker coefficient is the analogue of $f \mapsto a_1(f)$, i.e. the first Fourier coefficient for modular forms. As we will see in the following, higher Fourier coefficients will be recovered by enhanced Whittaker coefficient.

Denote $\psi\colon LN \to \mathbb{G}_\mathrm{a}$ the morphism induced as

$$\psi\colon LN \longrightarrow L(N/[N,N]) \longrightarrow N/[N,N] \cong \mathbb{G}_\mathrm{a}^k \longrightarrow \mathbb{G}_\mathrm{a}.$$

Defined the Ran version of Whittaker category for G as $\mathrm{Whit}(G)_{\mathrm{Ran}} = \mathcal{D}(\mathrm{Gr}_{G,\mathrm{Ran}})^{LN,\psi}$. For example, an object in the Whittaker category is a $\mathcal{D}$-module $\mathcal{F}$ on $\mathrm{Gr}_{G,\mathrm{Ran}}$ such that under the pullback along $p\colon LN \times \mathrm{Gr}_{G,\mathrm{Ran}} \to \mathrm{Gr}_{G,\mathrm{Ran}}$, we have $p^! \mathcal{F} \cong \mathcal{F} \boxtimes \psi^* \exp$. Then the coefficient functor has an enhancement as follows:

$$\mathrm{coeff}^{\mathrm{enh}}\colon \mathcal{D}_{\frac{1}{2}}(\mathrm{Bun}_G) \longrightarrow \mathrm{Whit}(G)_{\mathrm{Ran}},$$

defined as $\mathrm{Av}_* \circ \pi^!$, where the diagram is

$$\mathrm{Bun}_G \xleftarrow{\pi} \mathrm{Gr}_{G,\mathrm{Ran}} \xrightarrow{\mathrm{Av}} \text{“}(LN,\psi)\backslash \mathrm{Gr}_{G,\mathrm{Ran}}\text{”}.$$

Remark 5.6. The enhanced Whittaker coefficient is an analogue of the collection of all Fourier coefficients of a modular form $\{a_n(f)\}_{n\geq 1}$.

(4) The geometric Casselman–Shalika formula.

Theorem 5.7 (Geometric Casselman–Shalika)**.** *There exists an equivalence of factorization category (for the notion of factorization, see* [GLC2, appendix B]*):*

$$\mathrm{CS}_G\colon \mathrm{Whit}(G)_{\mathrm{Ran}} \xrightarrow{\cong} \mathrm{Rep}(\check{G})_{\mathrm{Ran}}.$$

Moreover, it is compatible with the action of derived geometric Satake equivalence on both sides.

Moreover, we have a commutative diagram (part of the Fundamental Commutative Diagram in [GLC2]):

$$
\begin{array}{ccc}
\mathrm{Whit}(G)_{\mathrm{Ran}} & \xrightarrow{\mathrm{CS}_G} & \mathrm{Rep}(\check{G})_{\mathrm{Ran}} \\
{\scriptstyle \mathrm{coeff}^{\mathrm{enh}}}\uparrow & & \uparrow{\scriptstyle \Gamma^{\mathrm{Spec}}_{\check{G}}} \\
\mathcal{D}_{\frac{1}{2}}(\mathrm{Bun}_G) & \xrightarrow{\mathbb{L}_G} & \underline{\mathrm{IndCoh}}_{\mathrm{Nilp}}(\mathrm{LS}_{\check{G}})
\end{array}
$$

Here are also some tools that are special for GL_n which is crucial in the construction.

(1) Schur–Weyl duality from last section.

(2) Geometrization of mirabolic trick. Recall Bun'_n as the stack classifying a rank n vector bundle V with an embedding $\Omega^{\otimes(n-1)} \to V$. Then we have the following isomorphism

$$\mathrm{Mir}\colon \mathcal{D}_{\mathrm{cusp}}(\mathrm{Bun}'_n) \cong \mathrm{Whit}(\mathrm{GL}_n)_{\mathrm{Ran}}.$$

It is compatible with Whittaker coefficient in the following sense. Denote the morphisms as $\mathrm{Bun}'_n \xrightarrow{p} \mathrm{Bun}_n$. The object constructed above satisfies the following diagram

$$
\begin{array}{ccc}
\mathcal{D}_{\mathrm{cusp}}(\mathrm{Bun}_n) & \xrightarrow{p^!} & \mathcal{D}_{\mathrm{cusp}}(\mathrm{Bun}'_n) \\
{\scriptstyle \mathrm{coeff}^{\mathrm{enh}}}\downarrow & \swarrow{\scriptstyle \mathrm{Mir},\cong} & \\
\mathrm{Whit}(\mathrm{GL}_n)_{\mathrm{Ran}} & &
\end{array}
$$

