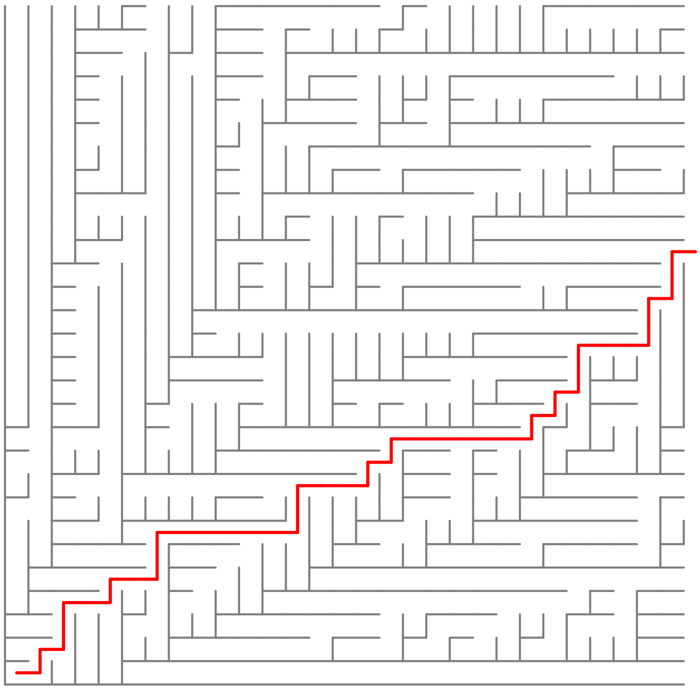
Research Domain Introduction
IMT Nord Europe - Lille

Last-Passage Percolation: Infinite Geodesics through Busemann functions



M2 thesis Supervised by David COUPIER (IMT Nord Europe) and Viet Chi TRAN (Université Gustave Eiffel)

Tom GARCIA-SANCHEZ

September 27, 2024

Context and structure

This document aims to introduce the problem of infinite geodesics in last-passage percolation, subject of my M2 thesis. The goal was to prepare my PhD which focuses on the problem of infinite geodesics in the Directed Spanning Forest. The context of last-passage percolation is one of the most simple settings where the behavior of infinite geodesics is non-trivial. Understanding the tools and ideas involved in its study is crucial as it can be adapted to many other contexts and it was the main motivation.

We will start by introducing the problem briefly in Section 1. Then we will focus on a particular case which is completely solved today in Section 2. It will allow us to understand roughly what is challenging in the general case and why tools like Busemann functions were developed. We will then set some notations in Section 3 and introduce properly a shape Theorem in Section 4. Next we will introduce the notion of cocycle in Section 5 and explain why such objects are interesting for the problem of infinite geodesics, motivating the construction of Busemann functions, that we will introduce in Section 6. Finally we will state results obtained from Busemann functions in Section 7 and 8. Note that we will admit a lot of results to avoid getting too technical, and we will focus on giving the ideas and intuitions behind the study of infinite geodesics in that context.

1 Introduction

Last-passage percolation (LPP) is a very classical area of research in probability. The definition of the model is quite simple and yet understanding its behavior remains challenging, a lot of conjectures about it are still open. The goal is usually to understand a *random distance* on a fixed *graph*, where random *weights* are assigned to its *edges* or its *vertices*. Generally, one wants to understand how the random geometry behaves in large scales or to study the paths with the biggest weight between two points, known as *geodesics*.

Let us begin with the definition of the LPP model in our case. In this work, we consider the *lattice* $\mathbb{Z}^2$, where we have an edge from x to y if and only if $|y-x|=1$, where $|\cdot|$ denotes the standard euclidean norm. We consider $(\omega_x)_{x\in\mathbb{Z}^2}$ a collection of *i.i.d. real-valued random variables*, attached to the vertices of $\mathbb{Z}^2$ that we call *weights*. Note that these weights are not necessarily non-negative. We define $e_1 \coloneqq (1,0)$ and $e_2 \coloneqq (0,1)$ and for all $m \le n$ in $\mathbb{Z}$, we say that a sequence $(\pi_k)_{k=m}^n$ is an *up-right path* from π_m to π_n, or simply a *path*, if $\pi_{k+1}-\pi_k \in \{e_1,e_2\}$ for every $k \in [\![m, n-1]\!]$. We then denote by $T(\pi)$ its *passage-time*, defined by

$$T(\pi) \coloneqq \sum_{k=m}^{n-1} \omega_{\pi_k}. \tag{1.1}$$

Note that the weight of the endpoint of the path is not taken into account for convenience as we will see later on. For any points x, y in $\mathbb{Z}^2$ we define the *last-passage percolation time* $G_{x\,y}$ from x to y as the biggest passage-time of a path going from x to y, i.e.

$$G_{x\,y} \coloneqq \max_{\pi} T(\pi), \tag{1.2}$$

where the maximum is taken over all paths π going from x to y, with the convention $\max\emptyset = +\infty$. A path realizing this maximum is called a *geodesic*. For $m \in \mathbb{Z}$, an infinite sequence $(\pi_k)_{k\ge m}$ of vertices is called an *infinite geodesic* if for all $m \le n$, $(\pi_k)_{k=m}^n$ is a (finite) geodesic. We are interested in studying the properties of such infinite geodesics. We will see that the family of geodesics out of 0 forms an *up-right tree* rooted at 0 that covers $\mathbb{N}^2$. Then we shall ask ourselves the following questions:

- Do infinite geodesics have an asymptotic direction?
- For each direction α, do we have the existence of an α-directed infinite geodesic?
- Do we have uniqueness of infinite geodesics with a fixed asymptotic direction?

