Liouville conformal field theory

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Abstract

This article is an introduction to the recent developments on Liouville conformal field theory due to Colin Guillarmou, Antti Kupiainen, Rémi Rhodes, and Vincent Vargas.

Contents

1 Segal's axioms for CFT

For our purposes, a conformal field theory (CFT) is a "projective" functor from the cobordism category of marked Riemann surfaces to the category of Hilbert spaces. More precisely, consider the following category:

- An object of the category is a disjoint union of circles. It can be empty.

- A morphism from an object C_1 to an object C_2 is a tuple $(\Sigma, \mathbf{x}, \boldsymbol{\alpha}, \boldsymbol{\zeta})$ (up to biholomorphism in the obvious sense) where:
  - Σ is a compact Riemann surface with boundary $\partial\Sigma = C_1 \sqcup C_2$.
  - $\mathbf{x} = (x_1, \dots, x_m)$[1] are distinct marked points on $\Sigma \setminus \partial\Sigma$ with real weights $\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_m)$. Here m can be 0.
  - $\boldsymbol{\zeta} = (\zeta_1, \dots, \zeta_b)$ are real-analytic parametrizations of the circles in $\partial\Sigma$ from the unit circle $\mathbb{T}$.
  - We endow C_1, C_2 with the orientation induced by $\boldsymbol{\zeta}$ and require that the induced boundary orientation is $\partial\Sigma = \overline{C}_1 \sqcup C_2$ where the bar denotes reversal of orientation. Thus we call C_1 (resp. C_2) the *incoming* (resp. *outgoing*) boundary.
- Composition of morphisms is by gluing along the common boundary in the obvious way.

Fix a complex Hilbert space $\mathcal{H}$. A (2D) CFT is a functor-like object that does the following on the category above:

- To an object $\bigsqcup_{i=1}^{b} \mathbb{T}$, it associates the Hilbert space $\mathcal{H}^{\otimes b}$.
- To a morphism $(\Sigma, \mathbf{x}, \boldsymbol{\alpha}, \boldsymbol{\zeta})$ from $\bigsqcup_{i=1}^{b^-} \mathbb{T}$ to $\bigsqcup_{i=1}^{b^+} \mathbb{T}$ and a conformal Riemannian metric g on Σ, it associates a Hilbert–Schmidt operator $\mathcal{A}_{\Sigma,g,\mathbf{x},\boldsymbol{\alpha},\boldsymbol{\zeta}} : \mathcal{H}^{\otimes b^-} \to \mathcal{H}^{\otimes b^+}$.

We impose the following axioms:

- (Conformal/Weyl covariance) For $\omega \in C^\infty(\Sigma)$, $\mathcal{A}_{\Sigma,e^\omega g,\mathbf{x},\boldsymbol{\alpha},\boldsymbol{\zeta}}$ is a scalar multiple of $\mathcal{A}_{\Sigma,g,\mathbf{x},\boldsymbol{\alpha},\boldsymbol{\zeta}}$.
- (Gluing) For $(\Sigma, \mathbf{x}, \boldsymbol{\alpha}, \boldsymbol{\zeta}) = (\Sigma^1, \mathbf{x}^1, \boldsymbol{\alpha}^1, \boldsymbol{\zeta}^1) \circ (\Sigma^2, \mathbf{x}^2, \boldsymbol{\alpha}^2, \boldsymbol{\zeta}^2)$ and g a conformal Riemannian metric on Σ, $\mathcal{A}_{\Sigma,g,\mathbf{x},\boldsymbol{\alpha},\boldsymbol{\zeta}}$ is a scalar multiple of $\mathcal{A}_{\Sigma^1,g|_{\Sigma^1},\mathbf{x}^1,\boldsymbol{\alpha}^1,\boldsymbol{\zeta}^1} \circ \mathcal{A}_{\Sigma^2,g|_{\Sigma^2},\mathbf{x}^2,\boldsymbol{\alpha}^2,\boldsymbol{\zeta}^2}$.

