Perfectoid Spaces and some applications

Baojun Wu

24th June 2018

Abstract

The aim of this paper is to give a short introdution to the work of Peter Scholze on Perfectoid Spaces [Sch12], including almost purity theorem and some natural applications in number theory related Langlands Program. We study a beautiful object diamond who lives on pro-étale site.Finally we introduce Fontaine-Fargues curve and geometric view of untilts.

Contents

1 Introduction

In commutative algebra, algebraic geometry and number theory we study the mixed character case from character p case by Faltings almost purity method. The basic result , due to Fontaine and Wintenberger ,states that the absolute galois group of $\mathbb{Q}_p(p^{\frac{1}{p^\infty}})$ is isomorphic to the absolute galois group of $\mathbb{F}_p((t))$.Scholze generalizes this result,he proves a tilting equivalence for perfectoid field and perfectoid affinoid K-algebra. Globally there is an tilting equivalence for perfectoid spaces in both analytic topology and étale site. The inverse is also quite interesting, it can be interpreted by Fargues-Fontaine Curve, and nowadays development on diamond in pro-étale site.

By analogy with complex analytic geometry,we develop the theory of rigid analytic geometry by using Grothendieck topology. There are many surprised treasures, such as: the rigid GAGA theorem, comparison theorem as well as some shortcomings.

One is from topology,in rigid analytic geometry an open mapping with bijection on points level may not be an isomorphism. This suggests some points are missing. Another is it only focuses on affinoid K-algebras studies by Tate. For example the rigid generic fibre of $Spf(\mathbb{Z}_p[[T]])$ is rigid analytic open disk but not affinoid.
So we start from Huber's original ideal on contionuous valuation. We study the 'valuation geometry' and introduce the so called Adic Space. In particular,the Perfectoid Space. To do this, we need some settings on perfectoid tate ring. When we have some mixed character objects in perfectoid world,we can define a tilting map to make them have character p.The inverse of the tilting is not quite natural.Fortunately, Witt rings,the fundamental curve and p-divisible group can afford us some parameterization to know the untilting.
We also consider different sites on perfectoid space,étale sites,pro-étale sites and v-sites(in Scholze's paper étale cohomology of diamonds),we want some finitness property on cohomology and let the rigid adic space looks locally contractible,more precisely covered by perfectoid space.With this we can develop some p-adic Hodge theory to compare cohomology with Galois action.
Finally, all comes into a core, the Langlands program.This paper only introduce some Scholze's work on mod p Langlands correspondence. Scholze also use these geometric object get an approch to local langlands correspondence for reductive group as proving the rank n bundles on the fundamental curve Bun_n is an 'dimension 0 smooth Artin stack' by certain finite property of Rapoport-Zink Space.

2 Classical setting on adic space

In this section,we start with some introduction on rigid analytic spaces.

2.1 Overview on adic spaces

Consider A an algebra of finite type over an algebraically closed field k, and let X^0 be MaxSpec(A), and $X = \text{Spec}(A)$. Regard them simply as topological spaces with their Zariski topology. Let us recall the following:

Definition 2.1.1 A *constructible subset* C of a Noetherian topological space is a finite union of locally closed subsets (or, equivalently, a finite Boolean expression in open subsets).

Let X and X^0 be as above. Given a constructible set C in X, the assignment

$$C \mapsto C^0 := X^0 \cap C$$

yelds an inclusion-preserving bijection

$$\{\text{constructible sets in } X\} \leftrightarrow \{\text{constructible sets in } X^0\}$$

in both directions, and such $C \subset X$ is open, resp. closed, if and only if $C^0 \subset X^0$ is open, resp. closed in the Zariski topology. This works the same for any scheme X locally of finite type over a field, with X^0 its subspace of closed points. A sheaf theory on a topological space is developed on a basis for the topology, and is well posed once one has enough inclusion relations and, transition maps.

Proposition 2.1.2 *Let X be a scheme locally of finite type over a field, and X^0 its subspace of closed points. We have an equivalence of categories of sheaves of sets*

$$\mathbf{Shv}(X) \simeq \mathbf{Shv}(X^0)$$

defined by $\mathscr{F} \mapsto \mathscr{F}|_{X^0}$, where the categories of sheaves on X and X^0, respectively, are to be considered with respect to the Zariski topology.

Definition 2.1.3 Let S be a set, Σ a collection of non-empty subsets of S. We say $F \subset \Sigma$ is a *prime filter* if the following properties are satisfied:

(1) given $U, U' \in F$, then $U \cap U' \in F$ (so in particular $U \cap U' \neq \varnothing$).

(2) given $U \in F$, and $U' \supseteq U$, with $U \in \Sigma$, then also $U' \in F$.

(3) given $U_1, \ldots, U_n \in \Sigma$ such that $\cup U_i \in F$, then some U_i is in F.

As soon as Σ is non-empty, a Zorn's Lemma argument ensures prime filters on Σ exist. It T is a topological space, and F is a prime filter of open sets, then for any $\mathscr{F}$ sheaf of sets on T, the *F-stalk of $\mathscr{F}$* is defined to be

$$\varinjlim_{U \in F} \mathscr{F}(U).$$

One has the following::

Theorem 2.1.4 *Let X be a scheme locally of finite type over a field k, and X^0 be its subspace of closed points. Then the correspondence*

$$X \to \{\text{prime filters of non-empty open subsets of } X^0\}$$

given by $x \mapsto \{\text{open } U^0 \subset X^0 \mid x \in U\}$ *is bijective.*

For a refernce, see [BC]

Definition 2.1.5 A topological space T is *sober* if every irreducible closed subset has a unique generic point.

Remark 2.1.6 Examples of sober topological spaces are locally Hausdorff spaces and schemes. The easiest non-sober space is an infinite set with the cofinite topology, as the space itself is irreducible and it has no generic point.

Remark 2.1.7 The theorem 2.1.6, essentially says that if we have a sober topological space T, we can reconstruct t from its sheaf theory, that is, from knowledge of $\mathbf{Shv}(T)$.first, if X and Y are topological spaces, recall that for any morphism

$$f : X \to Y$$

we have an induced functor

$$f_* : \mathbf{Shv}(X) \to \mathbf{Shv}(Y)$$

and in fact an adjoint pair (f_*, f^{-1}), where f^{-1} is the inverse image functor

$$f^{-1} : \mathbf{Shv}(Y) \to \mathbf{Shv}(X),$$

and f^{-1} is exact (in the sense that it commutes with fiber products and equalizers, or equivalently with all finite limits). A *morphism* $\mathbf{Shv}(X) \to \mathbf{Shv}(Y)$ is a pair of functors (h', h), with

$$h : \mathbf{Shv}(X) \to \mathbf{Shv}(Y)$$

and

$$h' : \mathbf{Shv}(Y) \to \mathbf{Shv}(X)$$

for which h' is left adjoint to h, and h' is exact.

Theorem 2.1.8 *If X and Y are sober topological spaces, then the natural map*

$$\mathrm{Hom}_{\mathbf{Top}}(X, Y) \to \mathrm{Mor}(\mathbf{Shv}(X), \mathbf{Shv}(Y))/\sim$$

is a bijection, where $\sim$ denotes natural equivalence for adjoint pairs.

As an example, one may consider $X = \{*\}$ to be a single point, yelding

$$\mathrm{Mor}(\mathbf{Set}, \mathbf{Shv}(Y)) = |Y|,$$

the set underlying Y (via stalks and skyscraper sheaves of sets).

Towards adic spaces

In light of our previous discussion, we mention that Huber shows that for an affinoid algebra A over a complete nonarchimedean field k there is a naturally associated quasi-compact sober space $\mathrm{Spa}(A)$ containing $\mathrm{Sp}(A)$ as a subset so that the inclusion induces an equivalence of categories

$$\mathbf{Shv}(\mathrm{Spa}(A)) \simeq \mathbf{Shv}(\mathrm{Sp}(A)),$$

where we regard $\mathrm{Sp}(A)$ with the usual topology as in Tate's theory.