To answer theses questions, some assumptions on the weights are needed. The most important example is that if the distribution is continuous, then almost surely we have uniqueness of (finite) geodesics between any two points because two different paths cannot have the same passage time, while it is no longer the case if the distribution has an atom.

In what follows, we focus on studying the infinite geodesics going out of 0 in the model using *Busemann functions* under some general assumptions on the weight distribution.

2 The case of exponential weights

In this Section, we resume the history of the exactly solvable case of exponential weight in order to understand what is challenging and why Busemann functions emerged. We will explain the strategies behind the proofs and where the need for tools provided by *Busemann functions* arises.

When the weights are exponentially distributed, the behavior of infinite geodesics is fully understood. It is in fact the only example where it is the case.

Throughout this Section, we assume that ω_0 has an Exp(1) distribution.

Let us consider the set of all geodesics started at 0. A simulation is presented in Figure 1. Since the exponential distribution has no atom, almost surely theses geodesics can not intersect out of 0 and thus they form a tree that covers all the quarter grid $\mathbb{N}^2$. Note that for each $i \in \{1, 2\}$, $(me_i)_{m\geq 0}$ is always an infinite geodesic as there is only one admissible path between any two of its points.

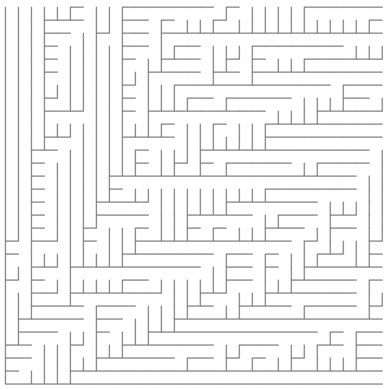


Figure 1: Geodesics starting at 0 (at the bottom left) in the model with exponential weights.

One key point for the study of the model is to understand how does the random balls $\mathcal{B}_t := \{x \in \mathbb{N}^2 \ : \ G_{0\,x} \leq t\}$ looks like when t is large. In the exponential case, it is known that there is an asymptotic deterministic shape whose expression is explicit. More precisely, denoting $\mathcal{B} := \{(x, y) \in [0, 1]^2 \ : \ \sqrt{x} + \sqrt{y} \leq 1\}$ and fixing $\epsilon \in (0, 1)$, we have almost surely

$$(1 - \epsilon)\mathcal{B} \subset \frac{1}{t}\mathcal{B}_t \subset (1 + \epsilon)\mathcal{B} \quad \text{for all } t \text{ large enough.}$$

In fact since $\mathcal{B}$ is compact, we also have that $\frac{1}{t}\mathcal{B}_t$ converges almost surely toward $\mathcal{B}$ as $t \to +\infty$ for the *Hausdorff distance*. In word we have some kind of uniform convergence toward an explicit deterministic limit shape. A simulation to see this shape Theorem is given Figure 2. Historically, this result was first given by [Ros81] in 1981. Roughly, they proved it exploiting the fact that the distribution is memoryless to construct a nice coupling with a particle system known as the TASEP (Totally Asymmetric Simple Exclusion Process), where they are able to do computations.

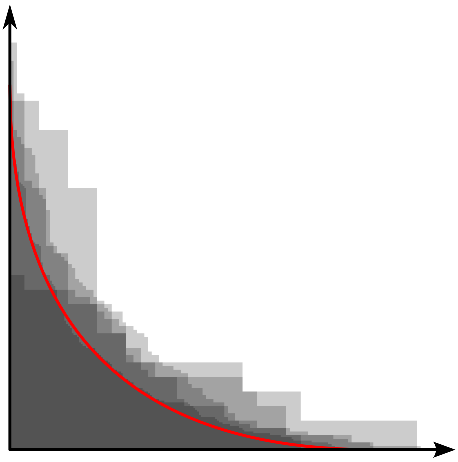


Figure 2: Illustration of the shape Theorem in the case of exponential weights. In grey the sets $\{x \in \mathbb{N}^2 \ : \ G_{0x} \leq n\}$ (one point correspond to a pixel) rescaled by n, for $n = 25, 50, 100, 200, 400$ and in red the explicit scale limit.

In the general case, with only some moment assumptions, there is also a shape Theorem given by [Mar04] in 2004, but the main challenge is that the limit shape is not explicit. In particular the question of knowing when its boundary is strictly convex is still open and plays a crucial role as we will see.

