SYMPLECTIC TOPOLOGY AND PERSISTENCE MODULES

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ABSTRACT. We study topological properties of the group of Hamiltonian diffeomorphisms of certain closed surfaces by applying barcodes. Topological data analysts developed the concept of persistence modules and barcodes to study topological aspects of data. L. Polterovich and E. Shelukhin came up with an idea of bridging symplectic topology and topological data analysis which are apparently very different two fields and proved some fundamental results in symplectic topology. In this article, we roughly explain this idea without assuming any specific background in symplectic geometry.

CONTENTS

1. Introduction

Organization of the paper. In Section 2, we explain very roughly tools in symplectic topology that are used in this article such as (filtered) Floer homology.

In Section 3, we define barcodes and persistence modules.

In Section 4, we define Floer barcodes which is the most important object of this article.

In Section 5, we see how to apply barcodes to answer to questions in Hofer geometry.

2. Preliminaries in symplectic topology: Hofer geometry and Floer homology

In this section, we will explain basic notions of symplectic topology that will be used in the following sections.

2.1. **Symplectic manifolds.**

Definition 1. *(1) A symplectic form ω on a manifold M is a 2-form that is closed and non-degenerate.*
(2) A symplectic manifold (M, ω) is a pair of a smooth manifold M and a symplectic form ω.

Example.
   (1) Euclidean spaces: $(\mathbb{R}^{2n}, \omega_{st} := dx_1 \wedge dy_1 + dx_2 \wedge dy_2 + \cdots + dx_n \wedge dy_n)$
   (2) Cotangent bundles: $(T^*M, d\lambda)$ where λ is the Liouville form of the cotangent bundle.
   (3) Closed surfaces: (Σ, ω_{area}) where ω_{area} denotes the area-form on the surface Σ.

Next, we will introduce symplectic diffeomorphisms.

Definition 2. *(1) Assume (M, ω) and (N, η) are symplectic manifolds. A symplectomorphism $\phi : (M, \omega) \to (N, \eta)$ is a diffeomorphism between M and N satisfying $\phi^*\eta = \omega$.*
(2) Two symplectic manifolds (M, ω) and (N, η) are symplectomorphic if there exists a symplectomorphism between them.

Definition 3. *(1) Let (M, ω) be a symplectic manifold. A Hamiltonian is a time-dependent smooth function $S^1(= \mathbb{R}/\mathbb{Z}) \times M \to \mathbb{R}$.*
(2) The Hamiltonian vector field associated to the Hamiltonian H is the vector field X_{H_t} defined by the following: $\omega(X_{H_t}, \cdot) = -dH_t$.

    (3) The Hamiltonian flow ϕ_H^t of the Hamiltonian H is the flow of the Hamiltonian vector field X_{H_t}.

REMARK 4. We often call the time-one map of the Hamiltonian flow a Hamiltonian diffeomorphism.

Proposition 5. *Let (M, ω) be a symplectic manifold and H a Hamiltonian. The Hamiltonian flow of H is a symplectomorphism for all t.*

    The main subject of this paper is the study of the group of symplectomorphisms and its special subgroup called the group of Hamiltonian diffeomorphisms.

Definition 6. *Let (M, ω) be a symplectic manifold. We denote the group of symplectomorphisms on (M, ω) by $Symp(M, \omega)$.*

Proposition 7. *Let (M, ω) be a symplectic manifold. The set $Ham(M, \omega)$ of Hamiltonian diffeomorphisms on (M, ω) is a group (in particular, it is a subgroup of $Symp(M, \omega)$).*

Definition 8. *Let (M, ω) be a symplectic manifold. We denote the group of Hamiltonian diffeomorphisms on (M, ω) by $Ham(M, \omega)$.*

REMARK 9. We introduce following notations.
    (1) $Fix(\phi) := \{x \in M : \phi(x) = x\}$
    (2) $\mathcal{P}(H) := \{$time-1 periodic orbits of the Hamiltonian flow of $H\}$

REMARK 10. $Ham(M, \omega)$ is a subset of $Symp_0(M, \omega)$ where $Symp_0(M, \omega)$ denotes the set of elements in $Symp(M, \omega)$ belonging to the same path-connected component as the Id.

2.2. **Hofer geometry.** In Hofer geometry, we study topological aspects of the group $Ham(M, \omega)$ with respect to the Hofer metric d_H. We will define Hofer metric in this subsection.