Definition 2.1.9 A topological $K-algebra$ A is called affinoid over K,if there exists a surjective algebra homomorphism

$$K < T_1, T_2, ..., T_n > \to A$$

Definition 2.1.10

1. We note $Con(A)$ is the equivalence class of continous valuation on A.
2. Let $Spa(A, A^0) = \{v \in \mathrm{Con(A^0)} \mid v(A^0) \leq 1\}$
3. For A is an affinoid algebra, $Sp(A)$=$\{X = Maxspec(A), Grothendieck\ topology,\ \mathcal{O}_X\}$

We have the following:

Proposition 2.1.11 *The map $f : A \to B$ as above is flat (in the commutative-algebraic sense) if and only if the morphism $\mathrm{Sp}(B) \to \mathrm{Sp}(A)$ is flat (that is, is flat on stalks of the respective structure sheaves).*

Now the question is, given that f is flat, whether or not $f_K : K\widehat{\otimes}_k A \to K\widehat{\otimes}_k B$ is flat as well. The answer is yes, but to prove it it's used Raynaud's theory on formal models for rigid spaces. The metric completion inherent in such scalar extension is the main source of difficulties, but at a more geometric level one has the annoyance that, roughly speaking, that in the following diagram the dotted arrows do not exist:

$$\begin{array}{ccc} \mathrm{Sp}(A_K) & \xleftarrow{\mathrm{Sp}(f_K)} & \mathrm{Sp}(B_K) \\ \downarrow & & \downarrow \\ \mathrm{Sp}(A) & \xleftarrow[\mathrm{Sp}(f)]{} & \mathrm{Sp}(B) \end{array}$$

2.2 Formal models

Let k be a nonarchimedean field, with valuation ring $\mathcal{O}$. Roughly, a formal model for a rigid space X over k is a quasi-compact formal scheme $\mathfrak{X}$ over $\mathcal{O}$, which is locally isomorphic to

$$\mathrm{Spf}\left(\mathcal{O}\{t_1, \ldots, t_n\}/(f_1, \ldots, f_m)\right).$$

If $0 < |\pi| < 1$, then $\mathcal{O}\{t_1, \ldots, t_n\}[\frac{1}{\pi}]$ is a Tate algebra, so one can associate to such $\mathfrak{X}$ a quasi-compact quasi-separated rigid space over k by gluing affinoids

$$\mathrm{Sp}\left(k\langle t_1, \ldots, t_n\rangle/(f_1, \ldots, f_m)\right),$$

yelding a "generic fiber" $\mathfrak{X}^{\mathrm{rig}}$ of the formal scheme $\mathfrak{X}$ over $\mathcal{O}$.

More in detail, if X is a quasi-compact quasi-separated rigid space over k, then Raynaud proved X has the form $\mathfrak{X}^{\mathrm{rig}} := \mathfrak{X} \otimes k$ for some formal scheme $\mathfrak{X}$ as above flat over $\mathcal{O}$, and in the affinoid case, explicitly, if we let

$$A := k\langle t_1, \ldots, t_n\rangle/I$$

for some ideal I of $k\langle t_1, \ldots, t_n\rangle$, then for $X = \mathrm{Sp}(A)$ we can choose

$$\mathfrak{X} = \mathrm{Spf}\left(\mathcal{O}\{t_1, \ldots, t_n\}/(\mathcal{O}\{t_1, \ldots, t_n\} \cap I)\right).$$

(If $|k^\times| \subset \mathbf{R}^\times_{>0}$ is not discrete then some work is needed to show $\mathcal{O}\{t_1, \ldots, t_n\} \cap I$ is finitely generated).

Theorem 2.2.1 *Let X be a quasi-compact, quasi-separated rigid space over k. Then*

$$\varprojlim_{\mathfrak{X}^{\mathrm{rig}} \simeq X} |\mathfrak{X}|$$

is homeomorphic to the adic space attatched to X.

Theorem 2.2.2 *Let K be a complete nonarchimedean field There is a fully faithful functor from rigid analytic space over K to adic space over $Spa(K, K^0)$ which sends $Sp(R)$ to $Spa(R, R^0)$.*

Theorem 2.2.3 *There is a fully faithful functor from locally noetherian scheme to adic space which sends $Spf(R)$ to $Spa(R, R^0)$ .This is caller gneric fiber construction.*

For a refernce, we recommend Huber's Étale cohomology [HE].

3 Huber rings

A huber ring is a topological ting A containning an open subring A_0,which is adic with respect to a finitely generted ideal $I \subset A_0$. For a Huber ring A,we always write A^0 for power bounded element of A, A^{00} is the ideal of topological nilpotent elements and A^+ shall be any open and integrally subring of A^0.

Example 3.0.1 For $F/\mathbf{Q}_p$ a complete nonarchimedean extension with ring of intergers $\mathcal{O}_F$ consider tate algebra $F\langle X_1, , , , X_n\rangle$ is a Huber ring.

Proposition 3.0.2 *For A a Huber ring, $Con(A)$ is spectral.*

Proof. (sektch,for detail see [H1])
Since $Con(A) = \{v \in Spv(A, A^{00}A) \mid v(a) < 1 \ for \ all \ a \in A^{00}\}$ is closed in $Spv(A, A^{00}A)$ and notice that a closed subspace of a spectral space is spectral. □

Definition 3.0.3

We let $Spa(A, A^+) = \{v \in Con(A^0) \mid v(A^+) \leq 1\}$

Since the pro-constructible subset of spectral space is again spectral,so we have

Proposition 3.0.4 *The topological space $Spa(A, A^+)$ is spectral*

Example 3.0.5 1. Let K be an nonarchimedean field, then it's easy to see $Spa(K, \mathcal{O}_K)$ is a single point.

2. Let K be an nonarchimedean field,complete with respect to a discrete valuation $|\cdot|_K$ with the residue field k.Then $Spa(\mathcal{O}_K, \mathcal{O}_K)$ has two points,given by

$$|\cdot|_K : \mathcal{O}_K \to \mathbb{R}_{\geq 0}$$

$$and\ |\cdot|_k : \mathcal{O}_K \to \mathcal{O}_K/\varpi \to \mathbb{R}_{\geq 0}$$

the last arrow is trivial valuation.the first point corresponds a morphism

$$\eta : Spa(\mathcal{O}_K, \mathcal{O}_K) \to Spa(K, \mathcal{O}_K)$$

the adic generic fibre of a formal scheme $\mathfrak{X}/\mathcal{O}_K$ is defined as

$$\mathfrak{X}_\eta^{ad} = \mathfrak{X}^{ad} \times_{Spa(\mathcal{O}_K, \mathcal{O}_K)} Spa(K, \mathcal{O}_K)$$

The topology of $X = Spa(A, A^{+)}$ is generated by so called rational subsets

$$X(\frac{T_1}{s_1}, , , \frac{T_n}{s_n}) = \{x \mid |f_i(x)| \leq |s_i(x)| \neq 0 \mid f_i \in Ti\}$$

Where $T_i \subset A$ is a finite subset such that T_iA is open in A. We put a natural structure presheaf $\mathcal{O}_x$ on X as follows. We equip the ring $R[\frac{1}{s_1}, , , \frac{1}{s_n}]$ with a topology making $R_0[\frac{T_1}{s_1}, , , \frac{T_n}{s_n}]$ open equipped with the $J = IR_0[\frac{T_1}{s_1}, , , \frac{T_n}{s_n}]$ adic topology.This defines a ring topology on $R[\frac{1}{s_1}, , , \frac{1}{s_n}]$ and turns into a huber ring. Define

$$R\langle\frac{T_1}{s_1}, , , \frac{T_n}{s_n}\rangle = J - adic\ completion\ of\ R[\frac{1}{s_1}, , , \frac{1}{s_n}]$$

Now for $U = X(\frac{T_1}{s_1}, , , \frac{T_n}{s_n})$,

$$\mathcal{O}_X(U) = R\langle\frac{T_1}{s_1}, , , \frac{T_n}{s_n}\rangle$$

and $\mathcal{O}_X^+(U)$ as the completion of the integral closure of$R^+[\frac{T_1}{s_1}, , , \frac{T_n}{s_n}]$ in $\mathcal{O}_X(U)$.

Definition 3.0.6 A Huber ring is called sheafy if $\mathcal{O}_X$ is a sheaf on $X = Spa(A, A^0)$.

Remark 3.0.7 $\mathcal{O}_X$ is a sheaf on $X = Spa(A, A^+)$ for some choice of A^+ if and only if for any choice of A^+

Definition 3.0.8 A Huber ring A is called

1. Tate if it contains a topological nilpotent unit ϖ.
2. uniform if A^0 is a bounded subset of A.
3. stably uniform if $\mathcal{O}_X(U)$ is uniform for all rational subsets.
4. strongly noetherian if A is Tate and $A_1,,,X_n\rangle$ are noetherian for every n.
5. perfectoid is A is a complete uniform tate ring satisfies some conditions we define later.

Theorem 3.0.9 *A Tate ring A is sheafy if*

1. *(Huber) A is strongly noetherian.*
2. *(Scholze)A is perfectoid.*
3. *(Buzzard-Verberkmoes)A is stably uniform.*

Definition 3.0.10

1. An affinoid adic space is

$$X = (X, \mathscr{O}_X, \{|\cdot|_x\}_{x\in X}),$$

associated with sheafy huber pair.