In 2001, for the sake of studying geodesics in a model of random geometry based on a Poisson process, the concept of *straightness* for a tree was introduced by [HN01]. The definition is the following. Fix an *oriented rooted* tree T embedded in $\mathbb{R}^2$ and a function $f : \mathbb{R}_+ \to \mathbb{R}_+$. For a vertex u of the tree, denote by $T(u)$ the *subtree* of T rooted at u. For every $x \in \mathbb{R}^2 \setminus \{0\}$ and $\epsilon > 0$, we denote by $C(x, \alpha) \coloneqq \{y \in \mathbb{R}^2 \ : \ \Theta(x, y) \leq \epsilon\}$ the cone of angle width 2ϵ going out of 0 in direction x, where $\Theta(x, y)$ denotes the *absolute angle* between x and y. We say that T is f*-straight* if for all but finitely many vertex u, one has

$$T(u) \subset C(u, f(|u|)).$$

In word, T is said to be straight if the direction of its branches is asymptotically controlled. The interest of this definition is the following deterministic result ([HN01, Proposition 2.8]). If T has asymptotically points in every directions (see *asymptotically omnidirectional* in [HN01]) and is f-straight for some $f : \mathbb{R}_+ \to \mathbb{R}_+$ such that $f(x) \to 0$ as $x \to 0$, then every *infinite branch* of T has an asymptotic direction and moreover every direction is reached by some infinite branch. In the paper, they then show that the tree of geodesics in their model is indeed f-straight almost surely for a good function f, and they apply the deterministic result to deduce that almost surely, all infinite geodesics are directed and all directions are reached by some infinite geodesics.

This works inspired [FP05] in 2005, where they managed to show that the tree of geodesics of the LPP model with exponential weights is straight, and thus deduced the same properties on infinite geodesics. To do so, they did some estimations on the fluctuations of geodesics from straight lines using the shape Theorem. Note that their arguments rely strongly on the fact that the explicit limit $\sqrt{x} + \sqrt{y} = 1$ is *strictly convex*.

From here, we already see that we need to use other tools in the general case. Indeed, even with strong assumptions on the distribution, we can not tell if the limit shape is strictly convex or not, and thus we can not mimic the strategy of [FP05]. It is believed that strict convexity should hold for a quite general class of distribution, but yet it's still an open conjecture. The only distributions where it is known to be the case is the exponential and geometric ones, because we can check that the explicit expression of the limit is convex.

The question of uniqueness of directional infinite geodesics going out of 0 was finally closed by [Cou11] in 2011, refining some results of [FP05]. Here is a sum up of the behavior of infinite geodesics from 0 in the exponential weights case.

- Almost surely, each infinite geodesic has an asymptotic direction.
- For each direction α, almost surely there exists a unique α-directed geodesic.
- Almost surely, there exist a (non-deterministic) countable dense set D of exceptional directions such that for all direction $\alpha \notin D$, there exists exactly one α-directed infinite geodesic, while for all $\alpha \in D$, there is exactly 2 α-directed infinite geodesics. In particular, almost surely there is no direction α with 3 or more α-directed infinite geodesics.

Note the two last points do not contradicts each other as the set of all possible directions is uncountable. For each α we have uniqueness of the α-directed geodesics but we do not have the statement almost surely for all α at once. For any fixed direction α, we have $\mathbb{P}(\alpha \in D) = 0$.

The aim of this work is to generalize this characterization for more general weight distributions.

Busemann functions were introduced in 1994 by [New95], borrowing ideas from classical metric geometry. The principal advantage of these objects is that they allow to study infinite geodesics in all directions at once on a single event of full probability, whereas with other tools some direction-dependent events usually appear. Busemann functions also allow to work with general weight distribution, even when strict convexity do not hold and thus arguments of [FP05] fail to apply. It also allow to go further in the exponential case. It was indeed used by [JRAS22] in 2020 to finish the complete characterisation of the behavior of all the infinite geodesics going out of all the lattice points of $\mathbb{Z}^2$. Note that studying geodesics emanating from all points of $\mathbb{Z}^2$ is more intricate than only those going out of 0. Busemann functions has now become a principal tool for studying infinite geodesics.

3 Setup and Notation

In this Section we introduce some notation.

We denote $\hat{e} := e_1 + e_2$ and for all $k \in \mathbb{Z}$ we define the set of vertices at level k by $\mathbb{V}_k := \{x \in \mathbb{Z}^2 : x \cdot \hat{e} = k\}$, where $\cdot$ denotes the standard scalar product.

We denote the set of internal directions by $\mathcal{U} := \{te_1 + (1-t)e_2,\ t \in (0,1)\}$ and the whole set of directions by $\overline{\mathcal{U}} := \mathcal{U} \cup \{e_1, e_2\}$. Note that we did not chose to have unit vectors as directions, that way we can have a set that is is convex, it will be important later. When x and y are points of $\mathbb{Z}^2$, we write $x \le y$ if $x \cdot e_i \le y \cdot e_i$ for each $i \in \{1, 2\}$.

For all $h, h' \in \mathbb{R}^2$, we write $h \preceq h'$ if $h \cdot (e_1 - e_2) \le h' \cdot (e_1 - e_2)$. If $m \le n$ in $\mathbb{Z}$ and $(\pi_k)_{k=m}^n$, $(\pi_k)_{k=m}^n$ are two paths, we write $\pi \preceq \pi'$ if $\pi_k \preceq \pi'_k$ for all $m \le k \le n$, in that case we usually say that π is to the *left* of π'. Note that we will use $\preceq$ to compare elements of $\mathbb{R}^2$ that represent some directions as well as for elements of $\mathbb{Z}^2$, that are vertices.

