Introduction au domaine de recherche: Determination of functions and weak KAM theory

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Introduction

In [BCD18], Boulmezaoud and collaborators established that two convex functions $u, v \in \mathcal{C}^2(\mathcal{H})$, defined on a Hilbert space $\mathcal{H}$, and bounded from below, sharing the same modulus of the gradient, are equal up to an additive constant. That is, the equation

$$|\nabla u| = |\nabla v|$$

admits unique solutions up to an additive constant within the appropriate class of functions.

This result has been significantly generalized. First, the smoothness assumption can be relaxed in [BCD18], replacing the modulus of the gradient by the local slope. The ambient space was generalized to Banach spaces in [TZ23]. Furthermore, the local slope can be replaced by any descent operators (see definition 2.1), for general coercive functions as it was shown in [DLS23]. Finally, under suitable asymptotic conditions, the result also holds for unbounded convex functions on general Hilbert spaces [DSTG25].

We refer to these developments as the *determination theory*. It can be interpreted as studying uniqueness questions for the Eikonal equation, or more generally, as a particular case of Hamilton-Jacobi equations with mechanical Hamiltonian

$$H(x, p) = \frac{|p|^2}{2} - \frac{|\nabla u(x)|^2}{2}.$$

In the 90s, Albert Fathi introduced the weak KAM theory as a variational approach for studying such equations. While it was first stated on the torus, it has been generalized to noncompact manifolds [FM07] and to discrete settings in [BB08], [Zav21]. In the non-compact case, uniqueness results for weak KAM solutions were established by Contreras in [Con01]. In the following, we present the weak KAM theory, motivating its relevance to non-compact determination theory.

Contents

1 Weak KAM Theory on Riemannian Manifolds

In this section, we introduce some fundamental notions of weak KAM theory in the classical setting. While several approaches exist, this presentation is directly inspired from the text of Rifford [Rif12]. Our first objective is to give a meaning to a solution to the Halmiton-Jacobi equation

$$H(x, d_x u) = c$$

in the weak KAM theory. Here, H denotes an Hamiltonian satisfying some conditions that provide well-posed Lagrangian flows. Then, we will introduce the Aubry set $\mathcal{A}$ and highlight its key role in determination.

1.1 Weak KAM solution

In the following, we let (M, g) be a smooth connected Riemannian manifold without boundary of dimension $n \geq 2$, and $H : T^*M \to \mathbb{R}$ be a $\mathcal{C}^k$ Tonelli Hamiltonian (with $k \geq 2$). For the precise assumptions, we make on H, one can refer to the introduction of [Con01]. We define the Lagrangian $L : TM \to \mathbb{R}$ associated with H via the Legendre-Fenchel transform:

$$L(x, v) = \sup_{p \in T_x^*M} \left\{ \langle p, v \rangle - H(x, p) \right\}.$$

Since H is a $\mathcal{C}^k$ Tonelli Hamiltonian, this supremum is attained, and the resulting Lagrangian L is a $\mathcal{C}^k$ Tonelli Lagrangian.

A function $u : M \to \mathbb{R}$ is called a weak KAM solution (see below definition 1.5) whenever it satisfies two properties: a domination condition and the existence of a so-called semi-calibrated curve. To make this definition precise, we introduce the key quantities involved. We begin with a classical object from variational analysis: the minimal action time. It characterizes the minimal cost that we have to pay by going from x to y in time t.

Definition 1.1 (Minimal action time)
For $t > 0$ fixed, $k \in \mathbb{R}$, we define the minimal action to go from x to y in time t, $h_{t,k} : M \times M \to \mathbb{R}$ by

$$h_{t,k}(x, y) \stackrel{\text{def}}{=} \inf \left\{ \int_0^t [L(\gamma(s), \dot{\gamma}(s)) + k] \, ds \text{ such that } \gamma : [0, t] \to M \text{ is piecewise } \mathcal{C}^1 \text{ with } \gamma(0) = x \text{ and } \gamma(t) = y \right\}.$$

We can replace γ piecewise $\mathcal{C}^1$ by γ absolutely continuous or even piecewise $\mathcal{C}^\infty$. Thanks to the assumptions of the preliminaries on the Hamiltonian, the infimum is always achieved by an extremal curve. Moreover, this curve satisfies the Euler-Lagrange equation:

$$\frac{\partial}{\partial x} L(\gamma(s), \dot{\gamma}(s)) = \frac{d}{ds} \frac{\partial}{\partial v} L(\gamma(s), \dot{\gamma}(s)).$$