This functorial formulation of CFT is due to G. Segal [Seg88].

[1] We write them as a tuple for notational convenience. Of course, the order here of the marked points does not matter. The same applies to the boundary parametrizations.

In physics, the operator $\mathcal{A}_{\Sigma,g,\mathbf{x},\boldsymbol{\alpha},\boldsymbol{\zeta}} : \mathcal{H}^{\otimes b^-} \to \mathcal{H}^{\otimes b^+}$, often called *amplitude*, is given by a Feynman path integral of the form

$$\mathcal{A}_{\Sigma,g,\mathbf{x},\boldsymbol{\alpha},\boldsymbol{\zeta}}(\widetilde{\boldsymbol{\varphi}}) = \int_{\phi|_{\partial\Sigma}=\widetilde{\boldsymbol{\varphi}}} \prod_{i=1}^{m} V_{\alpha_i}(x_i)\, e^{-S_\Sigma(g,\phi)}\, \mathrm{D}\phi,$$

where:

- $\widetilde{\boldsymbol{\varphi}} \in \mathcal{H}^{\otimes(b^-+b^+)}$, viewing $\mathcal{A}_{\Sigma,g,\mathbf{x},\boldsymbol{\alpha},\boldsymbol{\zeta}}$ as a linear functional on $\mathcal{H}^{\otimes(b^-+b^+)}$.
- $V_\alpha(x) = e^{\alpha\phi(x)}$ is the *vertex operator*.
- $S_\Sigma(g,\phi)$ is the *action*, a nonlinear functional of ϕ, the *field*.
- The integration is over the space of fields, typically a Hilbert space of sections of a fiber bundle over Σ.

The mathematical difficulty of such integrals lies in the definition of $\mathrm{D}\phi$, for want of a good theory of measures on general infinite-dimensional spaces.

2 LCFT via probability

In Liouville conformal field theory (LCFT), fields are just functions on Σ and the action is the Liouville functional defined by

$$S_\Sigma(g,\phi) = \frac{1}{4\pi}\int_\Sigma \left(|d\phi|_g^2 + QK_g\phi + 4\pi\mu e^{\gamma\phi}\right) \mathrm{dv}_g,$$

where:

- K_g is the curvature and dv_g is the volume form of g.
- $\mu > 0$, $0 < \gamma < 2$ are the parameters of the theory, $Q = \frac{\gamma}{2} + \frac{2}{\gamma}$.

Thus the goal is to give a rigorous definition of the integral

$$\int_{\phi|_{\partial\Sigma}=\widetilde{\boldsymbol{\varphi}}} \prod_{i=1}^{m} e^{\alpha_i\phi(x_i)}\, e^{-\frac{1}{4\pi}\int_\Sigma(|d\phi|_g^2+QK_g\phi+4\pi\mu e^{\gamma\phi})\mathrm{dv}_g}\, \mathrm{D}\phi.$$

To this end, we shall invoke two tools from probability theory: Gaußian free field (GFF) and Gaußian multiplicative chaos (GMC).

2.1 Gaußian free field (GFF)

The first step is to make sense of an integral of the form

$$\int F(\phi)\, e^{-\frac{1}{4\pi}\int_\Sigma |d\phi|_g^2 \mathrm{dv}_g}\, \mathrm{D}\phi.$$

Fix $s < 0$. We take the space of fields to be the Sobolev space $H^s(\Sigma)$. Write

$$H_0^s(\Sigma) = \begin{cases} \{\phi \in H^s(\Omega) : \int \phi\, \mathrm{dv}_g = 0\}, & \partial\Sigma = \varnothing, \\ \{\phi \in H^s(\Omega) : \phi|_{\partial\Sigma} = 0\}, & \partial\Sigma \neq \varnothing. \end{cases}$$