Definition 11.   (1) Let $\phi \in Ham(M, \omega)$. Define the Hofer energy of ϕ by

$$E(\phi) := \inf_{H \in \mathcal{H}, \ \phi = \phi_H} [\int_0^1 \sup_{x \in M} H_t(x) - \inf_{x \in M} H_t(x) dt].$$

    (From now on, we will denote $\mathcal{E}(H) := \int_0^1 \{\sup_{x \in M} H_t(x) - \inf_{x \in M} H_t(x)\} dt$.)
    (2) Let $\phi, \psi \in Ham(M, \omega)$. Define d_H by $d_H(\phi, \psi) := E(\phi^{-1}\psi)$.

Surprisingly, this function d_H will turn out to be a distance (i.e. metric) on $Ham(M, \omega)$.

Theorem 12. *d_H defines a metric on $Ham(M, \omega)$.*

REMARK 13. From now on, we will refer this metric to the Hofer metric.

2.3. **Why do we study** *Ham***?** The following theorem tells us how much information the group Ham carries.

Theorem 14. *Let $(M_1, \omega_1), (M_2, \omega_2)$ be closed symplectic manifolds where M_1, M_2 are diffeomorphic manifolds.*

(1) *If $Symp(M_1, \omega_1)$ and $Symp(M_2, \omega_2)$ are isomorphic as a group, then $(M_1, \omega_1), (M_2, \omega_2)$ are symplectomorphic up to a constant i.e. there exists a diffeomorphism $f \in Diff(M_1, M_2)$ s.t. $f^*\omega_2 = c \cdot \omega_1$ for some constant c.*

(2) *If $Ham(M_1, \omega_1)$ and $Ham(M_2, \omega_2)$ are isomorphic as a group, then $(M_1, \omega_1), (M_2, \omega_2)$ are symplectomorphic up to a constant i.e. there exists a diffeomorphism $f \in Diff(M_1, M_2)$ s.t. $f^*\omega_2 = c \cdot \omega_1$ for some constant c.*

2.4. **Floer homology.** In this subsection, we sketch the construction of the Floer homology. Floer homology was introduced by A. Floer originally to solve the Arnold conjecture, stated in the 1950's, as an analogue of Morse theory. The conjecture has been one of the main landmarks in the study of symplectic geometry.

Definition 15. (1) *A Hamiltonian diffeomorphism ϕ is non-degenerate if for every $x \in Fix(\phi)$, $d\phi(x) : T_xM \to T_xM$ does not have 1 as an eigenvalue.*

(2) *A Hamiltonian H is non-degenerate if it generates a non-degenerate Hamiltonian diffeomorphism ϕ_H.*

Conjecture 16. *(The Arnold conjecture)*
If $H : S^1 \times M \to \mathbb{R}$ be a non-degenerate Hamiltonian, then

$$\#Fix(\phi_H) \geqslant \sum_k dim H_k(M; \mathbb{Z}/2).$$

In this subsection, we only consider aspherical symplectic manifolds which is defined as follows.

Definition 17. *A symplectic manifold (M, ω) is called aspherical (or symplectically aspherical) if the following two conditions are satisfied:*

(1) $\int_{S^2} w^*\omega = 0$ *for any map* $w : S^2 \to M$.

(2) $< c_1(TM), \pi_2(M) >= 0$ *where* $c_1(TM)$ *is the first Chern number of* (TM, J).

EXAMPLE. Closed surfaces of positive genus equipped with an area-form.

Why people thought of Morse theory when they think of the Arnold conjecture is that periodic orbits of a Hamiltonian H are exactly critical points of the action functional $\mathcal{A}_H$ defined on the set of contractible loops in M:

Definition 18. *Define the action functional* $\mathcal{A}_H : \mathcal{L}M \to \mathbb{R}$ *as the following where* $\mathcal{L}M$ *denotes the set of contractible loops in M:*

$$\mathcal{A}_H(z) := \int_0^1 H(t, z(t))dt - \int_{D^2} \overline{z}^*\omega$$

where $\overline{z} : D^2 \to M$ *denotes the capping of* $z : S^1 \to M$.

REMARK 19. The action functional is well-defined (i.e. it does not depend on the choice of the capping of z) thanks to the aspherical assumption.