The domination condition takes the following form: any function u is called dominated by $L + k$ if

$$u(x) - u(y) \leq h_{t,k}(x, y), \quad \forall t \in \mathbb{R}_+, \quad \forall (x, y) \in M^2.$$

We rewrite this condition trough the introduction of the action potential:

Definition 1.2 (Action potential)
Let $(x, y) \in M^2$, $k \in \mathbb{R}$ we define the action potential

$$\phi_k(x, y) = \inf_{t \in \mathbb{R}_+} h_{t,k}(x, y) = \inf_{\gamma \in \mathcal{AC}(x,y)} \int_0^t [L(\gamma(s), \dot{\gamma}(s)) + k] \, ds,$$

where $\mathcal{AC}(x, y)$ denotes the set of absolutely continuous curves going from x to y.

Hence, a function $u : M \to \mathbb{R}$ is said dominated by $L + k$ if we have

$$u(x) - u(y) \leq \phi_k(y, x), \quad \forall (x, y) \in M^2.$$

The role played by the constant k is very crucial. If k is close enough to $-\infty$, there are no dominated function. Moreover, if u is dominated by $L+k$, then u is dominated by $L+k'$ for all constant k' bigger than k. Hence, under our assumptions on the Lagrangian, there exists a critical value c, named Mañé critical value, for which

$$k \geq c \Longleftrightarrow [\{u : M \to \mathbb{R} \quad | \quad u \preceq L + k\} \neq \emptyset] \,.$$

The choice of notation is not hazardous. The critical value c is precisely the one of the Hamilton-Jacobi equation $H(x, d_x u) = c$. This motivates the definition of sub-solutions:

Definition 1.3
We call sub-solution any function dominated by $L + c$.

There can be many sub-solutions. For the mechanical Hamiltonian, from the introduction, we will see below that $c = \inf \frac{|\nabla u(x)|^2}{2}$ so that $L(x, v) + c \geq 0$ for any $(x, v) \in T_x M$. Hence, the constant function 0 is sub-solution of the mechanical Hamiltonian. The second condition in the definition of weak KAM solutions will choose some particular sub-solutions. We can not expect the inequality defining the sub-solution to be an equality everywhere. However, we will ask it to be equal among some precised curves.

Definition 1.4 (Semi-static curve)
An absolutely continuous curve $\gamma : [0, t] \to M \in \mathcal{C}(x, y)$ is called semi-static if

$$\int_0^t L(\gamma(s), \dot{\gamma}(s))ds + ct = \phi_c(x, y).$$

A semi-static curve is solution to the Euler-Lagrange equation.

We can now define the notion of weak KAM solution.

Definition 1.5 (Weak KAM solution)
A function $u : M \to \mathbb{R}$ is called a weak KAM solution (of negative type) to the Hamilton-Jacobi equation $H(x, d_x u) = c$ if

- *u is a weak KAM subsolution*
- *for every $x \in M$, there exists $\gamma :]-\infty, 0] \to M$ a semi-static curve such that $\gamma(0) = x$ and*

$$u(x) - u(\gamma(t)) = \int_t^0 L(\gamma, \dot{\gamma}) + c, \quad \forall t \leq 0.$$

It looks relevant to clarify that the notion of weak KAM solution is consistent with the notion of viscosity solution. One of the main results in weak KAM theory is the weak KAM theorem. It is proved, in this setting (non-compact manifold), by Albert Fathi in [FM07]. It states the existence of weak KAM solutions:

Theorem 1.6 (Weak KAM theorem)
There exists a weak KAM solution to the Hamilton-Jacobi equation $H(x, d_x u) = c$.

1.2 The Mather and Aubry sets

Let us address the issue of uniqueness. The idea we follow here is rather intuitive. Suppose first time that M is compact and let u be a weak KAM solution. For any $x \in M$, there exists a semi-static curve $\gamma_x :]-\infty, 0] \to M$ such that $\gamma_x(0) = x$ and

$$u(\gamma_x(0)) - u(\gamma_x(t)) = \int_t^0 L(\gamma_x, \dot{\gamma}_x) + c.$$

Therefore, if we know the value of u at a point along γ_x, we can compute its value at x. The earlier on this curve this known point lies, the more information we obtain about u. This motivates considering the omega-limit set of γ_x. If we know the values of u on the union of these omega-limits for all $x \in M$, we can recover u everywhere.