2. An adic space is locally affinoid adic space.
3. A rigid analytic space over K is an adic space X admitting an open covering by $Spa(A_i, A_i^0)$ with all A_i are K-affinoid alebras.

The most famous example of Adic space is the Fargues-Fontaine Curve, which is defined in section 8.

4 Perfectoid rings

For a Huber ring A,we always write A^0 for power bounded element of A,and A^+ shall be any open and integrally subring of A^0

Definition 4.0.1 A *perfectoid ring* is a complete Tate ring A (huber with topologically nilpotent unit $\varpi \in A$) satisfying the following properties:

(1) A^0 is bounded.

(2) There exists a topologically nilpotent unit ϖ with $\varpi^p \mid p$ in A^0.

(3) the frob $\Phi : A^0/\varpi A^0 \to A^0/\varpi A^0$ is surjective.

Remark 4.0.2 For any complete Tate ring A and nonzero pseudo-uniformizer ϖ satisfying $\varpi^p \mid p$ in A^0, the Frobenius map $\Phi : A^0/\varpi \to A^0/\varpi^p$ is necessarily injective. The surjectivity condition is independent of the choice of such ϖ.Acutually,it's equivalent to the frob Φ from A^0/p to itself is an isomorphism (surjective)

Example 4.0.3 Here is an example of perfectoid ring which doesn't arise as an algebra over a field:

$$A = \mathbf{Z}_p^{\mathrm{cyc}}\langle (p/T)^{1/p^\infty}\rangle^\wedge[1/T].$$

One can take $\varpi := T^{1/p}$, as $\varpi^p = T$ divides p in A^0.

There are two ways to introduce Perfectoid field which are equivalent.

Definition 4.0.4 A *perfectoid field* is a perfectoid ring K that is a field and its topology is defined by a rank 1 valuation

$$|\cdot| : K \to \mathbf{R}_{\geq 0}.$$

Remark 4.0.5 This definition is hard to describe whether a field is perfectoid or not.another definition asks the valuation group is non-discrete since it's p-divisible.

Proposition 4.0.6 *Let K be a nonarchimedean field.Then K is perferctoid field if and only if the folllowing conditions hold*

1. *K is not discrete valued*

2. *$|p|$<1*

3. the Frob $\Phi : A^0/p \to A^0/p$ is surjective.

Proposition 4.0.7 *Let A be a complete uniform Tate ring.*

(1) *If there exists a pseudo-uniformizer $\varpi \in A$ such that $\varpi^p \mid p$ and $\Phi : A^0/p \to A^0/p$ is surjective, then A is perfectoid.*

(2) *If A is perfectoid, then $\Phi : A^0/p \to A^0/p$ is surjective under the additional assumption that the ideal $pA^0 \subset A^0$ is closed.*

Example 4.0.8

(1) The field $\mathbf{Q}_p$ and it's finite extensions are not perfectoid field.

(2) We consider $\mathbf{Q}_p(p^{1/p^\infty})^\wedge$ (with $\varpi = p^{1/p}$) and $\mathbf{Q}_p(\zeta_{p^\infty})^\wedge$, with ϖ coming from the $\mathbf{Z}/p\mathbf{Z}$ piece of $\mathbf{Q}_p(\zeta_{p^2})/\mathbf{Q}_p$ (a $(\mathbf{Z}/p^2\mathbf{Z})^\times$-extension). Both are perfectoid fields.

(3) We consider:

$$\mathbf{Q}_p\langle T^{1/p^\infty}\rangle = \varinjlim_{n\geq 1} \mathbf{Q}_p\langle T^{1/p^n}\rangle^\wedge := (\varinjlim_{n} \mathbf{Z}_p[T^{1/p^n}])^\wedge[1/p].$$

and this is *not* perfectoid. However,

$$\mathbf{Q}_p^{\mathrm{cyc}}\langle T^{1/p^\infty}\rangle = \mathbf{Q}_p\langle T^{1/p^\infty}\rangle \widehat{\otimes}_{\mathbf{Z}_p} \mathbf{Z}_p^{\mathrm{cyc}}$$

is. This is also obtained as $A[1/p]$, A being the p-adic completion of $\mathbf{Z}_p^{\mathrm{cyc}}[T^{1/p^\infty}]$.

Perfectoid rings of char p are not very interesting, by using Banach open mapping theorem we can prove that

Proposition 4.0.9 *Let A be a topological ring with pR=0.Then the following are equivalent:*

1. A is perfectoid

2. A is a perfect complete tate ring.

4.1 The Tilting functor

There is a functor

$$\begin{array}{ccc} \{\text{perfectoid rings}\} & \longrightarrow & \{\text{perfectoid } \mathbf{F}_p\text{-algebras }\} \\ \uparrow & & \uparrow \\ \{\text{perfectoid fields}\} & \longrightarrow & \{\text{perfectoid fields of char. } p\} \end{array}$$

denoted $A \mapsto A^\flat$. Note that the above diagram contains many nontrivial statements in it. There exists such assignment $A \mapsto A^\flat$ sending a perfectoid ring to a perfectoid $\mathbf{F}_p$-algebra, and such assignment is such that such ring A is a field if and only if $A^\flat$ is. More than this, such A is a *perfectoid* field if and only if $A^\flat$ is, which means we'll have to keep track how this functor interacts with valuations.

Theorem 4.1.1 *For K a perfectoid field of characeristic 0, there exists an equivalence of categories:*

$$\{\text{perfectoid } K\text{-algebras}\} \to \{\text{perfectoid } K^\flat\text{-algebras }\}.$$

We shall refer to such equivalence with the name of "tilting equivalence", from now on.

The inverse functor depends on the "untilt" K of $K^\flat$, as for different K one can obtain the same $K^\flat$. Fontaine gave an exhaustive description of all the characteristic 0 fields that give a particular $K^\flat$.

Remark 4.1.2 For a perfectoid field K of character 0, the equivalence

$$\{\text{finite separable } L/K\} \cong \left\{\text{finite separable } L'/K^\flat\right\}$$

is a Theorem due to Fontaine and Wintenberger. The equivalence respects degrees in both directions, and in fact the Galois theories on both sides. We shall expand on this later, as this will turn out to be essential in the sequel.

Choose $\varpi \in A$ a pseudo-uniformizer such that $\varpi^p \mid p$, so that $\varpi \in A^0$ and A^0 has the ϖ-adic topology.

Definition 4.1.3 We define:

$$A^{0\flat} = \varprojlim_{\Phi} A^0/\varpi A^0 = \{(\overline{a_n})_{n\geq 0} \mid \overline{a_{n+1}}^p = \overline{a_n}\}.$$

Note that on $A^{0\flat}$, we have a *canonical* p-th root: if $a = (\overline{a_n})_{n\geq 0}$ then $a^{1/p} := (\overline{a_{n+1}})_{n\geq 0}$.

Lemma 4.1.4 *The multiplicative map:*

$$\varprojlim_{a\mapsto a^p} A^0 \to \varprojlim_{\Phi} A^0/\varpi A^0 =: A^{0\flat}_{\varpi}$$

sending $(a^{(n)}) \mapsto (a^{(n)} \mod \varpi)_{n\geq 0}$ *is a homeomorphism.*

Remark 4.1.5 We note that the left side is independent of ϖ, and we define $A = A^0[1/\varpi]$

Proof. The key is to build a continuous 0-th component of the inverse. We can then apply this construction to the canonical p-th root extraction on the right side.

□

As a first step towards the tilting equivalence, we seek to define some $\varpi^\flat \in A^{\flat 0}$, not a zero-divisor, satisfying the following properties:

(1) $A^{0\flat}[1/\varpi^\flat]$ is perfectoid using $\varpi^\flat$ with $A^{0\flat}$ the subring of power-bounded elements.

(2) There is a natural isomorphism:

$$A^{0\flat}/\varpi^\flat \simeq A^0/\varpi A^0$$

using the 0-th projection $A^{0\flat} \to A^0/\varpi A^\flat$.

we want $\varpi^\flat = (\varpi, \varpi^{1/p}, \varpi^{1/p^2}, \ldots)$.We notice that after multiplying ϖ a unit it has a compatible sequence of p-power rooots in A

Lemma 4.1.6 $\varpi^\flat$ *is* not *a zero-divisor in* $A^{0\flat}$ *and is topologically nilpotent.*

Example 4.1.7 If K is a perfectoid field, we are going to see that $K^\flat := K^{0\flat}[1/\varpi^\flat]$ is a perfectoid field of characteristic p with $|K^{\flat\times}| = |K^\times|$, and $|\varpi^\flat| = |\varpi|$.

Definition 4.1.8 Define the *tilt* of A to be:

$$A^\flat = A^{0\flat}[1/\varpi^\flat] \supset A^{0\flat}$$

with the $\varpi^\flat$-adic topology on $A^{0\flat}$. This is a complete Tate ring with $\varpi^\flat$ as a pseudo-uniformizer and $A^{0\flat}$ the ring of definition.