Lemma 20. *For any* $X \in T_z\mathcal{L}M$, *we have*

$$d\mathcal{A}_H(z)(X) = \int_0^1 \omega(\dot{z} - X_H(z), X)dt.$$

Proof. Let $X \in z^*TM (= T_z\mathcal{L}M)$. Take $u : (-\epsilon, \epsilon) \times S^1 \to M$ s.t. $u(0, t) = z(t), \dfrac{\partial u}{\partial s}(0, t) = X(t)$. We can calculate

$$d\mathcal{A}_H(z)(X) = \frac{d}{ds}|_{s=0}\mathcal{A}_H(u(s, \cdot)) = \int_0^1 \omega(\dot{z} - X_H(z), X)dt.$$

$\square$

We construct the Floer chain complex out of periodic orbits. For simplicity, we consider $\mathbb{Z}/2-$coefficients.

Definition 21. *Let* H *be a non-degenerate Hamiltonian. We define the Floer chain complex by*

$$CF_k(H) := \bigoplus_{z\in\mathcal{P}(H),\mu_{CZ}(z)=k} \mathbb{Z}/2 \cdot z$$

REMARK 22. The grading of the Floer chain complex is given by the Conley-Zehnder index μ_{CZ} which assigns an integer to every non-degenerate fixed point of ϕ_H. Here we omit explanations. See [AD] for details.

As in Morse theory, we count numbers of anti-gradient flows connecting two periodic orbits.
First we need a bilinear positive definite map which corresponds to a Riemannian metric in Morse theory.

$$\forall X, Y \in z^*TM, \ \langle X, Y \rangle := \int_{S^1} \omega(X(t), J(z(t))Y(t))dt$$

Now, since

$$d\mathcal{A}_H(z)(X) = \int_0^1 \omega(\dot{z} - X_H(z), X)dt,$$

we obtain the following equation of the anti-gradient flow.

$$u : \mathbb{R} \to \mathcal{L}M, \ \frac{du}{ds}(= -grad\mathcal{A}_H(z)) = -J((\dot{z} - X_H(z)))$$

By regarding u as a map $u : \mathbb{R} \times S^1 \to M$, we can express this equation as follows.

$$\frac{\partial u}{\partial s} + J \circ u \frac{\partial u}{\partial t} + \nabla H_t \circ u = 0$$

This PDE is called the Floer equation.

Now, we count the number of these anti-gradient flows connecting two periodic orbits z_-, z_+.

Definition 23. *For two (contractible) periodic orbits $z_-, z_+ : S^1 \to M$,*

$$\mathcal{M}((H, J); z_-, z_+) := \{u : \mathbb{R} \times S^1 \to M : \frac{\partial u}{\partial s} + J \circ u \frac{\partial u}{\partial t} + \nabla H_t \circ u = 0,$$

$$u(s, \cdot) \to z_-(s \to -\infty), u(s, \cdot) \to z_+(s \to +\infty)\}.$$

Fredholm analysis of the Floer equation shows that the moduli space $\mathcal{M}((H, J); z_-, z_+)$ is a (Banach) manifold of dimension $\mu_{CZ}(z_-) - \mu_{CZ}(z_+)$ when (H, J) is *regular*. Here we will not explain what a pair (H, J) being *regular* means. See [AD].
Taking the $\mathbb{R}$-action on $\mathcal{M}((H, J); z_-, z_+)$ (i.e. $s' \cdot u(s, t) := u(s + s', t)$) into consideration, we think of the $\mathbb{R}$-quotient $\widehat{\mathcal{M}}((H, J); z_-, z_+) := \mathcal{M}((H, J); z_-, z_+)/\mathbb{R}$ which decreases the dimension by 1.
Thus, if $\mu_{CZ}(z_-) - \mu_{CZ}(z_+) = 1$, then $\widehat{\mathcal{M}}((H, J); z_-, z_+)$ is a 0-dimensional manifold. They turn out to be compact by additional arguments.

Definition 24. *For a periodic orbit $z(= \phi_H^t(x))$ of index k,*

$$\partial_{(H,J)}(z) := \sum_{w \in \mathcal{P}(H), \mu_{CZ}(w) = \mu_{CZ}(z)-1=k-1} \#\widehat{\mathcal{M}}((H,J); z, w) \cdot w$$

for every $k \in \mathbb{Z}$. We extend this definition linearly to $CF_k(H)$.