(3) Geometrization of Whittaker pattern. We introduce the stack $\mathcal{Q}$ and its compactification $\overline{\mathcal{Q}}$. The latter classifies a rank n bundle $\mathcal{F}$ and a collection of injective morphisms

$$s_i\colon \Omega^{(n-1)+\cdots+(n-i)} \longrightarrow \wedge^i \mathcal{F}$$

such that generically this collection defines a complete flag of subbundles in $\mathcal{F}$. The former is a substack of the latter classifying those defining a complete flag of subbundles entirely. Separating the degree of $\mathcal{F}$, we get their connected components $\mathcal{Q}^d$ and $\overline{\mathcal{Q}}^d$. A glimpse that makes involving of Whittaker pattern (hence also the coefficient functor): $\mathcal{Q}^0$ is exactly $\mathrm{Bun}_N^{\rho(\Omega_X)}$ and $\overline{\mathcal{Q}}^0$ is the Drinfeld–Laumon compactification $\overline{\mathrm{Bun}}_N^{\rho(\Omega_X)}$. The precise relation with Whittaker coefficient is described as follows. Denote $\widetilde{\mathcal{Q}}$ the stack parametrizing a datum in $\mathcal{Q}$, which consists of a special flag on a rank n bundle V with an injection from V to another rank n bundle

W. There is a diagram

$$\begin{tikzcd} \mathrm{Coh}_0 & \widetilde{\mathcal{Q}} \arrow[l, "r"'] \arrow[r, "\psi"] \arrow[d, "\pi"] \arrow[dr, "\pi'"] & \mathbb{A}^1 \\ & \mathrm{Bun}_n & \mathrm{Bun}'_n \arrow[l, "p"'] \end{tikzcd}$$

Then we can form a functor $\mathcal{W}\colon \mathcal{D}(\mathrm{Bun}_n) \longrightarrow \mathcal{D}(\mathrm{Coh}_0)$ given by $\mathcal{W}(\mathcal{F}) := r_*(\pi^!\mathcal{F} \otimes \psi^* \exp)$. Then this functor will recover enhanced Whittaker coefficient after the next point.

(4) The compatibility between Schur–Weyl duality, Whittaker pattern and mirabolic trick. We use a commutative diagram to summarize the relations

$$\begin{tikzcd} \mathcal{D}(\mathrm{Bun}_n) \arrow[r, "p^!"] \arrow[dr, "\mathrm{coeff}^{\mathrm{enh}}"'] \arrow[ddrr, bend right, "\mathcal{W}"'] & \mathcal{D}(\mathrm{Bun}'_n) \arrow[d, "\mathrm{Mir}"] & \\ & \mathrm{Whit}(\mathrm{GL}_n)_{\mathrm{Ran}} \arrow[r, "\mathrm{CS}"] \arrow[dr, "\mathrm{SW}\circ\mathrm{CS}"'] & (\mathrm{Rep}\,\mathrm{GL}_n)_{\mathrm{Ran}} \arrow[d, "\mathrm{SW}"] \\ & & \mathcal{D}(\mathrm{Coh}_0) \end{tikzcd}$$

Now we put everything together. Forget about the Laumon sheaf $\mathcal{L}_E$ for a while. We firstly do the Poincaré sheaf (i.e. structure sheaf) construction. Denote the character as $\psi\colon \widetilde{\mathcal{Q}} \longrightarrow \mathbb{A}^1$ as above. Define

$$\mathrm{Poinc}_! = \pi_!\psi^* \exp \in \mathcal{D}(\mathrm{Bun}_n),\ \mathrm{Poinc}'_! = \pi'_!\psi^* \exp \in \mathcal{D}(\mathrm{Bun}'_n).$$