To conclude that $A^\flat$ is perfectoid first We know $A^\flat$ is independent of the choice of ϖ and $\varpi^\flat$ and that its ring of power-bounded elements is precisely $A^{\flat 0}$.

Proposition 4.1.9 *For any two $\varpi, \varpi' \in A^0$ and $\varpi^\flat, (\varpi')^\flat$ are associated choices in $A^{0\flat}$, then $A^{0\flat}[1/\varpi^\flat] \simeq A^{0\flat}[1/(\varpi')^\flat]$.*

It remains to show that $A^{0\flat}$ is actually the ring of power-bounded elements in $A^\flat$.

Proposition 4.1.10 *We have $A^{0\flat} = (A^\flat)^0$.*

We saw that

$$A^{0\flat}/\varpi^\flat A^{0\flat} \simeq A^0/\varpi A^0.$$

Definition 4.1.11 Let $x = (\overline{x_n}) \in A^{\flat 0}$. We pick *any* sequence of lifts (x_n). Then we define

$$x^\# = \varinjlim_{n\to\infty} x_n^{p^n}.$$

It's obviously the map is well define.

Proposition 4.1.12 *If K is a perfectoid field, then* $\mathrm{Cont}(K) \cong \mathrm{Cont}(K^\flat)$ *via*

$$v \mapsto v^\flat : x \mapsto v(x^\#).$$

Remark 4.1.13 Given a perfectoid ring R,we define $R^\flat = \varprojlim_\Phi R$,A priori this is a topological monoid,but it has a ring structure given by

$$(x_0, x_1, , ,) + (y_0, y_1, , ,) = (z_0, z_1, , ,)$$

where $z_i = \lim_{x\to\infty}(x_{i+n} + y_{i+n})^{p^n}$,luckly,exist and make $R^\flat$ into a perfectoid ring.

5 Tilting equivalence

Definition 5.0.1 A ring R is *intergral perfectoid* if it satisfies the following conditions: it is p-adically separated and complete, it is ϖ-adically complete and separated for some element $\varpi \in R$ such that ϖ^p divides p in R, the pth power map $\varphi : R/pR \to R/pR$ is surjective, and the kernel of $\vartheta : \mathbb{A}_{\mathrm{inf}}(R) \to R$ is principal.

Proposition 5.0.2 *Let S be a ring that is p-adically separated and complete as well as ϖ-adically complete and separated with respect to some element $\varpi \in S$ such that ϖ^p divides p in S. The following are equivalent:*

(1) *Every element of $S/\varpi pS$ is a pth power.*

(2) *Every element of S/pS is a pth power.*

(3) *Every element of $S/\varpi^p S$ is a pth power.*

If the above equivalent conditions hold, then there exist units $u, v \in S^\times$ such that $u\varpi$ and vp admit compatible systems of p-power roots in S.

Proposition 5.0.3 *Let S be a ring which is p-adically separated and complete as well as ϖ-adically complete and separated with respect to some element $\varpi \in S$ such that ϖ^p divides p in S. Then, the equivalent conditions in the last Proposition are also equivalent to the map $\vartheta : \mathbb{A}_{\text{inf}}(S) \to S$ being surjective.*

Theorem 5.0.4 *Let R-**Perf** be the category of integral perfectoid R-algebras, and likewise for $R^\flat$-**Perf**. Then the tilting functor $(\cdot)^\flat$ induces an equivalence of categories:*

$$R\text{-}\mathbf{Perf} \xrightarrow{\simeq} R^\flat\text{-}\mathbf{Perf}$$

whose quasi-inverse is denoted by $(\cdot)^\#$ and called "untilting".

the hard part is the following theorem which use deformation theory.For a ference, see [Bh]

Theorem 5.0.5 *Let R be a integral perfectoid ring.*

(1) *The reduction mod ϖ functor:*

$$R\text{-}\mathbf{Perf} \to (R/\varpi)\text{-}\mathbf{Perf}$$

is an equivalence of categories.

(2) *The reduction mod ϖ functor:*

$$R^a\text{-}\mathbf{Perf} \to (R^a/\varpi)\text{-}\mathbf{Perf}$$

is an equivalence of categories.

We will talk about it's global form when we develop geometric tools. We can also use some almost mathematics to prove the following result:

Theorem 5.0.6 *Let R be a Tate-perfectoid ring, $R^\flat$ its tilt. Then tilting induces an equivalence of categories:*

$$R\text{-}\mathbf{Perf} \to R^\flat\text{-}\mathbf{Perf}.$$

Remark 5.0.7 We finally note that, from the proof of the tilting equivalence for Tate-perfectoid rings, the untilting functor $(\cdot)^\#$ can be made explicit. Given a Tate-perfectoid R-algebra R', $R'^\flat$ its tilt, we have $R'^{\flat\#} = (\mathbb{A}_{\inf}(R'^0) \otimes_{\mathbb{A}_{\inf}(R^0)} R^0)[1/\varpi]$. This is exactly Fontaine's functor.

6 Perctoid space

We fix a Tate-perfectoid ring R, and call $S := \mathrm{Spa}(R, R^+)$ affinoid perfectoid space. We denote $S^\flat := \mathrm{Spa}(R^\flat, R^{\flat+})$. the tilting equivalence between perfectoid R-algebras and perfectoid $R^\flat$-algebras naturally extends to an equivalence between categories of perfectoid pairs over (R, R^+) and $(R^\flat, R^{\flat+})$. We shall denote by:

$$(R, R^+)\text{-}\mathbf{Perf}$$

the category whose objects are perfectoid pairs (R', R'^+) which come with a morphism of Huber pairs $(R, R^+) \to (R', R'^+)$, and whose morphisms are morphisms of Huber pairs. Likewise for $(R^\flat, R^{\flat+})$.

Proposition 6.0.1 *Tilting $(R', R'^+) \mapsto (R'^\flat, R'^{\flat+})$, and the continuous projection $x \mapsto x^\#$ induces an isomorphism*

$$R^{\flat+}/\varpi^\flat \simeq R^+/\varpi.$$

Moreover

$$R^{\flat+} = \varprojlim_{x \mapsto x^p} R^+.$$

Theorem 6.0.2 *Let (R, R^+) be a perfectoid pair, with tilt $(R^\flat, R^{\flat+})$. Define:*

$$X := \mathrm{Spa}(R, R^+) \quad \text{and} \quad X^\flat := \mathrm{Spa}(R^\flat, R^{\flat+}),$$

equipped with the presheaves $\mathscr{O}_X, \mathscr{O}_X^+$ and $\mathscr{O}_{X^\flat}, \mathscr{O}_{X^\flat}^+$ respectively.

(1) *We have a homeomorphism* $|X| \simeq |X^\flat|$ *given by sending* $x \in X$ *to the points* $x^\flat \in X^\flat$ *characterized by*

$$f \mapsto |f(x^\flat)| := |f^\#(x)|.$$

This homeomorphism identifies rational subsets in both directions. Moreover, the completed residue fields at x *and* $x^\flat$ *are perfectoid fields and are naturally tilts of each other (so in particular their value groups are naturally identified).*

(2) *Let* $U \subset X$ *be a rational subset, with tilt* $U^\flat \subset X^\flat$*. Then the complete Huber pair over* (R, R^+)*:*

$$(\mathscr{O}_X(U), \mathscr{O}_X^+(U))$$

is perfectoid, and its tilt is uniquely isomorphic to $(\mathscr{O}_{X^\flat}(U^\flat), \mathscr{O}_{X^\flat}^+(U^\flat))$ *functorially in* U*.*

(3) *The presheaves* $\mathscr{O}_X$ *and* $\mathscr{O}_{X^\flat}$ *are sheaves.*

(4) *The cohomology group* $H^i(X, \mathscr{O}_X^+)$ *is almost zero for all* $i > 0$*.*

We would like to know when the valuation on R sending $f \in R$ to $|f(x)|$, for $x \in X = \mathrm{Spa}(R, R^+)$, is close to being of the form $g \mapsto |g^\#(x)|$, because this last is supposed to induce the desired homeomorphism $X \simeq X^\flat$ by sending $x \mapsto x^\flat$. It turns out that one can find $g^\#$ approximating f, so that the two maps coincide on all but those points at which $|f(x)|$ and $|g^\#(x)|$ are small.