Definition 1.7 (Mather set)
The Mather set can be defined as

$$\mathcal{M} = \bigcup_{\gamma \text{ semi-static curve}} \omega - \lim(\gamma).$$

It is non-empty whenever the manifold M is compact.

In the compact case, the Mather set is a uniqueness set, that is:

Proposition 1.8
If M is compact and u and v are two weak KAM solutions, then

$$u = v \text{ in } \mathcal{M} \Rightarrow u = v \text{ in } M.$$

If the space M is non compact, it can happens that the Mather set is empty. However, this does not mean we can not classify the curves that come from $-\infty$. The work of Contreras in [Con01] is to use a Gromov like-compactification to address this issue. Nevertheless, the set that we will really consider in the following is the so-called Aubry set:

Definition 1.9 (Aubry set)
Let $h \in \mathcal{C}^0(X \times X, \mathbb{R}_+)$ be the Peierls barrier defined by

$$h(x,y) = \liminf_{t \to +\infty} \quad h_{t,c}(x,y).$$

The Aubry set is defined as

$$\mathcal{A} = \{x \in M \quad | \quad h(x,x) = 0\}.$$

If M is compact, then the Peierls barrier h is finite, but it can occur to not be finite in the non-compact setting. It can also happen that h is finite everywhere whereas the Aubry set is empty. Finally, we have $0 = \phi_c(x,x) \leq h(x,x), \quad \forall x \in M$ and $\phi_c(x,y) \leq h(x,y), \quad \forall (x,y) \in M^2$. The Aubry set is really important in the theory since it has many good characterizations in terms of regularity of the weak KAM solutions, existence of solutions and so on. One can see the Peierls barrier as the cost we have to pay from traveling from x to y, but we have to travel forever. Hence, the Aubry set can be seen as a set where we can stay an infinite amount of time (potentially moving inside this set) for a negligible cost. In the compact case, we have a decomposition of the Peierls barrier in this sense:

Proposition 1.10

$$h(x,y) = \inf_{p \in \mathcal{A}} \phi_c(x,p) + \phi_c(p,y).$$

We have the following inclusion that is always true: $\mathcal{M} \subset \mathcal{A}$. Hence, $\mathcal{A}$ is also a uniqueness set of the Hamilton-Jacobi equation. Although $\mathcal{A}$ is a larger set, this does not make the uniqueness result weaker. Let us explain why it is not the case: build a semi-metric d_c on M, by

$$d_c \stackrel{\text{def}}{=} \phi_c(x,y) + \phi_c(y,x).$$

The space $(M/_{(d_c=0)}, d_c)$ is a metric space. Moreover, if $d_c(x,y) = 0$, it means that going from x to y and coming back does not cost anything. With the interpretation of the Aubry set given above, it means that both x and y are in the Aubry set. It follows that all elements of M that are not in the Aubry set have singleton as equivalence classes for the relation $d_c(x,y) = 0$. We can rewrite $M/_{(d_c=0)} \cong \mathcal{A}/_{(d_c=0)} \cup (M \setminus \mathcal{A})$ and any family of representatives of $\mathcal{A}/_{(d_c=0)}$ is a set of uniqueness in the compact setting. The equivalence classes of this relation are called static classes.

Theorem 1.11 (Contreras [Con01])
Suppose that M is compact. Then the map

$$\begin{aligned} \{u : \mathcal{A} \to \mathbb{R} \mid u \prec L + c\} &\longrightarrow \{\textit{weak KAM solutions}\} \\ u &\longmapsto \inf_{p \in \mathcal{A}} (u(p) + \phi_c(p,x)) \end{aligned}$$

is a bijection.

This correspondence can be seen as a form of Perron's method, highlighting the connection between weak KAM solutions and viscosity solutions. Under this form, this theorem can not be extended in the non-compact setting: the Aubry set can be empty, and even if it is non-empty, there could be non bounded semi-static curves. The next section will be devoted to study this question.

1.2.1 Example: mechanical Hamiltonian

Let us consider (M, g) be a smooth connected compact Riemaniann manifold without boundary of dimension $n \geq 2$ and $l : M \to \mathbb{R}^+$ a $\mathcal{C}^2$ function. The mechanical Hamiltonian is defined as follows:

$$H(x,p) = \frac{1}{2}||p||_x^2 - l(x),$$

it corresponds to the energy of the mechanical system directed by the potential l. One can check that H is a $\mathcal{C}^2$ Tonelli Hamiltonian and the associated Lagrangian is