Then by formal adjointness, under the diagram

$$\begin{tikzcd} \mathrm{Whit}(\mathrm{GL}_n)_{\mathrm{Ran}} = \mathcal{D}_{\mathrm{cusp}}(\mathrm{Bun}'_n) \arrow[r] & \mathrm{Rep}(\mathrm{GL}_n)_{\mathrm{Ran}} \\ \mathcal{D}_{\mathrm{cusp}}(\mathrm{Bun}_n) \arrow[u, "\mathrm{coeff}^{\mathrm{enh}}"] \arrow[r, "\mathbb{L}_{\mathrm{GL}_n}"] & \underline{\mathrm{IndCoh}}_{\mathrm{Nilp}}(\mathrm{LS}_{\mathrm{GL}_n}) \arrow[u, "\Gamma^{\mathrm{Spec}}"'] \end{tikzcd}$$

the following objects are corresponded to each others:

$$\begin{tikzcd} \mathrm{Poinc}'_! \arrow[r] & 1_{\mathrm{Ran}} \\ \mathrm{Poinc}_! \arrow[u, "\mathrm{coeff}^{\mathrm{enh}}"] \arrow[r, "\mathbb{L}_{\mathrm{GL}_n}"] & \mathcal{O}_{\mathrm{LS}_{\mathrm{GL}_n}} \arrow[u, "\Gamma^{\mathrm{Spec}}"'] \end{tikzcd}$$

Now we add the information about E and $\mathcal{L}_E$. The sheaf Aut'_E constructed before, using Fourier–Deligne transform, has another description as follows:

$$\mathrm{Aut}'_E = \pi'_!(r^*\mathcal{L}_E \otimes \psi^* \exp) \in \mathcal{D}_{\mathrm{cusp}}(\mathrm{Bun}'_n).$$

Here all sheaves are holonomic so lower-! and upper-$*$ are defined. One can find the proof in [FGV] that this construction agrees with the original construction;

but intuitively, they agree because of unpacking the Fourier transform under the relation that:

$$(\mathrm{Coh}'_n \times_{\mathrm{Coh}_n} \mathrm{Coh}'_{n-1} \times_{\mathrm{Coh}_{n-1}} \cdots \times_{\mathrm{Coh}_1} \mathrm{Coh}'_1)\big|_{\text{open part given by bundles}} \cong \widetilde{Q}.$$

We admit the descent of Aut'_E to Aut_E, i.e. $\mathrm{Aut}'_E = p^! \mathrm{Aut}_E$. Then Aut_E is the desired Hecke eigensheaf.

In fact, by formal adjointness, $\mathcal{W}(\mathrm{Aut}_E) = \mathcal{L}_E \in \mathcal{D}(\mathrm{Coh}_0)$. Under Schur–Weyl duality, it is the image of the object in $(\mathrm{Rep}\,\mathrm{GL}_n)_{\mathrm{Ran}}$ determined by E. This object is exactly $\Gamma^{\mathrm{Spec}}(\delta_E)$ where $\delta_E \in \underline{\mathrm{IndCoh}}_{\mathrm{Nilp}}(\mathrm{LS}_{\mathrm{GL}_n})$ is given by the push-forward of $\mathcal{O}_{\mathrm{Spec}\,k}$ along $\mathrm{Spec}\,k \xrightarrow{E} \mathrm{LS}_{\mathrm{GL}_n}$; for this one could imagine as skyscraper sheaf on $\mathrm{LS}_{\mathrm{GL}_n}$ determined by E. Hence we obtain the correspondence of objects under the diagram as

$$\begin{array}{ccc} \mathrm{Aut}'_E & \longrightarrow & \Gamma^{\mathrm{Spec}}(\delta_E) \\ {\scriptstyle \mathrm{coeff}^{\mathrm{enh}}}\uparrow & & \uparrow{\scriptstyle \Gamma^{\mathrm{Spec}}} \\ \mathrm{Aut}_E & \xrightarrow{\mathbb{L}_{\mathrm{GL}_n}} & \delta_E \end{array}$$

This explains the construction of Hecke eigensheaf via Laumon sheaf under geometric Langlands conjecture.

6. Occurrence in local Langlands correspondence

Now we turn to the setting of local Langlands correspondence for GL_n. For simplicity let's take the local field to be $\mathbb{Q}_p$. Then the local Langlands correspondence gives a correspondence between the following two collection:

(1) Weil–Deligne representation of $\mathbb{Q}_p$ of dimension n.
(2) Smooth admissible irreducible representation of $\mathrm{GL}_n(\mathbb{Q}_p)$.