Let $(e_j)_j$ be an orthonormal basis of $H_0^s(\Sigma)$ consisting of eigenfunctions of the Laplacian Δ_g on (Σ, g) with corresponding eigenvalues $(\lambda_j)_j$. Writing $\phi = \sum_j a_j e_j$, we have $\int_\Sigma |d\phi|_g^2 \mathrm{dv}_g = \sum_j \lambda_j a_j^2$. Now, if F depends only on finitely many a_j's, then by a simple change of variables,

$$\int F\big((a_j)_j\big)\, e^{-\frac{1}{4\pi}\sum \lambda_j a_j^2} \prod \mathrm{d}a_j = \left(\prod \frac{2\pi}{\sqrt{\lambda_j}}\right) \int F\left(\left(\sqrt{\frac{2\pi}{\lambda_j}}\, a_j\right)_j\right) \prod \frac{e^{-\frac{1}{2}a_j^2}\, \mathrm{d}a_j}{\sqrt{2\pi}}.$$

This inspires us to define[2]

$$\int_{H_0^s(\Sigma)} F(\phi)\, e^{-\frac{1}{4\pi}\int_\Sigma |d\phi|_g^2 \mathrm{dv}_g}\, \mathrm{D}\phi = \sqrt{\frac{\mathrm{vol}_g \Sigma}{\det \Delta_g}}\, \mathbb{E}[F(X_g)],$$

where $\det \Delta_g$ is the regularized determinant of Δ_g (see [GKRV21] for the definition) and $X_g : \Omega \to \mathcal{D}'(\Sigma)$ is the random field on Σ defined by

$$X_g = \sqrt{2\pi} \sum_j a_j \frac{e_j}{\sqrt{\lambda_j}}, \qquad a_j \sim \mathcal{N}(0,1) \text{ i.i.d.},$$

which is a centered Gaußian random field with covariance $\mathbb{E}[X_g(x) X_g(y)] = 2\pi\, G_g(x,y)$, where G_g is the Green's function for Δ_g on $H_0^s(\Sigma)$. By Weyl's law, $\lambda_j = \mathrm{O}(j)$, so $X_g \in L^2(\Omega, H_0^s(\Sigma))$. This is the GFF of (Σ, g). One can show that the distribution of X_g depends only on the conformal class of g.

To pass to $H^s(\Sigma)$, define $\phi_g : \Omega \to H^s(\Sigma)$ by

$$\phi_g = \begin{cases} X_g + c, & \partial\Sigma = \varnothing, \\ X_g + P\widetilde{\boldsymbol{\varphi}}, & \partial\Sigma \neq \varnothing, \end{cases}$$

[2]The volume term is to ensure conformal invariance.

where c is a constant, $\widetilde{\boldsymbol{\varphi}} \in H^s(\partial\Sigma)$, $P : H^s(\partial\Sigma) \to H^{s+1/2}(\Sigma)$ is the Poisson operator for Δ_g. By repeating this construction on $H^s(\partial\Sigma) = H^s(\mathbb{T})^b$ and taking the Lebesgue measure on $\mathbb{R}$ for the constant parts, we can view ϕ_g as a random field inducing a product measure on

$$H^s(\Sigma) = \begin{cases} H_0^s(\Sigma) \oplus \mathbb{R}, & \partial\Sigma = \varnothing, \\ H_0^s(\Sigma) \oplus H_0^s(\mathbb{T})^b \oplus \mathbb{R}^b, & \partial\Sigma \neq \varnothing. \end{cases}$$

2.2 Gaußian multiplicative chaos (GMC)

Now that we have a rigorous definition of $\mathrm{D}\phi$, we still need to make sense of the nonlinear terms $e^{\alpha\phi}$ and $\int_\Sigma e^{\gamma\phi}\mathrm{dv}_g$. The problem is that $e^{\gamma\phi}$ does not make sense pointwise since ϕ_g takes values in H^s.