What we mean by a statement with some $\pm$ is a conjunction of two statements: one for the top sign, and another one for the bottom sign.

If π is an infinite path and $U \subset \overline{\mathcal{U}}$ is a set of directions, we say that π is U-directed if all limit points of the sequence $(\pi_n/n)_n$ lie in U.

Unless otherwise stated, all infinite geodesics have 0 as their starting point.

Now let us state the hypothesis we make on the weight distribution. We assume that

$$\omega_0 \in L^p \quad \text{for some real } p > 2. \tag{3.1}$$

This is the main hypothesis. It will be mainly used to get a shape Theorem. Any other assumption will be precised when needed.

We conclude the Section with the following observation. If $x \leq y$ in $\mathbb{Z}^2$ and π, π are two geodesic from x to y, then between two consecutive interSection points of π and π' we have a left and a right path by planarity, with the same passage-time by optimality. Then merging all these left (resp. right) paths we get a geodesic that is to the left (resp. right) of both π and π'. This lattice property ensures that there is always a leftmost and a rightmost geodesic from x to y.

4 The shape Theorem

In this Section, we first investigate properly the asymptotic behavior of $G_{0\,x}$ as $|x| \to +\infty$ stating a shape Theorem. Similarly as for the exponential case, this result is necessary to study infinite geodesics.

Recall definition (1.2) and observe that in (1.1) we have omitted the weight at the end-point of the path. That way when we merge paths we simply have to sum their passage times. In particular, if $x \leq y \leq z$ are points of $\mathbb{Z}^2$, we can merge a geodesic from x to y with one from y to z to get a path from x to z and obtain

$$G_{x\,z} \geq G_{x\,y} + G_{y\,z}. \tag{4.1}$$

This super-additive property is a key element. Let's fix a vector $\xi \in \mathbb{R}_+ \times \mathbb{R}_+$ representing a direction. Since $\omega_0 \in L^1$, $G_{0\,\lfloor n\xi \rfloor} \in L^1$ for all $n \in \mathbb{N}^*$ and we can define the deterministic quantity

$$\Lambda(\xi) := \limsup_{n \in \mathbb{N}^*} \frac{\mathbb{E}[G_{0\,\lfloor n\xi \rfloor}]}{n} \in (-\infty, \infty].$$

In fact, modulo some small perturbation caused by the floor function, using (4.1) we can apply Kingman's sub-additive ergodic Theorem (see [Kin68]) to get

$$\lim_{n \to +\infty} \frac{G_{0\,\lfloor n\xi \rfloor}}{n} = \Lambda(\xi) \quad \text{a.s. [and in } L^1 \text{ if } |\Lambda(\xi)| < +\infty]. \tag{4.2}$$

By super-additivity we can say that the function $\mathbb{R}_+^2 \ni \xi \to \Lambda(\xi)$ is concave and homogeneous of degree 1, that is $\Lambda(x + y) \geq \Lambda(x) + \Lambda(y)$ for all x, y in $\mathbb{R}_+^2$ and $\Lambda(\alpha x) = \alpha \Lambda(x)$ for all $x \in \mathbb{R}_+^2$ and $\alpha \in \mathbb{R}_+$. By symmetry it is also invariant under permutation of the coordinates. For the exponential case, from [Ros81] we have

$$\Lambda(x) = (\sqrt{x \cdot e_1} + \sqrt{x \cdot e_2})^2.$$

We can indeed check that it is concave, homogeneous of degree 1 and invariant under permutation of the coordinates.

In [Mar04], it is shown that thanks to assumption (3.1) (which in fact can be slightly weakened) we have that Λ is finite everywhere and continuous on all $\overline{\mathcal{U}}$. Note that concavity only ensures continuity on $\mathcal{U}$. Moreover we have a uniform convergence in the sense that

$$\lim_{n\to+\infty} \max_{x\in\mathbb{V}_n} \frac{|G_{0\,x} - \Lambda(x)|}{n} = 0 \quad \text{almost surely.} \tag{4.3}$$

It is this last result that we call the shape Theorem.

We conclude the Section by some notation that will be necessary to state that the infinite geodesics are well-directed.

Observe that $\mathcal{U} \ni \xi \mapsto \Lambda(\xi)$ is a concave function from an open segment of $\mathbb{R}^2$ to $\mathbb{R}$. Therefore we have the existence of one-sided derivatives at every $\xi \in \mathcal{U}$. Denoting $u := (e_1 - e_2)/\sqrt{2}$ we can thus define

$$\Lambda'(\xi\pm) := \lim_{\epsilon\to 0^\pm} \frac{\Lambda(\xi+\epsilon u) - \Lambda(\xi)}{\epsilon} \in \mathbb{R}. \tag{4.4}$$

From this, denote

$$\mathcal{U}_{\xi\pm} := \{\zeta \in \overline{\mathcal{U}} \ : \ \Lambda(\zeta) - \Lambda(\xi) = (\zeta - \xi)\cdot \Lambda'(\xi\pm)u\}. \tag{4.5}$$