$$L(x,v) = \frac{1}{2}||v||^2 + l(x).$$

Proposition 1.12
The Mather set

$$\mathcal{M} = \{x \in M \text{ such that } l(x) = -\inf l\}.$$

Proposition 1.13
The Aubry set is

$$\mathcal{A} = \{x \in M \text{ such that } l(x) = -\inf l\}.$$

1.2.2 Example: Mañé's Lagrangians

Let us consider (M, g) be a smooth connected compact Riemaniann manifold without boundary of dimension $n \geq 2$ and $l : M \to \mathbb{R}^+$ a $\mathcal{C}^2$ function. Let $f \in \mathcal{C}^2$ The Mañé's Lagrangian is defined as follows:

$$L(x,v) = \frac{1}{2}|v - l(x)|^2, \quad \forall (x,v) \in TM.$$

This Lagrangian is very interesting since the solutions of the equation $\dot{\gamma} = l(\gamma)$ are solutions of the Euler-Lagrange equation.

1.3 Uniqueness in the non-compact setting

In what follows, we adopt Mané's point of view, as presented in Contreras's book [CI99]. As we have seen intuitively, curves play a crucial role in this analysis, since the uniqueness conditions are established by following certain curves. A natural question arises: what can we do with curves that have empty omega-limit? Contreras addresses this question through his compactification construction.

The notion of semi-static curve naturally leads to that of a static curve, which we now define:

Definition 1.14 (Static curve)
An absolutely continuous curve $\gamma : [0,t] \to M$ with $\gamma \in \mathcal{C}(x,y)$ is called static *if*

$$\int_0^t L(\gamma(s), \dot{\gamma}(s))\, ds + ct = -\phi_c(x,y).$$

A static curve is also semi-static, since $\phi_c(x,y) + \phi_c(y,x) \geq 0$. In the definition of a weak KAM solution, the semi-static curves are defined on $\mathbb{R}_-$ and satisfy the Euler–Lagrange equation. To lighten notation, for $x \in M$ and $v \in T_xM$ we denote by

$$x_v(\cdot)\colon (-\infty, 0] \to M$$

the solution of the Euler–Lagrange equation satisfying

$$x_v(0) = x, \qquad \dot{x}_v(0) = v.$$

As previously mentioned, the Aubry set admits many equivalent characterizations. We can now give a new one in terms of static curves:

Proposition 1.15
The Aubry set $\tilde{\mathcal{A}}$ *can be expressed as*

$$\tilde{\mathcal{A}} = \{(x,v) \in TM \mid x_v : \mathbb{R} \to M \text{ is static}\}.$$

There is no conflict of notation with the previously introduced Aubry set, since $\pi(\tilde{\mathcal{A}}) = \mathcal{A}$, where $\pi : TM \to M$ denotes the canonical projection. A very important property of $\mathcal{A}$ is the *graph property*: the projection $\pi : \mathcal{A} \to M$ is injective. We now define two other sets, known as Mañé's sets:

Definition 1.16 (Mañé's sets)
We define the Mañé set as

$$\tilde{\mathcal{N}} = \Sigma = \{(x,v) \in TM \mid x_v : \mathbb{R} \to M \text{ is semi-static}\}.$$

We also define

$$\Sigma^- = \{(x,v) \in TM \mid x_v : (-\infty, 0] \to M \text{ is semi-static}\}.$$

The weak KAM theorem 1.6 shows that the projection $\pi(\Sigma^-)$ covers the whole manifold M. However, the projection $\pi : \Sigma^- \to M$may be not injective. We have the inclusions

$$\tilde{\mathcal{A}} \subset \tilde{\mathcal{N}} \subset \Sigma^- \subset TM.$$

It is important to notice that the graph property keeps true for Σ^- if the first coordinate is in the projected Aubry set. That is: the projection $\pi : \pi^{-1}(\mathcal{A}) \cap \Sigma^- \to M$ is injective.

The aim of the compactification is to construct a space $\mathfrak{M}^-$, larger than M, where semi-static curves can be identified as points. Consider the space

$$\mathcal{F} = \mathcal{C}^0(M, \mathbb{R})/\sim,$$

where

$$f \sim g \iff f - g \text{ is constant.}$$

We use this equivalence relation because the Hamilton–Jacobi equation involves the gradient of u, so functions are defined only up to an additive constant. The family of functions

$$z \mapsto \phi_c(x, z)$$

is Lipschitz and equicontinuous. Thus, we consider the closure of this family under the topology of uniform convergence on compact subsets and denote it by $\mathfrak{M}^-$. (This construction is close to the Gromov compactification of a metric space, where each point is identified with the function $x \mapsto d(x, \cdot)$.)