Define the g-regularization $\phi_{g,\varepsilon}$ of ϕ_g to be ϕ_g averaged on g-geodesic circles of radius $\varepsilon > 0$.[3] Then the vertex operator $V_\alpha(x)$ is approximated by

$$V_{\alpha,g,\varepsilon}(x) = \varepsilon^{\frac{1}{2}\alpha^2} e^{\alpha\phi_{g,\varepsilon}(x)}.$$

For the integral, consider the random measure on Σ defined by

$$M_\gamma^g(\phi_g, \mathrm{d}x) = \lim_{\varepsilon\to 0} \varepsilon^{\frac{1}{2}\gamma^2} e^{\gamma\phi_{g,\varepsilon}} \mathrm{dv}_g.$$

One can show that this converges in probability and weakly in the space of Radon measures to a nonzero random measure for $0 < \gamma < 2$. This is the GMC of (Σ, g). It is conformally invariant in the sense that for $\omega \in C^\infty(\Sigma)$,

$$M_\gamma^{e^\omega g}(X_{e^\omega g}, \mathrm{d}x) = e^{\frac{\gamma}{2}Q\omega(x)} M_\gamma^g(X_g, \mathrm{d}x).$$

For more on GFF and GMC, see [BP21].

2.3 Definition of LCFT amplitude

We are finally in a position to define the amplitudes of LCFT. Take $\mathcal{H} = H^s(\mathbb{T})$. Let F be a function of fields. If $\partial\Sigma = \varnothing$, define $\mathcal{A}_{\Sigma,g,\mathbf{x},\boldsymbol{\alpha},\varnothing}(F)$ to be

$$\sqrt{\frac{\mathrm{vol}_g \Sigma}{\det \Delta_g}} \lim_{\varepsilon\to 0} \int_{\mathbb{R}} \mathbb{E}\left[F(\phi_g) \prod_{i=1}^m V_{\alpha_i,g,\varepsilon}(x_i)\, e^{-\frac{1}{4\pi}\int_\Sigma \left(QK_g\phi_g \mathrm{dv}_g + 4\pi\mu M_\gamma^g(\phi_g,\mathrm{d}x)\right)} \right] \mathrm{d}c.$$

[3]If $\partial\Sigma \neq \varnothing$, the definition is modified near the boundary, see [GKRV21] for details.

If $\partial\Sigma \neq \varnothing$, assume g is *admissible* in the sense that $g = |dz|^2/|z|^2$ near $\partial\Sigma$ in annular charts extending $\boldsymbol{\zeta}$, then define $\mathcal{A}_{\Sigma,g,\mathbf{x},\boldsymbol{\alpha},\boldsymbol{\zeta}}(F,\widetilde{\boldsymbol{\varphi}})$ to be

$$\frac{e^{-\frac{1}{2}(\widetilde{\boldsymbol{\varphi}},(\mathbf{D}_\Sigma-\mathbf{D})\widetilde{\boldsymbol{\varphi}})_{L^2}}}{\sqrt{\det\Delta_g}}\lim_{\varepsilon\to 0}\mathbb{E}\left[F(\phi_g)\prod_{i=1}^m V_{\alpha_i,g,\varepsilon}(x_i)\,e^{-\frac{1}{4\pi}\int_\Sigma\left(QK_g\phi_g\mathrm{dv}_g+4\pi\mu M_\gamma^g(\phi_g,\mathrm{d}x)\right)}\right],$$

where $\mathbf{D}$ is the differential operator and $\mathbf{D}_\Sigma$ is the Dirichlet-to-Neumann operator on $H^s(\partial\Sigma)$. Here the expectations are over X_g. The LCFT amplitudes are the case $F \equiv 1$. One can show that $\mathcal{A}_{\Sigma,g,\mathbf{x},\boldsymbol{\alpha},\boldsymbol{\zeta}} \in L^2(H^s(\mathbb{T})^b)$ if the following *Seiberg bounds* are satisfied:

$$\begin{cases}\forall\, i,\ \alpha_i < Q,\\ \sum\alpha_i > Q\chi(\Sigma).\end{cases}$$