The coefficient of $\ell \neq p$ and $\ell = p$ make a difference due to the subtlity of choosing the topology. We work firstly in the ℓ-adic coefficient case, which is simpler, while later we turn to the p-adic coefficient case.

6.1. **In the ℓ-adic geometrization.** In the ℓ-adic case, by Grothendieck's ℓ-adic monodromy theorem, Weil–Deligne representation reduces to representation of the Weil group $W_{\mathbb{Q}_p}$. The good thing is that we already have a geometrization in [FS]. Namely, we consider the Fargues–Fontaine curve

$$\mathcal{X}_{\mathrm{FF}} := (\mathrm{Spa}\,\mathbb{A}_{\mathrm{inf}}(\mathcal{O}_{\mathbb{C}_p}) \backslash V(p[\varpi^\flat]))/\varphi,$$

which serves as the curve appearing in the geometric Langlands correspondence, then consider two objects, Bun_G and Div^1, which are considered as v-stacks, and study their relations. Fix Λ an ℓ-torsion or ℓ-adic ring. Precisely, a geometric conjecture is formulated as follows which is the last statement of [FS]:

Conjecture 6.1 (Vague version). *There is an equivalence of ∞-categories between*

$$\mathcal{D}(\mathrm{Bun}_G, \Lambda)^\omega \cong \mathcal{D}^{b,qc}_{\mathrm{Nilp,coh}}(\mathrm{LS}_{\mathrm{Div}^1, \check{G}}).$$

We won't discuss this conjecture, but only focus on the case $G = \mathrm{GL}_n$ and the "skyscraper case": this is to construct the Hecke eigensheaf for any irreducible local system σ on Div^1 of rank n, and we try to do the same construction of the Laumon sheaf.

For simplicity, we can even just consider the case of GL_2. But before that let's introduce the dictionary of Bun_G for Fargues–Fontaine curve.

Fix S perfectoid space over $\mathbb{F}_p$. Consider $\mathcal{Y}_{(0,\infty),S}$ and the Fargues–Fontaine curve $\mathcal{X}_S$. Its associate diamond is $\mathcal{X}_S^\diamond = S \times \mathrm{Spd}\,\mathbb{Q}_p/\varphi^{\mathbb{Z}} \times \mathbf{id}$. The mirror curve Div^1 is defined as $\mathrm{Spd}\,\mathbb{Q}_p/\varphi^{\mathbb{Z}}$. The first goal is to understand vector bundles on Fargues–Fontaine curve. Notice that there is a canonical embedding $\underline{\mathrm{Isoc}}_k \to \mathrm{Bun}(\mathcal{X}_S)$ for any S. Thanks to Dieudonne–Manin classification, it sends D_λ to $\mathcal{O}(-\lambda)$.

Proposition 6.2. *Over any geometric point C, any $\mathcal{E} \in \mathrm{Bun}(\mathcal{X}_C)$ is of the form $\bigoplus \mathcal{O}(\lambda)^{m_\lambda}$. Moreover, for any $\mathcal{E} \in \mathrm{Bun}(\mathcal{X}_S)$, the map associating C to the Harder–Narasimhan polygon is upper semicontinuous.*

Define $\mathrm{Bun}_G(\mathcal{X}_S)$ by Tannakian formalism. Define Bun_G as the v-stack sending S to $\mathrm{Bun}_G(\mathcal{X}_S)$. It is easy to show Bun_G is a small v-stack over $\mathrm{Spd}\,k$, hence we can talk about its associated topological space $|\mathrm{Bun}_G|$. Denote by $B(G)$ the isomorphism class of G-isocrystals. Kottwitz has constructed two maps $\nu\colon B(G) \to \left(X_*(T)_{\mathbb{Q}}^+\right)^\Gamma$ and $\kappa\colon B(G) \to \pi_1(G)_\Gamma$. Define a topology on $\to \left(X_*(T)_{\mathbb{Q}}^+\right)^\Gamma \times \pi_1(G)_\Gamma$ by order topology on the first coordinate and discrete topology on the second.

Theorem 6.3 ([FS, chapter II])**.** *There is a homeomorphism $B(G) \cong |\mathrm{Bun}_G|$.*

Therefore, Bun_G should have a stratification parametrized by $B(G)$. It is denoted by Bun_G^b for $b \in B(G)$. The canonical G-bundle gives a map $x_b\colon * \to \mathrm{Bun}_G^b$.