Proposition 1.17 (Busemann weak KAM solution)
For any $(x,v) \in \Gamma^-$*, define*

$$u_{x_v}(y) = \lim_{t \to -\infty} \big[\phi_c(x_v(t), y) - \phi_c(x_v(t), x)\big].$$

We call the functions u_{x_v} Busemann weak KAM solutions. *By construction, they are elements of* $\mathfrak{M}^-$.

We have an embedding

$$M/_{(d_c=0)} \hookrightarrow \mathfrak{M}^-,$$

and the classes of Busemann weak KAM solutions form a subset $\mathfrak{B}^- \subset \mathfrak{M}^-$. Since we want to obtain uniqueness results, it is natural to evaluate solutions at points of $\mathfrak{M}^-$. This is impossible for Busemann points corresponding to unbounded curves. So we associate finite representatives to each Busemann point (that is finite points on the curves x_v). More precisely, we consider an injective map

$$q : \mathfrak{B}^- \to M, \qquad \beta \mapsto q_\beta,$$

such that there exists a unique semi-static curve γ with $\gamma(0) = q_\beta$ and α-limit β. Lastly, we define the *extended static classes* as

$$\mathbb{A} = \{q_\beta\}_{\beta \in \mathfrak{B}^-}.$$

Note that if $(x,v) \in \tilde{\mathcal{A}}$, then $u_{x_v} \in \mathfrak{B}^-$ and hence $q_{x_v} \in [x]_{(d_c=0)}$, where $[x]_{(d_c=0)}$ denotes the static class containing x.

We can now state the analogue of theorem 1.11 in the non-compact setting. Before doing so, we comment briefly on dominated and strictly dominated functions. In the compact case, the theorem applies starting from dominated functions on the Aubry set. Here, this may fail (see example 1), so we replace the domination condition by the following:

A function $u : \mathbb{A} \to \mathbb{R}$ is said to be *strictly dominated* if

$$u(q_\alpha) < u(q_\beta) + u_{q_\beta, \dot{\gamma}_\beta(0)}(q_\alpha).$$

If the function is defined on the whole manifold M, we say it is strictly dominated if its restriction to $\mathbb{A}$ is strictly dominated. Note that if $q_\beta \in \mathcal{A}$, then

$$u_{q_\beta, \dot{\gamma}_\beta(0)}(q_\alpha) = \phi_c(q_\beta, q_\alpha),$$

and the inequality reduces to the usual domination condition with a strict inequality instead of a large one. This justifies the terminology.

Theorem 1.18 (Contreras [Con01])
The map

$$\{u : \mathbb{A} \to \mathbb{R} \mid u \text{ strictly dominated}\} \to \{\text{weak KAM strictly dominated solutions}\}$$
$$u \mapsto \inf_{\alpha \in \mathfrak{B}^-} \big(u(q_\alpha) + u_{q_\alpha, \dot{\gamma}_\alpha(0)}(x)\big)$$

is a bijection.

2 To the determination

2.1 Descent operator

As mentioned in the introduction, the determination theory came from smooth convex analysis, but leads to a more general abstract setting. This subsection mainly recalls result from [DLS23].

Definition 2.1 (Descent modulus [DLS23])
For a set X, we call descent modulus any operator from $\mathbb{R}^X$ to $\bar{\mathbb{R}}_+^{\mathbb{X}}$ satisfying:

- $f(x) = 0 \Rightarrow T_f(x) = 0$,
- $[\max(f(x) - f(z), 0) \geq \max(g(x) - g(z), 0), \forall z \in X] \Rightarrow [T_f(x) \geq T_g(x)]$,
- $0 < T_f(x) < +\infty \Rightarrow T_{rf}(x) > T_f(x), \quad \forall r > 1.$

In a complete compact space X, the determination result can be written as follows:

$$(T_f(x) = T_g(x), \quad \forall x \in X) \quad \Longrightarrow \quad (f = g \quad \text{up ton an additive constant}).$$

The idea followed by the authors mainly comes from the following lemma: if two functions f and g take the same value on the non-empty critical set $(\{x \mid T_f(x) = 0\}$, and $T_f > T_g$ outside this set, then f should grow faster than g, that would lead to $f > g$ outside the critical set.