For $\omega \in C^\infty(\Sigma)$, we have

$$\mathcal{A}_{\Sigma,e^\omega g,\mathbf{x},\boldsymbol{\alpha},\boldsymbol{\zeta}}(F,\widetilde{\boldsymbol{\varphi}}) = \mathcal{A}_{\Sigma,g,\mathbf{x},\boldsymbol{\alpha},\boldsymbol{\zeta}}\big(F\big(\cdot - \tfrac{Q}{2}\omega_\partial\big),\widetilde{\boldsymbol{\varphi}}+\tfrac{Q}{2}\omega_\partial\big)\times \\ e^{-\frac{Q}{2}(\mathrm{D}\omega_\partial,\widetilde{\boldsymbol{\varphi}})_{L^2}-\frac{Q^2}{8}(\mathrm{D}\omega_\partial,\omega_\partial)_{L^2}}\,e^{c_\mathrm{L}S_\mathrm{L}^0(\Sigma,g,\omega)-\sum\Delta_{\alpha_i}\omega(x_i)}$$

where:

- $\omega_\partial = \boldsymbol{\zeta}^*\omega \in C^\infty(\mathbb{T})^b$.
- $c_\mathrm{L} = 1+6Q^2$ is the *central charge* of the theory.
- $S_\mathrm{L}^0(\Sigma,g,\omega) = \dfrac{1}{96\pi}\displaystyle\int_\Sigma\left(|d\omega|_g^2+2K_g\omega\right)\mathrm{dv}_g$.
- $\Delta_\alpha = \frac{\alpha}{2}\big(Q-\frac{\alpha}{2}\big)$.

This verifies the axiom on conformal covariance and allows us to extend the definition to nonadmissible metrics.[4] As for the gluing axiom, the scalar for gluing along one circle is

$$\begin{cases}\sqrt{2}, & \partial\Sigma=\varnothing,\\ 1/\sqrt{2}\pi, & \partial\Sigma\neq\varnothing.\end{cases}$$

We also allow self-gluing, i.e., we can glue an incoming circle with an outgoing circle on the same surface, in which case we take the trace of the amplitude instead of a composition. With this, the definition of LCFT is complete.

The remaining sections discuss the structure of LCFT. Of particular interest are the amplitudes of closed surfaces, which are scalars since $\mathcal{H}^{\otimes 0} = \mathbb{C}$.

[4] In fact, one obtains the formula of the amplitude for general metrics by adding terms involving the geodesic curvature of $\partial\Sigma$, see [GKRV21] for details.

3 Sphere: DOZZ formula

On the Riemann sphere $\widehat{\mathbb{C}}$, the Seiberg bounds imply $m \geq 3$. In the case $m = 3$, the amplitude is determined up to a constant by Möbius invariance. This constant is called the *structure constant* of the theory. For LCFT, it is given by the following formula ([KRV20]):

Theorem (Dorn–Otto–Zamolodchikov–Zamolodchikov formula)**.**

$$\mathcal{A}_{\widehat{\mathbb{C}},e^{\omega}|dz|^2,(x_1,x_2,x_3),(\alpha_1,\alpha_2,\alpha_3)} = \prod_{i=1}^{3} |x_i - x_{i+1}|^{2(\Delta_{\alpha_{i+2}}-\Delta_{\alpha_i}-\Delta_{\alpha_{i+1}})} \times$$
$$\frac{1}{2}\, C^{\mathrm{DOZZ}}_{\gamma,\mu}(\alpha_1,\alpha_2,\alpha_3) \sqrt{\frac{\mathrm{vol}_g \widehat{\mathbb{C}}}{\det \Delta_g}}\, e^{-6Q^2 S^0_{\mathrm{L}}(\widehat{\mathbb{C}},e^{\omega}|dz|^2,-(\omega+4\log|\cdot|_+))-\sum \Delta_{\alpha_i}\omega(x_i)},$$

where the indices in the first term are modulo 3*,* $C^{\mathrm{DOZZ}}_{\gamma,\mu}(\alpha_1,\alpha_2,\alpha_3)$ *is an explicit constant,* $|\cdot|_+ = \max(1,|\cdot|)$*.*

The amplitude for $m \geq 4$ is computed by the conformal bootstrap method, to be discussed in Section 5 below.