Theorem 25. *$(CF(H), \partial_{(H,J)})$ is a chain complex i.e. $\partial_{(H,J)} \circ \partial_{(H,J)} = 0$. We will refer it to the Floer chain complex.*

REMARK 26.

(1) The homology defined from Floer chain complex is called Floer homology and will be denoted by $HF_k(H, J)$.

(2) The construction of the Floer homology, more precisely the definition of the boundary map $\partial_{(H,J)}$, is dependent on the choice of ($\omega-$compatible) almost complex structure J. However, it turns out that, given ($\omega-$compatible) almost complex structures J, J' which makes $(H, J), (H, J')$ *regular*, although $\partial_{(H,J)} \neq \partial_{(H,J')}$, $HF_k(H, J), HF_k(H, J')$ are isomorphic i.e. Floer homology does not depend on the choice of ($\omega-$compatible) almost complex structure. Thus we will denote it by $HF_k(H)$.

(3) By modifying the assumptions on the base manifold (M, ω), we can construct Floer homologies out of non-contractible periodic orbits in the same method. We will denote them by $HF_k(H)_\alpha$ where $\alpha \in \pi_0(\mathcal{L}M)$. See [PS].

(4) Define the filtered Floer chain complex by

$$CF_k^\lambda(H) := \bigoplus_{z \in \mathcal{P}(H), \mu_{CZ}(z)=k, \mathcal{A}_H(z) \leqslant \lambda} \mathbb{Z}/2 \cdot z.$$

Since the boundary map $\partial_{(H,J)}$ is decreasing the action (i.e. $\mathcal{A}_H(\partial_{(H,J)}(z)) \leqslant \mathcal{A}_H(z)$), $(CF^\lambda(H), \partial_{(H,J)})$ is also a chain complex. We call it a filtered Floer chain complex and its homology $HF^\lambda(H, J)_\alpha$ a *filtered Floer homology*.

2.5. **Continuation maps: comparing Floer homologies.** We discuss briefly how to compare two Floer homologies $HF^\lambda(H)_\alpha, HF^\lambda(G)_\alpha$ where H, G are non-degenerate Hamiltonians. In order to compare, we take a homotopy of non-degenerate Hamiltonians $H_s : \mathbb{R} \times S^1 \times M \to \mathbb{R}$ s.t. $H_s = \begin{cases} H \ (s \leqslant -1) \\ G \ (1 \leqslant s) \end{cases}$ and a family of almost complex structure $\{J_s\}_{s \in \mathbb{R}}$ such that (H_s, J_s) is *regular* for all $s \in \mathbb{R}$. We denote this homotopy by $(\mathcal{H}, j) := \{(H_s, J_s)\}$.

Then the Floer equation will be

$$\frac{\partial u}{\partial s} + J_s \circ u \frac{\partial u}{\partial t} + \nabla H_s \circ u = 0.$$

Let u be a solution of a Floer equation such that $u \to x$ $(s \to -\infty)$, $u \to y(s \to \infty)$ where $x \in \mathcal{P}(H), y \in \mathcal{P}(G)$.

So we define the following continuation map on Floer chains:

$$c_{(\mathcal{H},j)} : CF_k(H) \to CF_k(G),$$

$$c_{(\mathcal{H},j)}(z_-) := \sum_{\mu_{CZ}(z_-) - \mu_{CZ}(z_+)=0} \#\mathcal{M}((\mathcal{H},j); z_-, z_+) \cdot z_-.$$

In fact, $C_{(\mathcal{H},j)}$ turns out to be a chain map i.e. $C_{(\mathcal{H},j)} \circ \partial_{(H,J)} = \partial_{(G,J')} \circ C_{(\mathcal{H},j)}$ so it defines a map between Floer homologies:

$$(C_{\mathcal{H},j})_* : HF^\lambda(H, J) \to HF^{\lambda + \mathcal{E}^+(G-H)}(G, J').$$

We call $(C_{\mathcal{H},j})_*$ a continuation map.

REMARK 27. The definition of the continuation map on Floer chains depends on the choice of homotopy $(\mathcal{H}, j)$. However, they all induce the same map on Floer homology. Thus, we denote the continuation map by $C_{H,G} := (C_{\mathcal{H},j})_*$ from now on.