In words, $\mathcal{U}_{\xi\pm}$ corresponds to the maximal linear segment of Λ around ξ where the slope is given by $\Lambda'(\xi\pm)$. For $\xi \in \mathcal{U}$, we denote by $\mathcal{U}_\xi$ the possibly degenerate interval of directions $\mathcal{U}_{\xi-} \cup \mathcal{U}_{\xi+}$ and we set $\underline{\xi}$ and $\bar{\xi}$ to be respectively its leftmost and rightmost endpoint. Note that $\mathcal{U}_\xi$ is the set of direction $\zeta \in \mathcal{U}$ such that Λ is linear between ζ and ξ. An illustration is given Figure 3. In case 1 and 3, Λ is differentiable at ξ, whereas it is not the case in the other cases. In case 1 and 2, we have no linear segments around ξ, so $\underline{\xi} = \xi = \bar{\xi}$ and $\mathcal{U}_{\xi-} = \mathcal{U}_{\xi+} = \{\xi\}$. In case 3, ξ lies in the interior of a linear segment of Λ, we have $\underline{\xi} \prec \xi \prec \bar{\xi}$ and $\mathcal{U}_{\xi-} = \mathcal{U}_{\xi+} = [\underline{\xi}, \bar{\xi}]$. In case 4, ξ is at the right of a maximal linear segment and at the left of another one, we have $\underline{\xi} \prec \xi \prec \bar{\xi}$, $\mathcal{U}_{\xi-} = [\underline{\xi}, \xi]$ and $\mathcal{U}_{\xi+} = [\xi, \bar{\xi}]$. In case 5, ξ lies at the left of a maximal linear segment but there is no linear segment to its left, giving $\underline{\xi} = \xi \prec \bar{\xi}$, $\mathcal{U}_{\xi-} = \{\xi\}$ and $\mathcal{U}_{\xi+} = [\xi, \bar{\xi}]$. The only cases we have not draw are the three ones we obtain from 5 by applying a symmetry, adding differentiability at ξ, or both. Note that Λ is strictly concave around ξ in case 1 and 2 only.

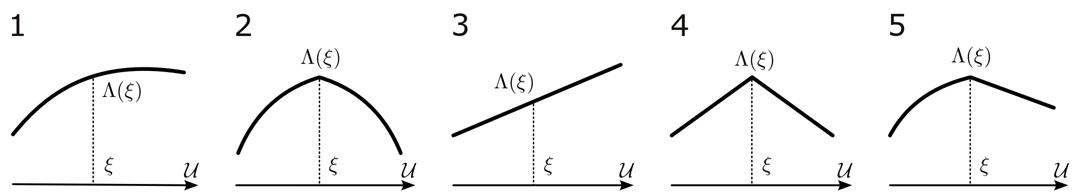


Figure 3: Illustration of the possible behaviors of Λ around ξ. The graph of $\zeta \mapsto \Lambda(\zeta)$ is represented in solid line.

The sets $\mathcal{U}_\xi$ will be used to show that we can generalize the results we have in the exponential case, we will see that we can get similar statements in the general case replacing "ξ-directed" by "$\mathcal{U}_\xi$-directed". Note that these statements will allow recover what we have in the exponential case as when Λ is strictly concave, there is no linear segments and thus $\mathcal{U}_\xi = \{\xi\}$ for all $\xi \in \mathcal{U}$.

5 The notion of cocycle

In this Section we introduce the algebraic notion of cocycles. We will see how it allows to construct infinite geodesics. It gives the motivation for constructing the Busemann functions.

Definition 5.1. We call *cocycle* any random function $B : \mathbb{Z}^2 \times \mathbb{Z}^2 \to \mathbb{R}$ (where we endow $\mathbb{R}^{Z^2 \times \mathbb{Z}^2}$ with the product Borel σ-algebra) satisfying the *additivity* property that $B(x, z) = B(x, y) + B(y, z)$ for all x, y, z in $\mathbb{Z}^2$. A cocycle B is said to be *stationary* if

$$B(x + z, y + z) = B(x, y) \circ T_z \quad \text{for all } x, y, z \text{ in } \mathbb{Z}^2,$$

integrable (or L^1) if

$$\mathbb{E}[|B(x, y)|] < +\infty \quad \text{for all } x, y \text{ in } \mathbb{Z}^2,$$

and we say that it *recovers the weights* if

$$\omega_x = \min_{i \in \{1,2\}} B(x, x + e_i) \quad \text{for all } x \in \mathbb{Z}^2.$$

The key property of cocycles is the following Proposition. Roughly, it says that that if we have a cocycle recovering the weights, then we can construct infinite geodesics by following its minimal increments. Note that recovery of the weight is essential as it makes the link with the LPP process.

Proposition 5.2. *Let B be a cocycle recovering the weights, then for all $x \leq y$ in $\mathbb{Z}^2$ we have*

$$G_{x\,y} \leq B(x, y).$$

If π is a path from x to y such that for all $x \cdot \hat{e} \leq i < y \cdot \hat{e}$

$$B(\pi_i, \pi_{i+1}) = \min_{i \in \{1,2\}} B(\pi_i, \pi_i + e_i) = \omega_{\pi_i},$$

then π is a geodesic of passage time $B(x, y)$, in that case we say that π is a B-geodesic. Furthermore, if $\pi_{i+1} - \pi_i = e_2$ (resp. $\pi_{i+1} - \pi_i = e_1$) whenever $B(\pi_i, \pi_i + e_1) = B(\pi_i, \pi_i + e_2)$ then π is the leftmost (reps. rightmost) geodesic from x to y.