Proposition 6.4. *The point x_b is surjective. As a result, $\mathrm{Bun}_G^b \cong [*/\widetilde{G}_b]$ of dimension $-\langle 2\rho, \nu_b\rangle$, where $\widetilde{G}_b$ is the automorphism of x_b. It has a filtration $\widetilde{G}_b^{\geq\lambda}$ such that $G_b(\mathbb{Q}_p) := \widetilde{G}_b^{\geq 0}/\widetilde{G}_b^{>0}$ is the automorphism of the corresponding G-isocrystal. If $\lambda > 0$, then $\widetilde{G}_b^{\geq\lambda}/\widetilde{G}_b^{>\lambda}$ is the Banach–Colmez space associated to $(\mathrm{ad}\,\mathcal{E}_b)^{\geq\lambda}/(\mathrm{ad}\,\mathcal{E}_b)^{>\lambda}$. Inside $B(G)$ we have those elements maximal under generalization, called basic elements. Moreover, the semistable part is described as $\mathrm{Bun}_G^{ss} = \bigsqcup_{b\ basic} \mathrm{Bun}_G^b$ with $\mathrm{Bun}_G^b = [*/G_b(\mathbb{Q}_p)]$.*

Now we return to the case $G = \mathrm{GL}_2$ and see how the Hecke eigensheaf is constructed. Starting from V a rank 2 local system on $\mathrm{Div}^1 = \frac{\mathcal{BC}(\mathcal{O}(1))\backslash\{0\}}{\mathbb{Q}_p^\times}$, we can form the following diagram:

$$\begin{array}{ccccc} \frac{\mathcal{BC}(\mathcal{O}(1))}{\mathbb{Q}_p^\times} & & & & \frac{\mathcal{BC}(\mathcal{O}(-1))}{\mathbb{Q}_p^\times} \\ & \searrow & & \swarrow & \\ & & \mathrm{B}\mathbb{Q}_p^\times & & \end{array}$$

The article [ALB] states that these are dual to each other in the setting of $\mathcal{D}(-,\Lambda)$, and we can do the Fourier transform for these categories. We extend the diagram to the following diagram

$$\begin{array}{ccccc} \frac{\mathcal{BC}(\mathcal{O}(1))\backslash\{0\}}{\mathbb{Q}_p^\times} \overset{j_1}{\subset} \frac{\mathcal{BC}(\mathcal{O}(1))}{\mathbb{Q}_p^\times} & & & & \frac{\mathcal{BC}(\mathcal{O}(-1))}{\mathbb{Q}_p^\times} \overset{j_2}{\supset} \frac{\mathcal{BC}(\mathcal{O}(\frac{1}{2}))\backslash\{0\}}{D^\times} \\ & \searrow & & \swarrow & \\ & & \mathrm{B}\mathbb{Q}_p^\times & & \end{array}$$

Then we define a sheaf Aut'_V on $\frac{\mathcal{BC}(\mathcal{O}(\frac{1}{2}))\backslash\{0\}}{D^\times}$ as follows:

$$\mathrm{Aut}'_V := j_2^* \,\mathrm{Four}(j_{1!}V).$$

The map $\frac{\mathcal{BC}(\mathcal{O}(\frac{1}{2}))\backslash\{0\}}{D^\times} \to \mathrm{B}D^\times$ is exactly the fiber of $\mathrm{Bun}'_2 \to \mathrm{Bun}_2$ over $*/D^\times \subset \mathrm{Bun}_2$. Hence, the descent of Aut'_V to a sheaf $\mathrm{Aut}_V \in \mathcal{D}(\mathrm{B}D^\times, \Lambda) = \mathrm{Rep}_\Lambda(D^\times)$ is the desired Hecke eigensheaf. This is exactly the attached smooth representation by composing the local Langlands correspondence with the local Jacquet–Langlands correspondence.

6.2. **In the p-adic geometrization.** Now we change our setting to p-adic representations of $\mathrm{Gal}_{\mathbb{Q}_p}$, which needs more thing involved to even state the correspondence, or to do the geometrization. The hard point comes from the topological issue. The category of Weil–Deligne representations is equivalent to the category of $(\varphi, N, \mathrm{Gal}_{\mathbb{Q}_p})$-modules. The point is that not every Galois representation can result a Weil–Deligne representation, but only those de Rham ones by p-adic monodromy theorem. Even then, this will lose informations, because we need add a filtration to recover de Rham representations. Before going into the construction of the Laumon sheaf, let's breifly introduce the geometrization procedure.