Now, we look at a particular homotopy defined by $H_s(t, x) = (1 - \beta(s))H(t, x) + \beta(s)G(t, x)$ where $\beta : \mathbb{R} \to \mathbb{R}$ *s.t.* $\beta = 0$ *when* $s \leqslant -1$, $\beta = 1$ *when* $s \geqslant 1$, $\beta(s)$ *is monotonely increasing and smooth.* This satisfies, $H_s = H$ when $s \leqslant -1$ and $H_s = G$ when $s \geqslant 1$.

We look at the difference of the energy between x and y. In fact, by focusing on the energy of this pseudo-holomorphic cylinder u, we achieve

$$E(u) \leqslant \mathcal{A}_H(x) - \mathcal{A}_G(y) + \mathcal{E}^+(G - H).$$

$E(u) \geqslant 0$ so, we have $\mathcal{A}_G(y) \leqslant \mathcal{A}_H(x) + \mathcal{E}^+(G - H)$.

Thus, we can define a continuation map between filtered Floer homologies with respect to the homotopy $(\mathcal{H}, j) := (H_s, j_s)$ as follows.

$$C_{(\mathcal{H},j)} : HF_k^\lambda(H)_\alpha \to HF_k^{\lambda + \mathcal{E}^+(G-H)}(G)_\alpha.$$

Note that continuation maps on homology does not depend on the choice of a homotopy as mentioned earlier but this is not always true when we take the filtertion into consideration.

REMARK 28.

    (1) Continuation maps provides the isomorphism between Floer homologies $HF(H), HF(G)$.

(2) In particular, if f is a C^2-small Morse function, then $HF(f) \simeq HM(f) \simeq H(M)$ where $HM(f)$ is the Morse homology defined by f. This gives us the proof of the Arnold conjecture.

(3) This estimate plays an essential role in proving proposition 45.

2.6. **Spectral invariants.** We are now ready to define *spectral invariants* which is one of the most important tool in symplectic topology. (Ideas are due to [Vit],[Sch],[Oh].)

Definition 29. *Define the spectral invariant* $\rho : \mathcal{H} \times H_*(M) \to \mathbb{R}$ *by*

$$\rho(H, \alpha) := \inf\{\lambda : \alpha \in Im(i_\lambda)_*, (i_\lambda)_* : HF^\lambda_*(H, J) \to HF_*(H, J) \simeq H_*(M)\}$$

where the map $(i_\lambda)_*$ *is the map induced by the inclusion in the chain (of the chain complex) and the isomorphism between the Floer homology and the singular homology is the correspondence we have seen in the previous section.*

Spectral invariants has been applied to prove a lot of important results in symplectic topology such as the energy-capacity inequality, the existence of a symplectic capacity, the non-degeneracy of d_H (the Hofer metric), etc. See [Hum],[Oh] for further information.

3. Persistence modules and barcodes

3.1. **Persistence modules.**

Definition 30. *A persistence module is a family of finite dimensional* $\mathbb{K}$*-vector spaces* $(V_t)_{t \in \mathbb{R}}$ *such that*

(1) $V_t = 0$ *for small enough* t

(2) *For any* $s \leqslant t$, *there exists a morphism* $i_{s,t} : V_s \to V_t$ *s.t.* $i_{t,t} = id$, $i_{s,t} \circ i_{t,r} = i_{s,r}$ *for any* $s \leqslant t \leqslant r$.

(3) *There exist a finite set* $Spec(V) := \{a_1, a_2, \cdots, a_N\} \subset \mathbb{K}$ *s.t. if* $a_k < s < t \leqslant a_{k+1}$, *then* $i_{s,t} : V_s \to V_t$ *is an isomorphism.*

EXAMPLE. Let $I := (a, b]$ be an interval $(b \in (-\infty, +\infty])$. Define $Q(I)$ by

$$Q(I)_t := \begin{cases} \mathbb{K} \ (t \in I) \\ 0 \ (t \notin I) \end{cases} , i_{s,t} := \begin{cases} id \ (s, t \in I) \\ 0 \ (otherwise) \end{cases}$$

Definition 31. (1) *A morphism* A *between two persistence modules* V, W *is a family of maps* $\{A_t\}_{t \in \mathbb{R}}$ *that satisfies* $A_t \circ i_{s,t} = i_{s,t} \circ A_s$ *for any* $s \leqslant t$.