In a smooth enough setting, the modulus of the gradient $|\nabla f|$ is a descent operator. If the function f is convex, we can take the distance of its sub-differential $\partial f(x)$ to 0, denoted as $|\partial^0 f(x)|$. Indeed, we know from convex analysis that

$$0 \in \partial f(x) \Leftrightarrow f(x) = \min f$$

and so the quantity $|\partial^0 f(x)|$ measures how close we are to satisfying the equivalence above. In the smooth setting, the results are mainly achieved by integrating the descent modulus along the gradient flows (curves solving $\dot{\gamma} = -\nabla f(\gamma)$ or $\dot{\gamma} \in -\partial f(\gamma)$). In the general setting, the proof is carried out by reasoning on the maximum of the function $f-g$ and showing it is critical by induction.

If we remove the compacity assumption, similarly to what we have seen for weak KAM theory, there is a price to pay. The proofs, smooth or non smooth, fall. Mainly, because the critical set can be empty. One can ask for coercivity to solve this issue, but we want to have a better result. The authors introduced this following definition for X being a metric space:

Definition 2.2 (Strong metric compatibility)
A descent modulus T is strongly metrically compatible, if there exists a strictly increasing continuous function $\theta : \mathbb{R}_+ \to \mathbb{R}_+$ with $\theta(0) = 0$ and $\lim_{t\to\infty} \theta(t) = +\infty$, such that for every functions f, g, and $\delta > 0$, we have:

$$T[f](x) < \delta < T[g](x) \implies \exists z \in X \ : \ \frac{\{f(x)-f(z)\}_+}{d(x,z)} < \theta(\delta) < \frac{\{g(x)-g(z)\}_+}{d(x,z)}$$

The idea behind the above definition is to guarantee our intuition above "f should grow faster than g". In the non-compact setting, one has to add something to the critical set. Contreras built an extended Aubry set, here the authors add some asymptotic critical sequences:

Definition 2.3 (Asymptotic critical sequences)
A sequence $\{z_n\}_{n\geq 1} \subset X^{\mathbb{N}}$ is called T-asymptotically critical for a function $g \in \mathcal{F}$ if it has no converging subsequence,

$$T_g(z_n) \neq 0, \forall n \in \mathbb{N} \quad \text{and} \quad \sum_{n=0}^{+\infty} T[g](z_n)\, d(z_n, z_{n+1}) < +\infty.$$

Then, the theorem is the following:

Theorem 2.4
Let X be a metric space and T a descent modulus strongly metrically compatible. If two functions f ang g satisfy:

$$\begin{cases} T_f(x) = T_g(x), & \forall x \in X, \\ f(x) = g(x), & \forall x \in Crit_f, \\ \liminf f(x_n) = \liminf g(x_n), & \forall (x_n) \ \textit{T-assymptotically critical}, \end{cases}$$

then,

$$f = g.$$

We come back to this theorem in the proposition 2.7. In particular, it will help the reader to have more intuitions about asymptotic critical sequences.

2.2 In a non-compact manifold

We come back here to the weak KAM theory on a manifold. The strict domination condition is not well fitted to study the determination results we want to consider. To show why we want to proceed differently, we give an example of weak KAM solutions that are not strictly dominated. We take the example of the mechanical Hamiltonian (subsection 1.2.1), but now assume the manifold is $\mathbb{R}$. Nevertheless, we still assume that l is bounded from below.

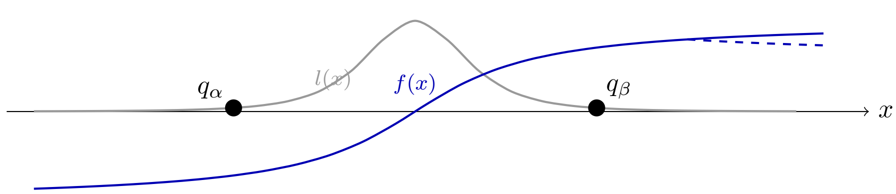

Figure 1: Two not strictly dominated functions

For instance, take $l(x) = \left|\frac{1}{1+x^2}\right|^2$, the functions $x \mapsto \arctan(x)$ and $x \mapsto \arctan(-x)$ are solutions. But, a continuous function equal to $\arctan(x)$ for x smaller than a real M and then equal to $\arctan(-x)$ up to an additive constant is solution. In this case, the compactification adds to more points to our space: one represents $-\infty$ and is called α and the one represents $+\infty$ and is called β. Whatever, the choice of q_α and q_β, we can find two different solutions that are not strictly dominated, weak KAM solutions and equal on the representatives as shown on Figure 1.