Proof. The proof is due to [GRAS17, proof of Lemma 4.1]. Let π^* be a geodesic from x to y, then

$$G_{x\,y} = \sum_{i=x\cdot\hat{e}}^{y\cdot\hat{e}-1} \omega_{\pi_i^*} \leq \sum_{i=x\cdot\hat{e}}^{y\cdot\hat{e}-1} B(\pi_i^*, \pi_{i+1}^*) = B(x, y)$$

where the first inequality follows from recovery of the weights and the last equality comes from additivity. Now that we have the first assertion, observe that by assumption

$$G_{x\,y} \leq B(x, y) = \sum_{i=x\cdot\hat{e}}^{y\cdot\hat{e}-1} B(\pi_i, \pi_{i+1}) = \sum_{i=x\cdot\hat{e}}^{y\cdot\hat{e}-1} \omega_{\pi_i} \leq G_{x\,y},$$

giving the second assertion. It remains to check the last point. By symmetry, we can assume that $\pi_{i+1} - \pi_i = e_2$ whenever $B(\pi_i, \pi_i + e_1) = B(\pi_i, \pi_i + e_2)$. Now let π' be any geodesic from x to y. Assume that k is an index such that $\pi_k = \pi'_k$ and $\pi_{k+1} - \pi_k = e_1$. By construction $B(\pi_k, \pi_k + e_1) < B(\pi_k, \pi_k + e_2)$, hence $\omega_{\pi_k} < B(\pi_k, \pi_k + e_2)$ by recovery of the weights. We also have $G_{\pi_k + e_2\,y} \leq B(\omega_{\pi_k} + e_2, y)$ by the first assertion of the Proposition. Combining the inequalities we get

$$\omega_{\pi_k} + G_{\pi_k + e_2\,y} < B(\pi_k, \pi_k + e_2) + B(\omega_{\pi_k} + e_2, y) = B(\pi_k, y) = G_{\pi_k, y},$$

where the two last equalities come respectively from additivity and from the second assertion of the Proposition. Therefore, recalling that $\pi'_k = \pi_k$, since π' is a geodesic, it shows that we must have $\pi'_{k+1} - \pi'_k = e_1$. This shows that π' always stays to the right of π, π is the leftmost geodesic. □

Now that we see how cocycles recovering the weight allow to construct infinite geodesics, we want to construct such functions. Note that we will also require stationnarity and integrability. The reason for that is that heuristically, the expectation of the cocycle controls in which direction the function grows faster, and it does not depend on the vertex we consider by stationnarity. We expect then that it will allow to control the asymptotic directions of the infinite geodesic we obtain by following minimal increments of the cocycles.

6 Busemann functions and geodesics

In this Section we state the existence of cocycle recovering the weights we nice properties.

There is a specific technical construction that allow to obtain a collection of cocycles recovering the weights that are stationary and integrable with explicit specific expectancy: *Busemann functions*. Roughly, the idea is to start from some natural approximations of such cocycles. Then a priori there is no reason for these approximations to converge toward something almost surely, so we tweak the approximations with an averaging trick to get tension in some proper extension of the probability space. Then we use Prokhorov Theorem to extract a weak limit and we finally check that the objects we get from that gives a collection of cocycle recovering the weight defined together with the last-passage percolation process on the extended probability space. The following Theorem summarizes the construction. The proof can be found in [JRA20, proof of Theorem 4.7]. Note that we simplified the statement and that some important properties are missing.

Theorem 6.1. *Up to extending the probability space, there exist two random functions* $\mathcal{U}\times\mathbb{Z}^2\times\mathbb{Z}^2 \ni (\xi, x, y) \mapsto \mathbf{B}^{\xi\pm}(x, y) \in \mathbb{R}$ *such that on a set of full probability:*

(i) For all $\xi \in \mathcal{U}$, $\mathbf{B}^{\xi\pm}$ *is a stationary* L^1 *cocycle recovering the weights, i.e.*

- $\mathbf{B}^{\xi\pm}(x, y) + \mathbf{B}^{\xi\pm}(x, y) = \mathbf{B}^{\xi\pm}(x, y)$ *for all* x, y, z *in* $\mathbb{Z}^2$. *(additivity)*
- $\mathbf{B}^{\xi\pm}(x + z, y + z) = \mathbf{B}^{\xi\pm}(x, y) \circ T_z$ *for all* x, y, z *in* $\mathbb{Z}^2$. *(stationarity)*
- $\omega_x = \min_{i\in\{1,2\}} \mathbf{B}^{\xi\pm}(x, x + e_i)$ *for all* $x \in \mathbb{Z}^2$. *(recovery of the weights)*

Moreover we have a specific explicit expression of $\widehat{\mathbb{E}}[\mathbf{B}^{\xi\pm}(0, e_i)]$ *for each* $i \in \{1, 2\}$.