Lemma 6.0.3 (Approximation) *Let* $\mathcal{O} = R\langle T_0^{1/p^\infty}, \ldots, T_n^{1/p^\infty}\rangle$*. Let* $f \in \mathcal{O}^0$ *be a homogeneous element of degree* $d \in \mathbf{Z}[1/p]$*. Pick any rational number* $c \geq 0$ *and any* $\varepsilon > 0$*. Then there exists an element*

$$g_{c,\varepsilon} \in \mathcal{O}^{\flat 0} = R^{\flat 0}\langle T_0^{1/p^\infty}, \ldots, T_n^{1/p^\infty}\rangle$$

homogeneous and of the same degree d*, such that for all points* $x \in X = \mathrm{Spa}(\mathcal{O}, \mathcal{O}^0)$*, we have*

$$|f(x) - g_{c,\varepsilon}^\#(x)| \leq |\varpi|^{1-\varepsilon} \max(|f(x)|, |\varpi|^c).$$

As a first consequence we make precise the intuition discussed above. If $\varepsilon < 1$, it means that $|f(x)|$ and $|g_{c,\varepsilon}^\#(x)|$ are small, and we have, for all $x \in X = \mathrm{Spa}(\mathcal{O}, \mathcal{O}^0)$,

$$\max(|f(x)|, |\varpi|^c) = \max(|g_{c,\varepsilon}^\#(x)|, |\varpi|^c).$$

Use this we can prove any rational subsets of X has form $\flat^{-1}(U)$ for some rational subset U of $X^\flat$. To prove $\mathscr{O}_X$ is a sheaf, we use (2), So $\mathscr{O}_X(U)$ is uniform for any rational open subset. Recall all stably uniform tate rings are sheafy. For(4) first use the sheafy condition and Tate acyclicity we know the Cech complex is exact. Then we consider the the integral level and use Banach open mapping theorem.

Definition 6.0.4 A perfectoid space over $S = Spa(R, R^+)$ is an adic space which is locally isomorphic to an affinoid perfectoid S-space. Morphisms of perfectoid spaces over S are morphisms of adic spaces over S. For an affinoid perfectoid space X over S, tilting yelds an affinoid perfectoid space $X^\flat$ over $S^\flat$.

We give the following definition, to establish what we mean when we say a perfectoid space $X^\flat$ over $S^\flat$ is the tilt of a perfectoid space X over S.

Definition 6.0.5 We say a perfectoid space $X^\flat$ over $S^\flat$ is the tilt of a perfectoid space X over S if and only if the following natural bijection holds true for all perfectoid pairs over (R, R^+):

$$\mathrm{Hom}(\mathrm{Spa}(R', R'^+), X) \simeq \mathrm{Hom}(\mathrm{Spa}(R'^\flat, R'^{\flat+}), X^\flat).$$

As a formal consequence of the results in the preceding sections, we obtain the following:

Theorem 6.0.6 *Any perfectoid space X over S admits a tilt $X^\flat$ over $S^\flat$, unique up to unique isomorphism. This induces an equivalence between the categories of perfectoid spaces over S and perfectoid spaces over $S^\flat$. The underlying topological spaces $|X|$ and $|X^\flat|$ are naturally homeomorphic, the homeomorphism preserving rational subsets in both directions. A perfectoid space X over S is affinoid perfectoid if and only if $X^\flat$ is affinoid perfectoid over $S^\flat$. Finally, for any affinoid perfectoid subspace $U \subset X$, the pair:*

$$(\mathscr{O}_X(U), \mathscr{O}_X^+(U))$$

is a perfectoid pair with tilt

$$(\mathscr{O}_{X^\flat}(U^\flat), \mathscr{O}_{X^\flat}^+(U^\flat)).$$

Moreover:

$$U \simeq \mathrm{Spa}(\mathscr{O}_X(U), \mathscr{O}_X^+(U)) \quad \text{and} \quad U^\flat \simeq \mathrm{Spa}(\mathscr{O}_{X^\flat}(U^\flat), \mathscr{O}_{X^\flat}^+(U^\flat)).$$

So why we study perfectoid space? Since the rigid analytic spaces are always locally perfectoid in asuitable sense. For A is affinoid $X = Spa(A, A^0)$,then $\pi_1^t(X)$ is quite large and $H^i_{\text{ét}}(X, \mathscr{O}_X^{0a}) = 0$ has no finitness result. But for perfectoid Tate ring everything become pleasant.

7 Étale topology and Pro-étale topology

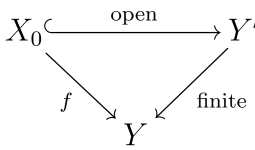
7.1 étale topology

First we recall some basic algebraic geometry:

Theorem 7.1.1 (Zariski's Main Theorem) *Suppose $f : X \to Y$ is a proper morphism of locally noetherian schemes.*

(1) *The set of points of X that are isolated in their fiber forms an open subset $X_0 \subset X$.*

(2) *The morphism $f|_{X_0} : X_0 \to Y$ factors into an open immersion followed by a finite morphism:*

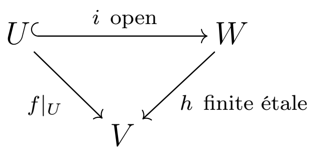


From now on,in this section, we assume all Huber rings are complete Tate and all spaces are adic.

Definition 7.1.2 A morphism $f : Spa(A, A^+) \to Spa(B, B^+)$ of affinoid adic space is finite étale if B is finite étale A algebra and B^+ is integral closure of A^+ in B. More generally,a morphism $f : X \to Y$ of adic spaces is finite étale if for all open affinoids $Spa(A, A^+) \subset Y$,$X \times_Y Spa(A, A^+) = Spa(B, B^+)$ is affinoid and $Spa(A, A^+) \to Spa(B, B^+)$is finite étale.

Definition 7.1.3 A morphism $f : X \to Y$ of locally noetherian adic spaces is called étale if for any point $x \in X$, there exist open neighbourhoods U and V of x and $f(x)$

respectively, and a factorization for $f|_U$



Definition 7.1.4

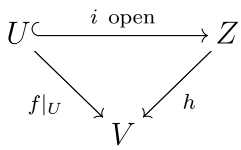
(1) A morphism of perfectoid pairs $(R', R'^+) \to (S, S^+)$ is called *strongly finite étale* if it is finite étale and, in addition, S^{0a} is a finite étale R'^{0a}-algebra.

(2) A morphism $f : X \to Y$ of perfectoid spaces is called strongly finite étale if there is a cover of Y by open affinoid perfectoid subspaces $V \subset Y$ such that the preimage $U := f^{-1}(V)$ is affinoid perfectoid, and the associated morphism of perfectoid pairs

$$(\mathscr{O}_Y(V), \mathscr{O}_Y^+(V)) \to (\mathscr{O}_X(U), \mathscr{O}_X^+(U))$$

is strongly finite étale.

(3) A morphism $f : X \to Y$ of perfectoid spaces is called *strongly étale* if for any point $x \in X$ there are open neighbourhoods U and V of x and $f(x)$ respectively, and a factorization for $f|_U$:



where h is strongly finite étale.

Remark 7.1.5 $f : X \to Y$ is strongly finite étale, resp. strongly étale, if and only if the tilt $f^\flat : X^\flat \to Y^\flat$ is

Theorem 7.1.6 (Almost Purity) *Let R' be a perfectoid R-algebra. For any finite étale cover S/R', S is perfectoid and S^{0a} is finite and étale over R'^{0a} in the sense of almost mathematics.*

to prove almost purity theorem,we need Gabber's henselian approximation method:

Lemma 7.1.7 *Let A be an R^0-algebra which is Henselian along (ϖ). Then the categories of finite étale $A[1/\varpi]$-algebras and $A^\wedge[1/\varpi]$-algebras are equivalent. $A^\wedge$ indicates the ϖ-adic completion of A.*

use this we get the following lemma:

Lemma 7.1.8 *Let (A_i) be a filtered direct system of complete R^0-algebras, and let A be the completion of the direct limit, which is again a complete R^0-algebra. Then we have an equivalence of categories:*

$$A[1/\varpi]_{\text{fét}} \simeq 2\text{-}\varinjlim A_i[1/\varpi]_{\text{fét}}.$$

In particular, if (R_i) is a filtered direct system of perfectoid R-algebras, and R' is the completion of the direct limit, then:

$$R'_{\text{fét}} \simeq 2\text{-}\varinjlim (R_i)_{\text{fét}}.$$

by this lemma,after few computation,we have

Lemma 7.1.9 *Fix $x \in X$. Then we have the following equivalence of categories:*

$$2\text{-}\varinjlim_{x\in U} \mathscr{O}_X(U)_{\text{fét}} \simeq \kappa(x)^{\wedge}_{\text{fét}}.$$

Now we also notice that The fully faithful functor $\kappa(x^\flat)^{\wedge}_{\text{fét}} \hookrightarrow \kappa(x)^{\wedge}_{\text{fét}}$ is an equivalence of categories. Finally, we use the fact that if $Y \to X$ is a strongly finite étale morphism of perfectoid spaces, for any affinoid perfectoid subspace $U \subset X$, its preimage $V \subset Y$ is affinoid perfectoid, and the morphism of Huber pairs over (R, R^+):

$$(\mathscr{O}_X(U), \mathscr{O}_X^+(U)) \to (\mathscr{O}_Y(V), \mathscr{O}_Y^+(V))$$

is strongly finite étale. Therefore, in particular, $\mathscr{O}_X(V)^{0a}_*[1/\varpi] = \mathscr{O}_X(V)$ is finite étale over $\mathscr{O}_X(U)$.