(2) *Persistence modules V, W are isomorphic if there exists morphisms $A : V \to W$ and $B : W \to V$ s.t. $A \circ B = id$, $B \circ A = id$.*

Theorem 32. *(The stability theorem)*
For any persistence module $(V_t)_{t \in \mathbb{R}}$, there exists a finite number of intervals $\{I_k\}_{k=1,2,\cdots,N}$ s.t. $V \simeq \bigoplus Q(I_k)$.

Definition 33.   *(1) We say that two persistence modules $V = (V_t)$ and $W = (W_t)$ are δ-interleaving if there exists morphisms $f = (f_t) : V \to W^\delta$ and $g = (g_t) : W \to V^\delta$ satisfying $g_t^\delta \circ f_t = i_{t,t+2\delta} : V \to V^{2\delta}$ and $f_t^\delta \circ g_t = j_{t,t+2\delta} : W \to W^{2\delta}$.*
   (2) For two persistence modules $V = (V_t)$ and $W = (W_t)$, we define
   $d_{int}(V, W) := \sup\{\delta \; : \; V \text{ and } W \text{ are } \delta - interleaving\}$
   and call it the interleaving distance between V and W.

3.2. **Barcodes.**

Definition 34. *A barcode is a finite collection of intervals $\{I_k\}_{k=1,2,\cdots,N}$ where $I_k := (a_k, b_k]$ $(a_k \in \mathbb{R}, b_k \in (-\infty, +\infty])$.*

Definition 35.   *(1) We say that two barcodes $B = \{I_k\}$ and $B' = \{I'_k\}$ are δ-matching if there exists a bijection $\rho : \{I_k \; : \; |I_k| \geqslant 2\delta\} \to \{I'_k \; : \; |I_k| \geqslant 2\delta\}$ s.t. $d(I_k, I'_{\rho(k)}) \leqslant \delta$.*
   (2) For two barcodes $B = \{I_k\}$ and $B' = \{I'_k\}$, we define
   $b_{bot}(B, B') := \inf\{\delta \; : \; B \text{ and } B' \text{ are } \delta - matching\}$
   and call it the bottleneck distance between B and B'.

Theorem 36. *(The isometry theorem)*
There exists an isometric map B from the set of persistence modules to the set of barcodes i.e. $d_{int}(V, W) = d_{bot}(B(V), B(W))$.

REMARK 37. B maps persistence modules to collections of intervals as in the stability theorem.

3.3. **Filtered Floer homology to barcodes.** In this subsection, we explain how to correspond a non-degenerate Hamiltonian diffeomorphism $\phi \in Ham(M, \omega)$ to a barcode. Throughout the section, we assume (M, ω) to be an aspherical symplectic manifold and α to be an arbitrary taken element of $\pi_0(\mathcal{L}M)$.

Proposition 38. *If two mean-normalized non-degenerate Hamiltonians H and G generates the same time-1 map (i.e. $\phi_H = \phi_G$), then $HF_k^\lambda(H)_\alpha = HF_k^\lambda(G)_\alpha$.*

REMARK 39. We can prove $HF_k^\lambda(H)_\alpha = HF_k^\lambda(G)_\alpha$ for any non-degenerate Hamiltonians H and G such that Hamiltonian paths generated by each Hamiltonian has the same homotopy type i.e. $\phi_H^t \sim \phi_G^t$

on a larger class of symplectic manifolds but proposition 38 uses the advantage of the symplectic manifold being symplectically aspherical.

Definition 40. *Let $\phi \in Ham(M, \omega)^{nondeg}$. Define the Floer barcode map $B_\alpha : Ham(M, \omega)^{nondeg} \to \mathcal{B}$ by $Bar(\phi) := Bar(HF^\lambda(H)_\alpha)$ where H is a mean-normalized non-degenerate Hamiltonian that generates ϕ.*

REMARK 41.
  (1) This is well-defined thanks to proposition 38.
  (2) Be careful that we take a mean-normalized Hamiltonian to define the barcode map.

Proposition 42. *The barcode map $B_\alpha : (Ham^{nondeg}(M, \omega), d_H) \to (\mathcal{B}, d_{bot})$ is 1-Lipschitz continuous. Here $Ham^{nondeg}(M, \omega)$ is the set of non-degenerate Hamiltonian diffeomorphisms and α is the homotopy type of the periodic orbits of the Floer homology.*

4. APPLICATION: BARCODES IN HOFER GEOMETRY

We introduce a theorem proven by L. Polterovich and E. Shelukhin in [PS]. In [PS], they deal with a larger class of symplectic manifolds but here we treat the easiest case for simplicity.