(ii) On $\widehat{\Omega}_0$, $(\mathcal{U}, \preceq) \ni \xi \mapsto \mathbf{B}^{\xi\pm}(0, e_1)$ *is non-decreasing and* $\xi \mapsto \mathbf{B}^{\xi\pm}(0, e_2)$ *is non-increasing. Moreover for all* $\xi \in \mathcal{U}$, *we have*

$$\lim_{\zeta\to\xi,\, \zeta\prec\xi} \mathbf{B}^{\zeta\pm} = \mathbf{B}^{\xi-} \quad \text{and} \quad \lim_{\zeta\to\xi,\, \zeta\succ\xi} \mathbf{B}^{\zeta\pm} = \mathbf{B}^{\xi+}.$$

In fact, for all $\xi \in \mathcal{U}$, $\mathbf{B}^{\xi-}$ and $\mathbf{B}^{\xi+}$ are respectively the left-continuous and right-continuous version of the same function. It comes from the fact that in the construction of Busemann functions we handle first a countable dense set of direction and then take some left and right limits to extend at all directions using some monotonicity properties.

7 Directions of geodesics

In this Section, we define Busemann geodesics constructed from Busemann functions and see what it allows to obtain on infinite geodesics. We will see that all these Busemann geodesics are well directed in some sense, and we will use it to deduce that in fact all infinite geodesics are well directed. The proofs can be found in [GRAS17].

Recall from Section 5 that we can construct infinite geodesics from Busemann functions by following the minimal increments of these cocycles. There is possibly multiple choices when there is a tie in the increments at some vertex, giving possibly multiple $\mathbf{B}^{\xi\pm}$-geodesics for a fixed $\xi \in \mathcal{U}$. We start by some notation.

Definition 7.1. For all $\xi \in \mathcal{U}$, we denote by $\mathbf{X}^{\xi-}$ the $\mathbf{B}^{\xi-}$-geodesic from 0 that takes e_2 steps in case of a tie and by $\mathbf{X}^{\xi+}$ the $\mathbf{B}^{\xi+}$-geodesic that takes e_1 steps in case of a tie. More precisely, $\mathbf{X}^{\xi\pm}$ is defined inductively with $\mathbf{X}_0^{\xi\pm} := 0$ and

$$\forall n \in \mathbb{N} \quad \mathbf{X}_{n+1}^{\xi\pm} := \begin{cases} \mathbf{X}_{n+1}^{\xi\pm} + e_1 & \text{if } \mathbf{B}^{\xi\pm}\left(\mathbf{X}_{n+1}^{\xi\pm}, \mathbf{X}_{n+1}^{\xi\pm} + e_1\right) < \mathbf{B}^{\xi\pm}\left(\mathbf{X}_{n+1}^{\xi\pm}, \mathbf{X}_{n+1}^{\xi\pm} + e_2\right) \\ \mathbf{X}_{n+1}^{\xi\pm} + e_2 & \text{if } \mathbf{B}^{\xi\pm}\left(\mathbf{X}_{n+1}^{\xi\pm}, \mathbf{X}_{n+1}^{\xi\pm} + e_1\right) > \mathbf{B}^{\xi\pm}\left(\mathbf{X}_{n+1}^{\xi\pm}, \mathbf{X}_{n+1}^{\xi\pm} + e_2\right) \\ \mathbf{X}_{n+1}^{\xi\pm} + e_\pm & \text{otherwise} \end{cases},$$

where $e_+ := e_1$ and $e_- := e_2$.

In word, for all $\xi \in \mathcal{U}$, $\mathbf{X}^{\xi+}$ is the $\mathbf{B}^{\xi+}$-geodesic that takes an e_1-step in case of a tie while following minimal increments of $\mathbf{B}^{\xi+}$ and $\mathbf{X}^{\xi-}$ follows minimal increments of $\mathbf{B}^{\xi+}$ and takes an e_2-step in case of a tie.

First, it can be shown that all these $\mathbf{X}^{\xi\pm}$-geodesics for $\xi \in \mathcal{U}$ are well-directed. It is summarized in the next Theorem. Note that when Λ is strictly concave, that is $\mathcal{U}_\xi = \{\xi\}$ for all $\xi \in \mathcal{U}$ it says that almost surely for every $\xi \in \mathcal{U}$, each $\mathbf{B}^{\xi\pm}$-geodesic has asymptotic direction ξ. In the exponential case, we have the strict concavity so it already says that every direction is reached as the asymptotic direction of an infinite geodesic, giving an existence result. Note that we are dealing with uncountably many directions at once.

Theorem 7.2. *There exists an event of full probability on which for each $\xi \in \mathcal{U}$, every $\mathbf{B}^{\xi\pm}$-geodesic is $\mathcal{U}_{\xi\pm}$-directed.*

Next, knowing that all Busemann geodesics are well-directed allow to show that it is also the case for any geodesic. The key idea is to compare any infinite geodesic with Busemann geodesic. With some work it is possible to show the following Theorem. Note that we do not know if all infinite geodesics are Busemann geodesics or not in general.