Theorem 7.1.10 *Let (R, R^+) be a perfectoid pair ,$X = Spa(R, R^+)$ with tilt $X^\flat$.*

1. *for any open affinoid perfectoid subspace $U \subset X$,we have a fully faithful functor*

$$\begin{Bmatrix} strongly\ finite\ tale\ cover \\ of U \end{Bmatrix} / \approx = \begin{Bmatrix} finite\ tale\ covers \\ of\ \mathscr{O}_X(U) \end{Bmatrix} / \approx .$$

by taking global sections

moreover,this functor is equivalence of categories.

2. *For any finite étale cover S/R, S is perfectoid and S^{0a} is finite and étale over R^{0a}. Moreover, S^{0a} is a uniformly almost finitely generated R^{0a}-module.*

We fix an affinoid perfectoid space $S := \mathrm{Spa}(R, R^+)$ and its tilt $S^\flat$.

We give the following:

Definition 7.1.11 Let X be a perfectoid space over S. The étale site of X is the category $X_{\text{ét}}$ of perfectoid spaces which are étale over X, and coverings are given by topological coverings. The associated topos is denoted by $X^\sim_{\text{ét}}$.

As soon as we have a morphism of perfectoid spaces $X \to Y$, we obtain an induced morphism of sites $X_{\text{ét}} \to Y_{\text{ét}}$, as well as of topoi.

Theorem 7.1.12 *the tilting operation not only induces a homeomorphism of topological spaces $|X| \simeq |X^\flat|$ which is functorial in X, but also an equivalence of sites:*

$$X_{\text{ét}} \simeq X^\flat_{\text{ét}}$$

which carries on to the étale topoi.

This is a key fact which turns out to be the full strength of the theory of perfectoid spaces, which will be made fruitful for the purpose of understanding p-adic étale cohomology of proper rigid-analytic varieties by the introduction of the pro-étale topology.

We conclude with a few vanishing results.

Proposition 7.1.13 *Let X be a perfectoid space over S. Then, for all $i > 0$, $H^i(X, \mathscr{O}_X^+)$ is almost zero. Moreover, the assignement*

$$U \mapsto \mathscr{O}_U(U)$$

is a sheaf on $X_{\text{ét}}$, and $H^i_{\text{ét}}(X, \mathscr{O}_X^{0a}) = 0$ for all $i > 0$ if X is affinoid perfectoid.

Proof. We prove the last statement, being the proof of the first identical. This is checked just proving exactness of the complex

$$(*) \quad 0 \to \mathscr{O}_X(X)^{0a} \to \prod_i \mathscr{O}_{V_i}(V_i)^{0a} \to \prod_{i,j} \mathscr{O}_{V_i \times_X V_j}(V_i \times_X V_j)^{0a} \to \cdots$$

for any finite covering $\{V_i\}$ of X, where each V_i is étale over X. The V_i's are rational subsets of some finite étale $V_i' \to U_i \subset X$, and U_i is a rational subspace. We are reduced to the case X is affinoid perfectoid and by tilting we reduce to the characteristic $p > 0$ case.

We first assume we know already $\mathscr{O}_X$ is an étale sheaf on X, and therefore the complex

$$0 \to \mathscr{O}_X(X) \to \prod_i \mathscr{O}_{V_i}(V_i) \to \prod_{i,j} \mathscr{O}_{V_i \times_X V_j}(V_i \times_X V_j) \to \cdots$$

is exact. We use the almost purity theorem and Banach open mapping theorem to deduce that a suitable power of ϖ kills the cohomology of $(*)$. By applying the inverse of the pth power map, we deduce such cohomology is almost zero, and we conclude.

We are left to show that $\mathscr{O}_X$ is a sheaf on $X_{\text{ét}}$. We reduce to the case X is affinoid perfectoid. For any étale cover $V \to X$, by the almost purity theorem we know V is again a perfectoid space, and then it makes sense to tilt it. By tilting, we reduce to checking exactness of the complex:

$$0 \to \mathscr{O}_X(X) \to \prod_i \mathscr{O}_{V_i}(V_i) \to \prod_{i,j} \mathscr{O}_{V_i \times_X V_j}(V_i \times_X V_j) \to \cdots$$

in characteristic $p > 0$, we reduce to the case X is a locally noetherian adic space over $\mathbf{F}_p((\varpi))(\varpi^{1/p^\infty})^\wedge$, for which $\mathscr{O}_X$ is indeed an étale sheaf. To prove the exactness we use some p-finite reduction method as in Scholze's original paper[Sch+12]. □

When we define the pro-étale topology on locally noetherian adic spaces, it will turn out that perfectoid spaces form a basis for such topology (in characteristic 0) and their well behavedness is due to the fact that they are contractible in the sense of almost mathematics, according to the above Proposition. with th same method we have

Proposition 7.1.14 *If X is affinoid perfectoid, $H^i_{\text{ét}}(X, \mathscr{O}_X^+/p)$ is almost zero for all $i > 0$*

7.2 Pro-étale topology

Definition 7.2.1 A morphism $f : X \to Y$ of perfectoid spaces is pro étale if locally on X it is of the form

$$Spa(A_\infty, A_\infty^+) \to Spa(A, A^+)$$

with

$$(A_\infty, A_\infty^+) = \widehat{\varinjlim_{i\in I}(A, A^+)}$$

where $\varinjlim_{i\in I}$ is a filtered direct limit and complection is taken with respect to the topology that makes $\varinjlim_{i\in I} A_i^+$ open and bounded,with A,A_i,A_∞ all perfectoid with each $Spa(A_i, A_i^+) \to Spa(A, A^+)$ étale.

Definition 7.2.2 (The big pro étale site) Let $\mathscr{P}erf$ denote the category of perfectoid spaces in characteristic p, We make this into a site by defining a pro étale covering to be a collection of pro étale morphism $f : X_i \to X_{i\in I}$ such that for all quasicompact opens$U \subset X$,there exists a finite subset $I_U \subset I$ and quasicompact opens $U_i \subset X_i$ such that $U = \bigcup_{i\in I_u} f_i(U_i)$

The main theorem we use to do computation in p-adic Hodge theory is

Proposition 7.2.3 *When X is locally noetherian analytic adic space over* $Spa(\mathbb{Q}_p, \mathbb{Z}_p)$ *then* X_{prot} *has a basis of affinoid perfectoid subsets.*

8 Diamond and untilting with a geometric explanation by Fontaine Fargues curves

8.1 Untilt of a Perfectoid field in char p

Definition 8.1.1 Let K be a perfectoid field of char p.An untilt of K is a pair $(K^\#, r)$,where $K^\#$ is perfectoid and $r : K \cong K^{\#\flat}$ is an isomorphism.

Given an untilt $(K^\#, r)$,the multiplication map $K^0 \to K^{\#\flat 0} \to K^{\#0}$ induces a surjective ring homomorphism

$$\theta_{K^\#} : W(K^0) \to K^{\#0}$$

$$\Sigma[f_n]p^n \mapsto \Sigma f_n^{\#} p^n$$

Where $ker\theta_{K\#}$ is an ideal which is primitive of degree 1.