The problem in this example is the following fact: γ_β is a semi-static curve, but is non necessarily a calibrated curve for u: we can have

$$u(q_\beta) - u(\gamma(-t)) < \int_{-t}^{0} (L(\gamma(s), \dot\gamma(s)) + c)ds, \quad \forall t \geq 0$$

but this curve could be calibrated somewhere

$$u(\gamma(-t')) - u(\gamma(-t)) = \int_{-t}^{-t'} (L(\gamma(s), \dot\gamma(s)) + c)ds, \quad \forall t \geq t', \quad t' > 0.$$

A way to fix it, is to keep in mind what would be the value of u on the calibrated curve if it we would follow it all along. This is what does the application $\tilde u$.

Definition 2.5 (The application $\tilde u$)
Let $u\colon M \to \mathbb{R}$ be a weak KAM solution of negative type, and define $\tilde u : \Sigma^- \to \mathbb{R} \cup \{+\infty\}$ by

$$\tilde u(x, v) \;=\; \liminf_{t\to-\infty} \Big[u\big(x_v(t)\big) + h_{-t}\big(x_v(t), x\big) \Big].$$

By construction, $\tilde u$ takes the value $+\infty$ at (x, v) if the curve x_v is never calibrated for u. We want to continue to take advantages of Contreras's compactification. First, let us remark that if x is in the Aubry set, there is a unique v such that $(x, v) \in \Sigma^-$. This means, the application $\tilde u$ can be viewed as an application from $\tilde M \overset{\text{def}}{=} \Sigma^-/_{(d_c\circ\pi=0)}$ to $\mathbb{R} \cup \{+\infty\}$. Similarly to the last subsection, the reader should think as $\tilde M$ as being

$$\tilde M \cong \mathcal{A}/_{(d_c=0)} \cup \left(\Gamma^- \setminus \tilde{\mathcal{A}}\right).$$

Inside this set, we also want to define a kind of extended Aubry set

$$\tilde{\mathbb{A}} \overset{\text{def}}{=} \{(q_\beta, \dot\gamma_\beta(0)\}_{\beta\in\mathfrak{B}^-}.$$

On this set, we have the following result of uniqueness:

Theorem 2.6
Let u and v be two weak KAM solutions. We have

$$\tilde u = \tilde v \text{ in } \tilde{\mathbb{A}} \Longrightarrow u = v \text{ in } M.$$

It is possible to write this result in term of bijections as it was done for the theorem 1.11.

2.3 Examples

In this section, we examine, through examples, the connections that can be drawn between weak KAM theory and determination results. In particular, we explore how the machinery of weak KAM theory could be applied fruitfully to determination.

Example 1. Take $M = \mathbb{R}^d$ with the mechanical Hamiltonian

$$H(x,p) = \tfrac{1}{2}|p|^2 \;-\; \tfrac{1}{2}|\nabla f(x)|^2, \quad f \in \mathcal{C}^3(\mathbb{R}^d), \; \inf_{x\in\mathbb{R}^d} |\nabla f(x)| = 0.$$

This is the example from subsection 1.2.1, now considered on the non-compact manifold $\mathbb{R}^d$. In smooth determination theory, the typical result asserts: if $|\nabla u| = |\nabla v|$ and $u = v$ at the "end" of the gradient flows, then $u \equiv v$. In weak KAM theory, we have a similar principle: two solutions are equal if they coincide at the "ends" of the semi-static curves.

These notions would coincide if the semi-static curves were the same as the gradient flow lines. This is indeed the case when u is a smooth solution. Along a semi-static curve γ, calibrated for u, one has:

$$\frac{d}{dt}u(\gamma(-t)) = -\langle \nabla u(\gamma(-t)), \dot\gamma(-t)\rangle = L(\gamma(-t), \dot\gamma(-t)) = \frac{|\nabla u(\gamma(-t))|^2 + |\dot\gamma(-t)|^2}{2}.$$

For instance, the following proposition, which is the smooth application of the subsection 2.1, is a corollary of Theorem 2.6.

Proposition 2.7
Let $u, v\colon \mathbb{R}^d \to \mathbb{R}$ be two negative weak KAM solutions of the equation $H(x, d_x u) = 0$, both bounded from below. Assume:

- *$u = v$ on $Crit_f = \{x \mid \nabla f(x) = 0\}$,*
- *for every sequence $(x_n) \subset \mathbb{R}^d$ such that*

$$|\nabla f(x_n)| \to 0 \quad \text{and} \quad \sum d(x_n, x_{n+1})\,|\nabla f(x_n)| < \infty,$$

one has $\liminf_{n\to\infty} u(x_n) = \liminf_{n\to\infty} v(x_n)$.