To get a such geometrization, we still work on the Fargues–Fontaine curve. Imagine this as the fundamental curve in geometric Langlands. the philosophy is that if we work in the p-adic coefficients, it is unavoidable to consider some $\mathcal{D}$-modules over the Fargues–Fontaine curve and its Bun_G stack. This was still an obstruction till recent work about analytic de Rham stack [RC], based on the theory of condensed mathematics and analytic stacks developped in [CS19], [CS20] and [CS22]. The point is that we can form the analytic de Rham stack of an arbitrary nonarchimedean analytic stack, and the solid sheaves on can be imagined to be identified with analytic $\mathcal{D}$-modules on this analytic stack. Here the theory of condensed mathematics is really involved because we are doing p-adic analytic geometry. Take the Fargues–Fontaine curve as example. Here the curve should be $\mathcal{X}_{\mathbb{Q}_p,\mathrm{FF}}$, which is by v-descent of the Fargues–Fontaine curve $\mathcal{X}_{\mathbb{C}_p,\mathrm{FF}}$. Then one can prove the following statement:

Theorem 6.5. *There is an equivalence of categories, essentially comes from local monodromy conjecture, that*

$$\{\textit{vector bundles on } \mathcal{X}^{\mathrm{dR}}_{\mathbb{Q}_p,\mathrm{FF}}\} \cong \{(\varphi, N, \mathrm{Gal}_{\mathbb{Q}_p})\textit{-modules over } \widehat{\mathbb{Q}_p^{\mathrm{un}}}\}.$$

This means that "$\mathcal{D}$-modules" on the fundamental curve $\mathcal{X}_{\mathbb{Q}_p,\mathrm{FF}}$ are classified by Weil–Deligne representations. However, the point is that in the local setting, the mirror of the curve, usually as Div^1, differs with the curve itself. This happens in ℓ-adic case as we have seen. In p-adic case, the version of Div^1 is exactly the analytic prismatization $\mathrm{Spa}\,\mathbb{Q}_p^{\Delta}/\varphi$, which is defined in [ALBRCS], but still much in progress. Nevertheless, we can prove the following statement:

Theorem 6.6. *There is an equivalence of categories*

$$\{\textit{vector bundles on } \mathrm{Spa}\,\mathbb{Q}_p^{\Delta}/\varphi\} \cong \{(\varphi,\Gamma)\textit{-modules over } \widetilde{B}_{\mathrm{rig},\mathbb{Q}_p}\}.$$

Hence we get a quite satisfied explanation of the phenomenon that Weil–Deligne representations differs from p-adic Galois representations, because the curve differs from its mirror. Moreover, by a result of Fontaine, (φ,Γ)-modules contains all p-adic representations as etale ones, so this is a reasonable category to work in. This is a summary for the spectral side.

Remark 6.7. We slightly deviate from the philosophy of geometric Langlands, because we are considering the vector bundle over Div^1, instead of local systems! However, this is not a wrong philosophy, and there also exists a similar phenomenon in the real geometrization, consulting [Sch].

The automorphic side is simpler to describe. The analytic version of Bun_G is described as follows. Compared to the Bun_G in ℓ-adic case, it also has a strata parametrized by $B(G)$, with each stratum $[*/\widetilde{G}_b]$, such that $\widetilde{G}_b$ has a filtration, with the first graded piece $G_b(\mathbb{Q}_p)^{\mathrm{la}} := \mathrm{AnSpec}\, C^{\mathrm{la}}(G_b(\mathbb{Q}_p),\mathbb{Q}_p)$ the locally analytic version of $G_b(\mathbb{Q}_p)$, and the other graded pieces the analytic Banach–Colmez spaces, which can be imagined as an enhancement of the classical Banach–Colmez spaces. Before going in detail, let's give an insight of this Bun_G.

Proposition 6.8. *The category of solid sheaves $D_{\blacksquare}([*/G(\mathbb{Q}_p)^{\mathrm{la}}])$ is the category of locally analytic representations of $G(\mathbb{Q}_p)$.*

Then we give some examples of analytic Banach–Colmez spaces.

Example 6.9. We compute three simplest cases.