4 Annuli: Spectral theory

Deformation of a boundary circle can be realized by gluing an annulus along it. Therefore, we turn next to the amplitude of annuli.

Annuli form a semigroup under gluing and amplitude is a projective representation of this semigroup. This is studied in detail in [BGKR24]. Here we only need a two-dimensional subsemigroup thereof to derive the spectrum of LCFT: For $q = e^{-t+i\theta}$ where $t > 0$, $\theta \in \mathbb{R}$, consider the annulus

$$\mathbb{A}_q = \{z \in \mathbb{C} : |q| \leq |z| \leq 1\}, \qquad \boldsymbol{\zeta}_q = (\mathrm{id}, \overline{q\,\mathrm{id}}), \qquad g_{\mathbb{A}} = |dz|^2/|z|^2.$$

Then $q \mapsto (\mathbb{A}_q, \varnothing, \varnothing, \boldsymbol{\zeta}_q)$ is homomorphic and

$$S(q) = \frac{1}{\sqrt{2\pi}}\, |q|^{\frac{c_{\mathrm{L}}}{12}} \mathcal{A}_{\mathbb{A}_q, g_{\mathbb{A}}, \varnothing, \varnothing, \boldsymbol{\zeta}_q}$$

defines a representation of the punctured open unit disk $\mathbb{D}^\circ \setminus \{0\}$ on $H^s(\mathbb{T})$. One can show that it is generated by two commuting self-adjoint operators $\mathbf{H}, \mathbf{\Pi}$ in the sense that

$$S(q) = e^{-t\mathbf{H}} e^{i\theta\mathbf{\Pi}}.$$

Here $\mathbf{H}$ is the *Hamiltonian* of the theory. It has the form

$$\mathbf{H} = -\frac{1}{2}\partial_c^2 + \mathbf{P} + \frac{1}{2}Q^2 + \mu e^{\gamma c} V$$

where $\mathbf{P}$ is a self-adjoint operator on $H_0^s(\mathbb{T})$ with spectrum $\mathbb{N}$ and V is an unbounded non-negative operator on $H_0^s(\mathbb{T})$.[5] The spectral theory of $\mathbf{H}, \mathbf{\Pi}$ is worked out in [GKRV20] using methods from microlocal analysis. Roughly speaking, the result is the following:

Theorem (Spectral resolution of LCFT)**.** $\mathbf{H}, \mathbf{\Pi}$ *have a complete set of generalized eigenfunctions* $\Psi_{P,\nu,\tilde{\nu}}$ *(*$P \in \mathbb{R}$*,* $\nu, \tilde{\nu}$ *Young diagrams) with*

$$S(q)\Psi_{P,\nu,\tilde{\nu}} = q^{\Delta_{Q+iP}+|\nu|}\bar{q}^{\Delta_{Q+iP}+|\tilde{\nu}|}\Psi_{P,\nu,\tilde{\nu}}.$$

It has the following algebraic structure. There are two commuting Virasoro algebras $(\mathbf{L}_n)_n, (\widetilde{\mathbf{L}}_n)_n$ of central charge c_{L} acting on $H^s(\mathbb{T})$ such that:

- $H^s(\mathbb{T})$ is a Verma module with highest weight states $\Psi_{P,0,0}$ ($P \in \mathbb{R}$) with conformal weight Δ_{Q+iP}.
- $\mathbf{H} = \mathbf{L}_0 + \widetilde{\mathbf{L}}_0$.
- $\Psi_{P,\nu,\tilde{\nu}} = \mathbf{L}_{-\nu(k)} \cdots \mathbf{L}_{-\nu(1)} \widetilde{\mathbf{L}}_{-\tilde{\nu}(\ell)} \cdots \widetilde{\mathbf{L}}_{-\tilde{\nu}(1)} \Psi_{P,0,0}$ for Young diagrams $\nu = (\nu(1), \ldots, \nu(k))$, $\tilde{\nu} = (\tilde{\nu}(1), \ldots, \tilde{\nu}(\ell))$.