Theorem 43. *Let (Σ, σ) be a closed surface having genus $\geqslant 4$ equipped with an area form. Then,*
  (1) $aut(\Sigma, \sigma) := \sup\limits_{\phi \in Ham(\Sigma, \sigma)} d_H(\phi, Aut) = +\infty$.
  (2) $power_k(\Sigma, \sigma) := \sup\limits_{\phi \in Ham} d_H(\phi, Power_k) = +\infty$ for any $k \geqslant 2$
  where $Power_k := \{\phi \in Ham : \exists \psi \text{ s.t. } \phi = \psi^k\}$

REMARK 44.
  (1) The first result is a direct consequence of the second one since $Aut \subset Power_k$ for all $k \in \mathbb{N}$.
  (2) We only need to prove the second statement in the case where k is a prime since $Power_{kp} \subset Power_p$ for all $k \in \mathbb{N}$ (where p is a prime).
  (3) In this section, we denote $Ham^{p-nondeg} := \{\phi \in Ham : \phi^p \text{ is non-degenerate}\}$.

Closed surfaces of genus $\geqslant 2$ satisfies all the properties to define Floer homology $HF^\lambda(\phi)_\alpha$ for any $\phi \in Ham^{nondeg}(\Sigma, \sigma)$ and any homotopy type $\alpha \in \pi_0(\mathcal{L}M)$ (precisely, they are symplectically aspherical and atoroidal). Hence, we can apply discussions of the preceding sections.

One of the significance of $\phi \in Power_p$ is the p-periodicity of their fix points: If $\phi = \psi^p$ and $x \in Fix(\phi)$, then $\{x, \psi(x), \psi^2(x), \cdots, \psi^{p-1}(x)\}$ are all fixed points of ϕ too.

Especially, in this case we can define the rotation map $R_p(\psi) : z(t) \mapsto z(t + \frac{1}{p})$ on Floer homology $HF^\lambda(\phi)_\alpha$.

Authors of [PS] pushed this observation forward in the following way.

(1) Given a $\phi \in Ham^{p-nondeg}$, we obtain a $\mathbb{Z}_p$-persistence module $(HF^\lambda(\phi^p)_\alpha, R_p(\phi))$. (We take a p-power of ϕ to be able to define a rotation map.)

(2) Because of the p-periodicity due to the rotation map, each interval I consisting the barcode $B_\alpha(\phi^p)$ has multiplicity $m(B_\alpha(\phi^p), I) \equiv 0 \bmod p$. Thus, we take a $\zeta-$eigenspace $L_\zeta(\phi^p) := Ker(R_p(\phi) - \zeta \cdot Id)$ where ζ is a p-th root of 1. This enables us to treat each p-orbit $\{x, \phi(x), \phi^2(x), \cdots, \phi^{p-1}(x)\}$ as one element.

(3) If $\phi \in Power_p$ (i.e. $\phi = \psi^p$ for some ψ), then this defines another rotation map $R_{p^2}(\psi) : HF^\lambda(\phi^p)_\alpha \to HF^\lambda(\phi^p)_\alpha, z(t) \mapsto z(t + \frac{1}{p^2})$. This actually descends to a map on $L_\zeta(\phi^p)$ and this implies that each interval I consisting the barcode $B(L_\zeta(\phi^p))$ has multiplicity $m(L_\zeta(\phi^p), I) \equiv 0 \bmod p$.

(4) We define
$$\mu_\alpha(\phi) := \begin{cases} \sup\{\delta > 0 : m(L_\zeta(\phi^p), I) = m(L_\zeta(\phi^p), I^{2\delta}) \neq 0 \bmod p\} \\ 0 \text{ when } \forall I, \ m(L_\zeta(\phi^p), I) = 0 \end{cases}$$
From the observation in step (3), if $\phi \in Power_p$, then $\mu_\alpha(\phi) = 0$. In other words, for $\phi \in Ham^{nondeg}$, if $\mu_\alpha(\phi) > 0$, then $\phi \notin Power_p$.