- $\mathcal{BC}(\mathcal{O}) = \mathbb{Q}_p^{\mathrm{la}}$. Here $\mathbb{Q}_p^{\mathrm{la}}$ is defined as an infinitesimal extension of $\mathbb{Q}_p$
$$0 \longrightarrow \mathbb{G}_{\mathrm{a}}^{\dagger} \longrightarrow \mathbb{Q}_p^{\mathrm{la}} \longrightarrow \underline{\mathbb{Q}_p} \longrightarrow 0.$$
- $\mathcal{BC}(\mathcal{O}(1)) = \widetilde{\mathbb{D}}$, where $\mathbb{D}$ is the open unit disc as a subgroup of $\mathbb{G}_{\mathrm{m}}$ via $T \mapsto 1+T$, and $\widetilde{\mathbb{D}}$ is the universal cover of $\mathbb{D}$.
- $\mathcal{BC}(\mathcal{O}(-1)) = \mathbb{G}_{\mathrm{a}}^{\mathrm{dR}}/\underline{\mathbb{Q}_p}$ which is the de Rham stack of $\mathbb{G}_{\mathrm{a}}$ quotient by smooth $\mathbb{Q}_p$.

As we already see in the example, the duality doesn't work as same as the ℓ-adic case. Indeed, as pointed out in [RC], here the duality between Banach–Colmez spaces is deduced from Cartier duality, and is of the form

$$\mathcal{BC}(\mathcal{O}(\lambda)) \longleftrightarrow \mathrm{B}(\mathcal{BC}(\mathcal{O}(1-\lambda))).$$

This is to say, there is a Fourier transform makes the category of solid sheaves on both sides to be equivalent.

Now we start the Laumon construction in the p-adic setting. Starting from V a rank 2 vector bundle on $\mathrm{Div}^1 = \frac{\mathcal{BC}(\mathcal{O}(1))\backslash\{0\}}{\mathbb{Q}_p^\times}$, and the following diagram

$$\begin{array}{ccccc} \frac{\mathcal{BC}(\mathcal{O}(1))\backslash\{0\}}{\mathbb{Q}_p^\times} \overset{j_1}{\subset} \frac{\mathcal{BC}(\mathcal{O}(1))}{\mathbb{Q}_p^\times} & & & & \frac{\mathrm{B}\mathcal{BC}(\mathcal{O})}{\mathbb{Q}_p^\times} \overset{j_2}{\supset} \mathrm{BMir}_2(\mathbb{Q}_p)^{\mathrm{la}} \\ & \searrow & & \swarrow & \\ & & \mathrm{B}\mathbb{Q}_p^\times & & \end{array}$$

Then we define a solid sheaf Aut'_V on $\mathrm{BMir}_2(\mathbb{Q}_p)^{\mathrm{la}}$ as follows:

$$\mathrm{Aut}'_V := j_2^* \,\mathrm{Four}(j_{1!}V).$$

Similarly, $\mathrm{BMir}_2(\mathbb{Q}_p)^{\mathrm{la}} \to \mathrm{BGL}_2(\mathbb{Q}_p)^{\mathrm{la}}$ is exactly the fiber of $\mathrm{Bun}'_2 \to \mathrm{Bun}_2$ on $\mathrm{BGL}_2(\mathbb{Q}_p)^{\mathrm{la}} \subset \mathrm{Bun}_2^{\mathrm{dR}}$. Then we desire to do the descent of Aut'_V to $\mathrm{Aut}_V \in D_{\blacksquare}(\mathrm{BGL}_2(\mathbb{Q}_p)^{\mathrm{la}})$. If we can do this, then the construction $V \mapsto \mathrm{Aut}_V$ will recover the construction of Colmez functor [Col]:

$$\mathrm{Col}\colon \{(\varphi,\Gamma)\text{-modules of rank } 2\} \longrightarrow \{\text{locally analytic representation of } \mathrm{GL}_2(\mathbb{Q}_p)\}.$$

Remark 6.10. We notice there is a difference between the ℓ-adic case and p-adic case in the construction of Hecke eigensheaf via Laumon sheaf, that in ℓ-adic case (resp. p-adic case) we care about its fiber over $\mathrm{B}D^\times$ (resp. $\mathrm{BGL}_2(\mathbb{Q}_p)^{\mathrm{la}}$). This can be explained as the difference of the dualizing sheaf in the ℓ-adic case and p-adic case. Precisely, in ℓ-adic case (resp. p-adic case) the dualizing sheaf is $\mathcal{O}$ (resp. $\mathcal{O}(1)$).

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