In particular, we see that LCFT has continuous spectrum.

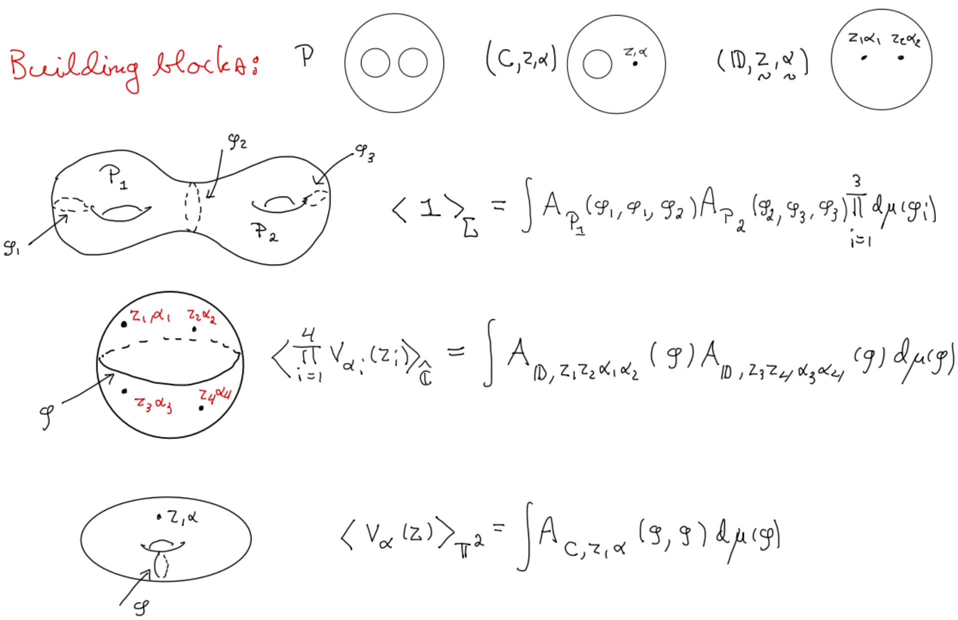
Finally, we remark that geometrically, $\Psi_{P,0,0}$ is essentially the amplitude $\mathcal{A}_{\mathbb{D},0,Q+iP,\mathrm{id}}$ of the unit disk with one marked point 0.

5 Conformal bootstrap

A *building block* is a Riemann surface obtained by removing b ($b = 1, 2, 3$) disjoint disks from the Riemann sphere with $3-b$ marked points. Concretely, there are the following types:

- $b = 1$: A disk with two marked points.
- $b = 2$: An annulus with one marked point.

[5] To be rigorous, $\mathbf{H}$ makes sense only in the weak sense due to the $e^{\gamma c} V$ term.



(Taken from a talk by A. Kupiainen)

- $b = 3$: A pair of pants with no marked point.

The method of conformal bootstrap is to obtain the amplitude of a general surface by gluing building blocks. The details are too complicated to go into here, but we shall sketch the main ideas.

First, we define the *conformal block* of a building block to be essentially the holomorphic part of the amplitude in the spectral resolution (i.e., its values on $\Psi_{P,\nu,0}$'s). By Ward's identities, the conformal block determines the whole amplitude. We then define the conformal block of a general surface by gluing conformal blocks of building blocks. The result depends on the chosen decomposition into building blocks, and one can show that it varies nicely under deformation of the cutting curves. More precisely, the conformal block can be realized as a holomorphic section of a holomorphic line bundle over the Teichmüller space of the surface.

6 Research directions

One direction of research is to study the full dependence of the conformal block on the decomposition into building blocks. More precisely, this should yield a representation of the mapping class group.

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