Theorem 8.1.2 *The map* $I \to (W(K^0)/I)[\frac{1}{p}]$ *is a bijection of between the set of primitive ideal of* $W(K^0)$ *of degree 1,and the set of isomorphism classes of untilts of K.*

Definition 8.1.3 (The adic Fargues-Fontaine curve) $\mathscr{X}_K, \mathscr{Y}_K$ Let

$$\mathscr{Y}_K = Spa(W(K^0)) \setminus \{|p[\varpi]| = 0\}$$

where ϖ is a pseudo uniformizer of K.The Frob automorphism on K^0 induces a properly discontinous automorphism $\phi : \mathscr{Y}_K \to \mathscr{Y}_K$; *We let* $\mathscr{X}_K = \mathscr{Y}_K/\phi^{\mathbb{Z}}$

We claim that $\mathscr{Y}_K$ is covered by rational subsets of form

$$U(a,b) = \{|[\varpi^b]| \ \leq |p| \ \leq |[\varpi^a]|\} \subset Spa(W(K^0))$$

For an interval $I = [a,b]$ with endpoints lying in $\mathbb{Z}[\frac{1}{p}]_{>0}$, let $\mathscr{Y}_{K,I}$be the rational subset defined above, and let $B_{K,I} = H^0(\mathscr{Y}_{K,I}, \mathscr{O}_{K,I})$, finally let $B_K = \varprojlim_I B_{K,I}$

Theorem 8.1.4 *(kedlaya)* $B_{K,I}$ *is strongly noetherian. thus the adic curves are actually adic space.*

Theorem 8.1.5 *Suppose K=C is algebraic closed.Then there is a bijection between the set of closed maximal ideals of* B_C *and te set of characteristic 0 untilts of C, given by* $I \mapsto B_C/I$

Remark 8.1.6 We can also define a schematic Fargues Fontaine curve $\mathbb{X}_C$,and the set of frobenius-equivalence classes of char 0 untilts of C, is bijection with closed points of the scheme $\mathbb{X}_C$. For the detail see [FF]

8.2 Untilts of Perfectoid Space in char p

Let us now talk about a Sheaf on $\mathscr{P}erf$ with respect to pro-étale topology. If X is a perfectoid space of Char p, we have representable presheaf $h_X = Hom(X,Y)$ For the theory of diamond, it was developed in Scholze's Berkeley lecture.

Proposition 8.2.1 *[SW] The presheaf* h_X *is a sheaf.*

Now we degine the diamond.It is meant to mimic the notion of algebraic spaces,which is the quotient of a scheme by étale equivalence relation.For details,see [SW]

Definition 8.2.2 A diamond is a sheaf on $\mathscr{P}erf$ which is a quotient of perfectoid space by a pro-étale equivalence.

Definition 8.2.3 If X is a perfectoid space, let $X^{\diamond} = h_{X^{\flat}}$,this is a diamond. In the case $X = Spa(K)$ for a perfectoid field K,we also write $Spd(K) = X^{\diamond}$

Proposition 8.2.4 *We define* $Spd(\mathbb{Q}_p) = h_{Spa(\mathbb{Q}_p^{\flat})} = Spa(\mathbb{Q}_p^{cycl,\flat})/\underline{\mathbb{Z}_p^{\times}}$, *then it is a partially proper diamond.*

Theorem 8.2.5 *[SW17] There is an equivalence of categories between perfectoid spaces over* $\mathbb{Q}_p$ *,and a category of perfectoid spaces of char p together with a 'structure morphism'* $X^{\diamond} \to Spd(\mathbb{Q}_p)$.

Definition 8.2.6 Let X be an analytic adic space on which p is topological nilpotent.let X be the functor on $\mathscr{P}erf$ which sendds object S to the set of equivalence pairs$(S^{\#} \to X, r)$.where $S \to X$ is a perfectoid space fibered over X,and $r : S^{\#,\flat} \to S$ is an isomorphism.

Theorem 8.2.7 *[SW17]* $X^{\diamond}$ *is a diamond.*

Theorem 8.2.8 *(diamond formula) Let C be an algebraically closed perfectoid field of char p,*

$$\mathscr{Y}_C^{\diamond} \cong Spd(C) \times Spd(\mathbb{Q}_p)$$

9 An application in mod p langlands correspondence

In the last section, we see some application of perfectoid geometry in Langlands program.

9.1 l-adic cohomology and motiviation for mod p langlands correspondence

We assume $L = \mathbb{C}$ or $L = \bar{\mathbb{Q}}_l$. Let $p \neq l$ be a prime number and $F/\mathbb{Q}_p$ be a finite ring extension with ring of integers $\mathscr{O}_F$,uniformizer$\varphi$and residue field $\mathbb{F}_q$. Let$G_F = Gal(\bar{F}/F)$.The classical local langlands correspondence for $GL_n(F)$ is an injection

$$\begin{Bmatrix} continuous\ representation \\ \rho : G_F \to GL_n(L) \end{Bmatrix} / \approx = \begin{Bmatrix} irreducible\ smooth \\ L - representation\ of\ GL_n(L) \end{Bmatrix} / \approx .$$

$$\rho \longmapsto \pi(\rho)$$

which is chracterised br certain identities of $L-$ and $\epsilon-$factors.If we enlarge the left side with all frobenius semisimple Weil-Deligne representations ,this is a bijection. Now let D/F a division algebra with center F and invariant $\frac{1}{n}$ and let D^*be the group of unit of D. The the local Jacquet Langlands correspondence is an injection

$$\begin{Bmatrix} irreducible\ smooth\ L - representation \\ of\ D^* \end{Bmatrix} / \approx = \begin{Bmatrix} irreducible\ smooth \\ L - representation\ of\ GL_n(F) \end{Bmatrix} / \approx .$$

$$\pi \longmapsto JL(\pi)$$

We remark that objects on left side are finite dimensional.

Both correspondence can be realized by l-adic cohomology of Lubin-tate tower$\mathscr{M}$,which is a tower of deformation of p-divisible groups. lubin tate tower provide a nonabelian analogue to class field theory.For $l \neq p$ one can associate to $\mathscr{M}$ the l-adic étale cohomology

$$H_c^i(\mathscr{M}, \bar{\mathbb{Q}}_l) = \varinjlim_n H_c^i(\mathscr{M}_n, \bar{\mathbb{Q}}_l)$$

to prove this one needs Berkovich's theory of vanishing cycles attached to formal scheme.So the space $H_c^i(\mathscr{M}, \bar{\mathbb{Q}}_l)$ realize canonical correspondence between representations of all three groups act on Lubin Tate space.

Theorem 9.1.1 *(Harris-Taylor,2002) (roughly)*

$$Hom_G(\pi, H_c^*(\mathscr{M}, \bar{\mathbb{Q}}_l) = JL(\pi) \otimes rec(\pi)$$

So it's natural to ask is there some geometry and cohomology theories for the case $L = \bar{\mathbb{F}}_p$ or $L = \bar{\mathbb{Q}}_p$. Since Lubin tate Tower livers on category of adic spaces. we define the sheaf $\mathscr{F}_\pi$ onétale site $\mathbb{P}^{n-1}_{\mathbb{C}_p}$ and prove

Theorem 9.1.2 *For any admissible representation* π *of* $GL_n(F)$*,the cohomology groups*

$$\mathscr{S}^i := H^i_t(\mathbb{P}^{n-1}_{\mathbb{C}_p}, \mathscr{F}_\pi), i \geq 0$$

are admissibe D^* *representations.They carry an action of* G_F *and vanish for* $i > 2(n-1)$.

Lubin Tate tower is a tower $\mathscr{M}_n$ of rigid analytic varieties parametrizing deformation space of a p-divisible group with some level structure. For a full definition see [RZ] the inverse limit $\varprojlim \mathscr{M}_n$can be equipped with perfectoid space structure.
Let $F = \mathbb{Q}_p$,$k = \mathbb{F}_p$ and fix a p-divisible group H/k of dimension 1 and height n.Then$D = End(H)$ is a division algebra with center $\mathbb{Q}_p$ and invariant $\frac{1}{n}$.

Definition 9.1.3 Let $Nilp$ be the categroy of $W(k) - algebra$ R in which p is nilpotent.A deformation of H to $R \in Nilp$ is a pair (G, ρ) where G is a $p - divisible$ group over R and

$$\rho : H \otimes_k R/p \to G \otimes_R R/p$$

is a qusasi-isogeny
define the functor Def_H from $Nilp$ to $Sets$ send R to the equivalence class of pair (G, ρ).

Theorem 9.1.4 *The functor* Def_H *is representable by a formal scheme* $\mathfrak{X}/W(k)$*. we have a decomposition*

$$\mathfrak{X} \cong \bigsqcup_{i \in \mathbb{Z}} \mathfrak{X}^i$$

according to the height i *of the quasi-isogeny and non-canonically*

$$\mathfrak{X}^i \cong Spf(W(k)[[t_1, , , t_{n-1}]])$$

Define $\check{F} = F \otimes_{W(\mathbb{F}_q)} W(k)$ be the completion of the unramified extension of F with residue field k.Now let $\mathscr{M}_0 = \mathfrak{X}_\eta^{ad} \times_{\check{F}} \mathbb{C}_p$ be the dic generic fibre.One can introduce the level structures to get spaces $\mathscr{M}_m$ with finite étale map $\mathscr{M}_m \to \mathscr{M}_0$
Now we cite the main result in Scholze's paper moduli of p-divisible groups [SW]

Theorem 9.1.5 *there exista unique up to isomorphism perfectoid space $\mathscr{M}_\infty$ over $\mathbb{Q}_p$ such that*

$$\mathscr{M}_\infty \sim \varprojlim \mathscr{M}_m$$

The infinite lubin tate tower $\mathscr{M}_\infty$ has action of groups D^*, $GL_n(\mathbb{Q}_p)$ and the Weil group $W_{\mathbb{Q}p}$

9.2 Gross-Hopkins period map

Roughly speaking the gross-hopkins periods map is the quotient of $\mathscr{M}_\infty$ by $GL_n(F)$

Theorem 9.2.1 *The gross-hopkins map*

$$\pi_{GH} : \mathscr{M}_\infty \to \mathbb{P}^{n-1}_{\mathbb{C}_p}$$

is an étale surjective map of rigid analytic spaces.Each fibre consists of a single isogeny class of lifts on H.