Proposition 45. *Let $\phi, \psi \in Ham^{nondeg}$. We have*

$$|\mu_\alpha(\phi) - \mu_\alpha(\psi)| \leqslant p \cdot d_H(\phi, \psi).$$

REMARK 46. Proposition 45 enables us to define μ_α for any Hamiltonian diffeomorphism including the degenerate ones. In fact, Ham^{nondeg} is dense in Ham with respect to the Hofer metric d_H so define $\mu_\alpha(\phi) := \lim_{n\to+\infty} \mu_\alpha(\phi_n)$ where $\phi_n \in Ham^{nondeg}, d_H(\phi_n, \phi) \to 0$.

Since $Ham^{p-nondeg}$ is dense in Ham with respect to the Hofer metric d_H so we have the following property.

Proposition 47. $\mu_\alpha(\phi) = 0$ *for any $\phi \in Power_p$.*

Authors of [PS], constructed the following sequence which completes the proof of theorem 43. This is the only part in the proof where we use the assumption that the number of genus $\geqslant 4$.

Proposition 48. *Let Σ be a closed surface having genus $g \geqslant 4$. There exists a sequence $\{n_k\}_{k\in\mathbb{N}} \subset \mathbb{N}$ that is strictly increasing (i.e. tending towards infinity), a sequence $\{\phi_k\} \subset Ham(\Sigma, \omega)$ and a family $\alpha_k \in \pi_0(\mathcal{L}\Sigma)$ satisfying following properties:*

  (1) $(\phi_k)^p$ has exactly 2^{2p} p-tuples $\{x, \phi_k(x), (\phi_k)^2(x), (\phi_k)^3(x), \cdots, (\phi_k)^{p-1}(x)\}$ of fixed points of homotopy type α_k.

  (2) There exists two constants C_1 and C_2 satisfying the following: For any ϕ_k and for any two fixed points x and x' of ϕ_k^p which belongs to different p-tuples (i.e. $\{x, \phi_k(x), (\phi_k)^2(x), (\phi_k)^3(x), \cdots, (\phi_k)^{p-1}(x)\} \cap$

  $\{x', \phi_k(x'), (\phi_k)^2(x'), (\phi_k)^3(x'), \cdots, (\phi_k)^{p-1}(x')\} = \emptyset$), we can estimate their action difference by $|\mathcal{A}_H(x) - \mathcal{A}_H(x')| \geqslant C_1 \cdot k + C_2$.

We finish the proof of theorem 43 as follows.

Proof. For each k, focus on the interval $I_k := (m_{k,1}, m_{k,2}]$ where $m_{k,1}$ is the least action $\min_{x\in Fix(\phi_k)} \mathcal{A}_{H_k}(x)$ and $m_{k,2}$ the second least action. Recall that H_k is a mean-normalized Hamiltonian s.t. $\phi_k = \phi_{H_k}$. Barcodes $Bar(L_{\zeta,\alpha_k}(\phi_k))$ satisfy $m(Bar(L_{\zeta,\alpha_k}(\phi_k)), I_k) = m(Bar(L_{\zeta,\alpha_k}(\phi_k), I_k^{2\delta}) = 1 \bmod p$ for any k. Therefore, we have

$$d_H(\phi_k, Power_p) \geqslant \frac{1}{p}\cdot\mu_\alpha(\phi_k) \geqslant \frac{1}{p}\cdot\frac{1}{2}\cdot|I_k| = \frac{1}{2p}\cdot(m_{k,2}-m_{k,1}) \geqslant \frac{1}{2p}\cdot(C_1\cdot k+C_2) \to +\infty$$

as $k \to +\infty$. Thus, $power_p(\Sigma) = +\infty$. $\square$

5. WHY ARE BARCODES FASCINATING?

(1) As we mentioned already, spectral invariants had been one of the main tools in symplectic topology. Floer barcodes contain all the information of spectral invariants. In fact, from the point of view of barcodes, spectral invariants are birth points of $half-infinite$ intervals appearing in Floer barcodes. Thus, Floer barcodes contain more information than spectral invariants. Indeed, the result in [PS] explained in the previous section makes use of the $finite$ intervals.

(2) Spectral invariants are not suitable to study fixed points of non-contractible type: $HF^\lambda(H)_\alpha$ vanishes for λ large enough and $\alpha \neq pt$ and thus spectral invariants are always $-\infty$. On the

other hand, barcodes constructed from $HF^\lambda(H)_\alpha$ do have finite bars. Thus, there is a chance that we can obtain information of $Fix_\alpha(\phi)$.

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