Theorem 7.3. *Almost surely, all infinite geodesics are $\mathcal{U}_\xi$-directed for some $\xi \in \overline{\mathcal{U}}$.*

To sum up the Section, we have seen that for all $\xi \in \mathcal{U}$, every $\mathbf{B}^{\xi\pm}$-geodesic is $\mathcal{U}_{\xi\pm}$-directed. We then saw that using Busemann geodesics to frame any arbitrary infinite geodesic, we get to the fact that every infinite geodesic is $\mathcal{U}_\xi$-directed for some $\xi \in \overline{\mathcal{U}}$. Note that the control on the direction of Busemann geodesic is a priori stronger than for arbitrary geodesics, as $\mathcal{U}_\xi$ might be a bigger set than $\mathcal{U}_{\xi+}$ or $\mathcal{U}_{\xi-}$.

In the exponential case, recall that $\mathcal{U}_{\xi\pm} = \{\xi\}$ for all $\xi \in \overline{\mathcal{U}}$ as Λ is strictly concave. In that case we have that every infinite geodesic admits an asymptotic direction. Moreover every $\xi \in \mathcal{U}$ is reached as the asymptotic direction of any $\mathbf{B}^{\xi\pm}$-geodesic. Note that as we have seen in Section 2, almost surely every direction is reached by at most two geodesic. In fact it can be shown that $\mathbf{X}^{\xi-}$ and $\mathbf{X}^{\xi+}$ are the only geodesic with asymptotic direction ξ. In particular, since all infinite geodesics are directed, all infinite geodesics are Busemann geodesic. In the general setting, the question of existence of infinite geodesics that are not Busemann geodesics is still open.

8 Uniqueness and non-uniqueness

In this Section we state some further result on uniqueness and non-uniqueness of well-directed infinite geodesic. We start by the following Theorem from [GRAS17]. It gives the existence of an exceptional direction of non-uniqueness. Note that this direction is a priori random.

Theorem 8.1. *Assume that the distribution of ω_0 has no atom. Then there exists almost surely a direction $\xi^* \in \mathcal{U}$ such that there is two $\mathcal{U}_{\xi^*}$-directed infinite geodesics from 0 intersecting only at the origin.*

In fact, there can not be too much exceptional directions of non-uniqueness. Let us explain the intuition. Take two infinite geodesics going out of 0 with the same asymptotic direction. If theses two geodesics are not the same, then they eventually separates. It means that at some point, one takes an e_1-step while the other takes an e_2-step. From this observation we deduce that if the weight distribution is continuous and there is a direction of non-uniqueness, then it is necessarily the asymptotic direction of the competition interface of some vertex, i.e. the infinite dual path that separates the sub-tree of geodesics going out of this vertex by taking an e_1-step from the ones that take an e_2-step. Summing up, if there is a direction of non-uniqueness and the distribution is continuous, then there is a vertex whose competition interface has this asymptotic direction. A simulation of the competition interface at 0 for exponential weights is presented in Figure 1. Since the set of vertex is countable, then the set of exceptional direction is at most countable. Theorem 8.1 gives that in fact each competition interface gives almost surely an exceptional direction of non-uniqueness. Note that knowing when the set of exceptional direction is dense requires more work.

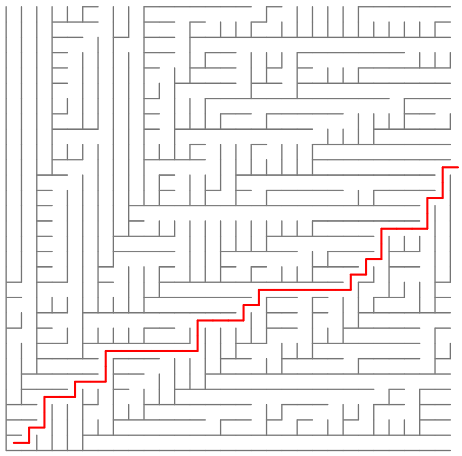


Figure 4: In red, the competition interface at 0 for exponentially distributed weights.

Now let us state a uniqueness result. The proof can be found in [GRAS17]. It gives some condition on Λ and $\xi \in \mathcal{U}$ to guarantee that there is almost surely only one $\mathcal{U}_\xi$-directed infinite geodesic.

Theorem 8.2. *Fix $\xi \in \mathcal{U}$ such that Λ is differentiable at $\bar{\xi}$ and $\underline{\xi}$, and assume that ω_0 has a continuous distribution. Then almost surely the is only one $\mathcal{U}_\xi$-directed infinite geodesic.*

In the exponential weight case, since Λ is differentiable everywhere and strictly concave, and Exp(1) has no atom, we deduce that for every fixed $\xi \in \mathcal{U}$, there is almost surely only one infinite geodesic out of 0 with asymptotic direction ξ.

Conclusion

Busemann geodesics is a very useful tool to study infinite geodesics. It allow to generalize what has been shown first in the exponential case with less robust techniques. Note that except for the fact that exceptional directions are dense and that there can not be more than two infinite geodesics with the same asymptotic direction, we saw in this work that it allow to recover the behavior of infinite geodesics in the exponential case while generalizing it.

The main problem that remains is that we still do not know how to show strict concavity or differentiability of the limit shape for general distribution. It is believed that strict concavity should hold under some general assumptions but the question is still open.

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