Proposition 9.2.2 *$\pi_{GH} is equivariant for D^*$, $GL_n(\mathbb{Q}_p)$ action.And it factors through a corresponding map at all finite levels*

$$\pi_{GH,m} : \mathscr{M}_m \to \mathbb{P}^{n-1}_{\mathbb{C}_p}$$

and all these maps are étale covering.

Remark 9.2.3 $\mathbb{P}^{n-1}_{\mathbb{C}_p}$ is far from being simple connected.

9.3 Construction of Scholze's functor

first we notice that $H^i_t(\mathscr{M}_\infty, \mathbb{F}_p)$ depend on the choice of complete algebraic closure of $\check{F}$ and they are not admissible. Let π be an admissible representation of $GL_n(F)$ on

the $\mathbb{F}_p$-vector space.
For the étale map $U \to \mathbb{P}^{n-1}_{\mathbb{C}_p}$ define

$$\mathscr{F}_\pi(U) = \mathscr{C}^0_{GL_n(F)}(|U \times_{\mathbb{P}^{n-1}_{\mathbb{C}_p}} \mathscr{M}_\infty|, \pi)$$

Proposition 9.3.1 *$\mathscr{F}_\pi$is a sheaf on $(\mathbb{P}^{n-1}_{\mathbb{C}_p})_t$ and it's an exact functor.*

Theorem 9.3.2

$$\mathscr{S} : \left\{ \begin{matrix} smooth \\ \mathbb{F}_p - representation\ of\ GL_n(F) \end{matrix} \right\} / \approx = \left\{ \begin{matrix} smooth\ \mathbb{F}_p - representation \\ of\ D^* \end{matrix} \right\} / \approx .$$

$$\pi \longmapsto \mathscr{S}(\pi)$$

The space $\mathscr{S}(\pi)$ carries action of G_F

9.4 The main theorem

Theorem 9.4.1 *For any admissible representation π of $GL_n(F)$,the cohomology groups*

$$\mathscr{S}^i := H^i_t(\mathbb{P}^{n-1}_{\mathbb{C}_p}, \mathscr{F}_\pi), i \geq 0$$

are admissible D^* representations.They carry an action of G_F and vanish for $i > 2(n-1)$.

Proof. (sketch) Fix $K \subset D^*$ a compact open subgroup (shrink if necessary) to show $H^i_t(\mathbb{P}^{n-1}_{\mathbb{C}_p}/K, \mathscr{F}_\pi) = continous\ K - cohomology\ on\ R\Gamma(\mathbb{P}^{n-1}_{\mathbb{C}_p}, \mathscr{F}_\pi)$ is finite.
Use Falting's almost strategy,we reduce it to prove $H^i(\mathbb{P}^{n-1}_{\mathbb{C}_p}/K, \mathscr{F}_\pi \otimes \mathcal{O}^+/p)$ is almost finitely generated. One picks $K - stable$ affinoid covers

$$\mathbb{P}^{n-1}_{\mathbb{C}_p} = \bigcup_{i \in I} U_i = \bigcup_{i \in I} V_i$$

where I is finite index set and $U_i \subset\subset V_i$ stirctly. Since π_{GH} admits local sections $V_i \hookrightarrow \mathscr{M}_0$ we claim that image of the transition map

$$H^j(V_i/K, \mathscr{F}_\pi \otimes \mathcal{O}^+/p) \to H^j(U_i/K, \mathscr{F}_\pi \otimes \mathcal{O}^+/p)$$

is almost finitely generated.
We note that $\mathscr{F}_\pi|_{\mathscr{M}_0}$ only depends on $\pi|_{GL_n(\mathcal{O})}$ as constructed from $GL_n(\mathcal{O}) - torsor$ $\mathscr{M}_\infty \to \mathscr{M}_0$
Let $\pi_{reg} = \mathscr{C}^\infty(GL_n(\mathcal{O}), \mathbb{F}_p)$ we choose a resolution

$$0 \to \pi \to \pi_{reg}^{n_0} \to \pi_{reg}^{n_1} \to \pi_{reg}^{n_2} \to$$

This can be guaranteed by Lazard's theorem that $\mathbb{F}_p[[GL_n(\mathcal{O})]]$ is noetherian.Compute the spectral sequence we can reduce to the case $\pi_0 = \pi_{reg}$
consider the diagram

$$\begin{array}{ccc} \mathscr{M}_{LH,\infty,\mathbb{C}_p} & \longrightarrow & \mathscr{M}_{LH,0,\mathbb{C}_p} \\ \uparrow & & \uparrow \\ V_{i,\infty} & \longrightarrow & V_i \end{array}$$

And use falting's result

$$\mathscr{M}_{LH,\infty} \cong M_{Dr,\infty}$$

Then we have

$$\begin{array}{ccc} \mathscr{M}_{Dr,\infty,\mathbb{C}_p} & \longrightarrow & \mathscr{M}_{Dr,K,\mathbb{C}_p} \\ \uparrow & & \uparrow \\ V_{i,\infty} & \longrightarrow & V_{i,K} \end{array}$$

when we write$\mathscr{M}_{Dr,\infty} \sim \varprojlim \mathscr{M}_{Dr,K}$,we have equivalence of étale sites

$$\mathscr{M}_{Dr,\infty}/K \cong \mathscr{M}_{Dr,K} \quad V_{i,\infty}/K \cong V_{i,K}$$

So

$$H^j(V_i/K, \mathscr{F}_{\pi_{reg}} \otimes \mathcal{O}^+/p) \cong H^j(V_{i,\infty}/K, \mathcal{O}^+/p) \cong H^j(V_{i,K}, \mathcal{O}^+/p)$$

Sum up to show

$$H^j(V_{i,K}, \mathcal{O}^+/p) \to H^j(U_{i,K}, \mathcal{O}^+/p)$$

has almost finitely generated image. Since now

$$U_{i,K} \subset\subset V_{i,K} \ \subset \mathscr{M}_{Dr,K,\mathbb{C}_p}$$

Are usual smooth affinoid space over $\mathbb{C}_p$. So we can use p-adic hodge theory to calculate directly.

□

Remark 9.4.2 There is also some local-global compatibility result in p-adic cohomology of Lubin-Tate Tower. See [Sch15]

(Acknowlegement)
I am grateful to my thesis supervisor Prof. Ariane Mázard for introducing me such a wonderful topic and for innumerable helpful discussion and suggestion, Including content, writing method and some grammar correction.
I also thank Prof. Matthew Morrow for his course on perfectoid space and some discussion after class.

References

[BH] B. Bhatt, An introduction to Perfectoid Space
[BC] B. Conrad, Seminar on Perfectoid Spaces
[H1] R. Huber, Continuous valuations.
[HE] R. Huber, *Étale cohomology of rigid analytic varieties and adic spaces.*

[HA]R. Huber, *A generalization of formal schemes and rigid-analytic varieties.*

[BV] K. Buzzard, A. Verberkmoes, *Stably uniform affinoids are sheafy.*

[ST] A. J. de Jong, *Stacks Project.*

[VP]M. van der Put - *Cohomology on affinoid spaces.* Compositio Math. 45 (1982), 165 - 198.

[RZ] M. Rapoport, Th. Zink. Period spaces for p-divisible group.

[TW] T. Wedhorn, *Adic Spaces*, unpublished lecture notes.

[Sch15]P. Scholze On p-adic cohomology of Lubin-Tate Space

[SW13]P. Scholze,J.Weinstein Moduli of p-divisible groups.

[SW17]P. Scholze,J.Weinstein P-adic geometry,2017 version.

[Sch12]P. Scholze, *Perfectoid spaces.*

[Sch13]P. Scholze P-adic hodge theory for rigid analytic varieties.

[GR] O. Gabber, L. Ramero, *Almost Ring Theory.*

[Fo] J. - M. Fontaine, *Perfectoides, presque pureté et monodromie-poids.* Séminaire Bourbaki.
[FF] L.Fargues, J.-M.Fontaine,*Courbes et fibre vectoriels en theorie de hodge p-adique*