Then $u \equiv v$ on $\mathbb{R}^d$.

The sequences considered in the proposition can be interpreted as discretizations of gradient flow curves. Here, determination theory introduces a kind of extended critical set, that is the set of critical asymptotic sequences. On the other hand, weak KAM theory defines an extended Aubry set, which represents the semi-static curves. Interestingly, in this context, the critical asymptotic sequences turn out to be discretizations of the semi-static curves.

Example 2. Crandall–Pazy directions [DSTG25]:

Conjecture 2.8
Let $\mathcal{H}$ be a general Hilbert space, and define the local slope of a function f at x as

$$s_f(x) = \limsup_{y\to x} \frac{\max(f(x) - f(y),\, 0)}{d(x,y)}.$$

For any convex function, let p_f be the unique vector such that all gradient flows satisfy $\dot\gamma \to p_f$. Then,

$$\begin{cases} |\nabla f| &= |\nabla g| \\ p_f &= p_g \end{cases} \quad \Rightarrow \quad f = g \text{ up to an additive constant.}$$

Here, the vector p_f is called the *Crandall–Pazy direction*, and it should be interpreted as a kind of Neumann condition for the eikonal equation $|\nabla u(x)|^2 = l(x)$. This result is known in some cases using determination techniques. To analyze it within the framework of weak KAM theory, we face a first issue compared to the previous presentation. If we consider the mechanical Hamiltonian, the condition $p_f > 0$ implies $c < 0$. In other words, the Hamilton–Jacobi equation takes the form

$$|p|^2 = |\nabla u|^2 + c,$$

whereas we aim to study solutions of

$$|p|^2 = |\nabla u|^2.$$

This relates to an aspect we had not mentioned earlier: in the non-compact setting, weak KAM solutions can exist for $k > c$, where c is the critical value, provided we replace every occurrence of c in the definition of a weak KAM solution with k. These solutions still satisfy the theorem stated above, but they are in fact easier to analyze, mainly because $h_k(x, y) = +\infty$ for all $(x, y) \in M^2$. It also holds that the semi-static curves, calibrated for u smooth, are gradient flows of u.

Example 3. Discounted global Eikonal equation [LTG24]:

As previously mentioned, determination theory can be applied to any descent operator, such as the global slope, in a general metric space:

$$G_f(x) = \sup_{y \in X} \frac{\max(f(x) - f(y), 0)}{d(x, y)}.$$

We refer to the Eikonal equation in which the gradient is replaced by this global slope as the *discounted global Eikonal equation*. This equation has been studied in [LTG24]. In this setting, the weak KAM theory discussed earlier does not apply, as the ambient space is not a manifold. However, a discrete version of weak KAM theory does apply, and in particular, it can be used to recover the integration formula established in the work of Minh Lê and Tapia-García.

2.4 Other perspectives

The connections between weak KAM theory and determination remain exploratory. However, the core ideas in the continuous setting are very similar and can be summed up by a simple mantra: following the right curves. In the discrete setting, while curves are no longer part of the weak KAM framework, the analogy persists. For example, the sequences in Proposition 2.7 can be interpreted as discrete counterparts of the relevant curves. We believe that the connection between asymptotically critical sequences and semi-static (or calibrated) curves is a promising direction for further investigation.

To explore the non-compact case, the question of compactification is crucial. The approach of Contreras consists mainly in selecting finite representatives for the points added to the manifold M. In [DGJTG24], the authors investigate whether a Gromov compactification is a compactification in the topological sense. This could lead to further studies on the topological properties of the Contreras compactification.

Although we have not touched on measure-theoretic aspects in this text, critical values, Aubry sets, and related objects can be defined using measures. In the non-compact setting, this approach has been pursued from a viscosity solution perspective, for example, by Ishii in [Ish07]. More broadly, weak KAM theory can be understood as the limit of ergodic control problems. To deal with non intregrable measures, one may, for instance, restrict attention to their values on bounded curves, as in [LQR16]. Such techniques could prove useful in the context we are considering.

Finally, determination theory is not restricted to metric spaces; it can be extended to more general topological settings. To our knowledge, weak KAM theory has not yet ventured beyond the metric framework. Might it be possible to adapt some of its ideas to more general spaces? One of the original motivations for developing weak KAM theory in metric spaces was optimal transport [BB08]. In light of the recent work of Bachir [Bac25], this appears to be a potentially reach area for further studies.

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