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Mathematics

Blow-up for the critical generalized Korteweg-de Viries equation

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BLOW-UP FOR THE CRITICAL GENERALIZED KORTEWEG-DE VIRIES EQUATION

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1. Introduction

1.1. L^2 **criticality for NLS.** The study of nonlinear dispersive[1] equations has attracted considerable attention recently. One of the most studied and physically significant is the nonlinear Schrödinger equation (NLS), which arises in the mathematical description of electro-magnetic wave propagation through nonlinear media. The focusing[2] nonlinear Schrödinger equation is written as follows

$$(NLS) \quad \begin{cases} iu_t = -\Delta u - |u|^{p-1}u, & (t,x) \in [0,T) \times \mathbb{R}^N, \\ u(0,x) = u_0(x), & u_0(x) : \mathbb{R}^N \to \mathbb{C}. \end{cases} \tag{1.1}$$

where t and x are time and space variable respectively, and u_t is the partial derivative with respect to t. It has the space-time translation symmetry as well as scaling symmetry: if $u(t,x)$ is a solution to NLS, then so is

$$u_\lambda(t,x) = \lambda^{\frac{2}{p-1}} u(\lambda^2 t, \lambda x) \tag{1.2}$$

for any $\lambda > 0$.

Assume the nonlinearity satisfies

$$1 < p < 2^* - 1 \quad \text{with} \quad 2^* = \begin{cases} +\infty & \text{for} \quad N = 1,2 \\ \frac{2N}{N-2} & \text{for} \quad N \geq 3 \end{cases} \tag{1.3}$$

the Cauchy problem is locally well posed in the energy space: for any $u_0(x) \in H^1(\mathbb{R}^N)$, there exits a unique maximal solution $u(t)$ of (1.1) in $C([0,T), H^1)$, and

$$T < +\infty \text{ implies } \lim_{t \to T} \|\nabla u(t)\|_{L^2} = +\infty.$$

Besides, the mass and energy of NLS are conserved by the flow: $\forall t \in [0,T)$,

$$M(u(t)) = \int |u(t)|^2 = M(u_0), \tag{1.4}$$

$$E(u(t)) = \frac{1}{2}\int |\nabla u(t)|^2 - \frac{1}{p+1}\int |u(t)|^{p+1} = E(u_0). \tag{1.5}$$

Moreover, the Gagliardo-Nirenberg-Sobolev interpolation estimate

$$\forall v \in H^1, \quad \int |v|^{p+1} \leq C(N,p) \left(\int |\nabla v|^2\right)^{\frac{N(p-1)}{4}} \left(\int |v|^2\right)^{\frac{N-2}{4}(2^*-1-p)} \tag{1.6}$$

together with the energy conservation implies global existence in the so-called subcritical cases:

[1]Dispersive equation is the equation $\partial_t u + Lu = 0$ which has the wave-type solution $e^{i(x\cdot\xi + h(\xi)t)}$, where $h(\xi)$ depends nonlinearly on ξ. That is to say, different frequencies have different propagation speeds.

[2]Focusing means same sign for diffusion and nonlinearity. If they are of different signs, the energy (1.5) is positive definite and therefore blow-up phenomena would not occur.

Theorem 1.1 (Global existence or blow-up in finite time [11]). *Assume p satisfies* (1.3),
(i), *if $p < 1 + \frac{4}{N}$ (subcritical), then for every $u_0 \in H^1$, the solution for the Cauchy problem* (1.1) *is global and bounded in H^1;*
(ii), *if $p = 1 + \frac{4}{N}$ (critical) or $p > 1 + \frac{4}{N}$ (supercritical), then the virial law* [3] *allows for the existence of finite time blow-up solutions, with negative energy.*

Remark. (i) In the linear case, formally the solutions disperse, as indicated by the *dispersive inequality*

$$\|e^{it\Delta}u_0\|_{L^{p'}} \leq C\frac{1}{|t|^{d\left(\frac{1}{p}-\frac{1}{2}\right)}}\|u_0\|_{L^p} \quad \text{for all } 1 \leq p \leq 2.$$

And this theorem states that: when the nonlinearity is sufficiently strong $p \geq 1+\frac{4}{N}$, the focusing phenomena introduced by the nonlinear media would dominate the linear dispersive effect, and lead to blow-up.
(ii) It is worth mentioning that for $p = 1 + \frac{4}{N}$, the scaling symmetry (1.2) leaves the L^2 norm invariant, that is the so-called L^2 criticality.

In between the dispersion effect and blow-up, there is a nontrivial solution which propagates without deformation (called the solitary wave), i.e. the solution of the form

$$u(t,x) = e^{it}Q(x) \tag{1.7}$$

where Q satisfies the elliptic equation

$$\Delta Q - Q + |Q|^{p-1}Q = 0, \quad Q \in H^1. \tag{1.8}$$

By elliptic PDE theories, there exists a unique solution of (1.8) which is positive and radially symmetric, called the ground state for NLS. Furthermore, $Q(x)$ is strictly positive, smooth and exponentially decay in infinity. Based on concentration compactness argument[4] T.Cazenave and P.-L.Lions [1] proved in the subcritical case, the solitary wave is (orbital) stable for NLS; but it is unstable by blow-up and scattering when $p \geq 1 + \frac{4}{N}$: take an arbitrarily small neighborhood of Q as the initial data, there always contain finite time blow-up solutions as well as global solutions which scatter as $t \to +\infty$, i.e. we have

$$\|u(t) - e^{it\Delta}u_+\|_{H^1} \to 0 \quad \text{as } t \to +\infty$$

for some $u_+ \in H^1$.

1.2. **Minimal mass blow-up for critical NLS problem.** Now we focus on the L^2 critical NLS problem, i.e. when $p = 1 + \frac{4}{N}$. In this case, the ground state $Q(x)$ satisfies $E(Q) = 0$ and that the Gagliardo-Nirenberg-Sobolev inequality (1.6) is an equality, hence gives the constant $C(N,p)$ for this sharp (or, optimal) inequality:

$$\forall v \in H^1, \quad \frac{1}{p+1}\int |v|^{p+1} \leq \left(\frac{\int |v|^2}{\int |Q|^2}\right)^{\frac{2}{N}} \left(\frac{1}{2}\int |\nabla v|^2\right).$$

This led to the first derivation of a criterion of global existence for critical NLS by Weinstein [12]. In particular, H^1 initial data with $\|u_0\|_{L^2} < \|Q\|_{L^2}$ generate

[3] If $s_c = \frac{N}{2} - \frac{2}{p-1} \geq 0$ and $xu_0 \in L^2$, then

$$\frac{d^2}{dt^2}\int |x|^2|u(t,x)|^2 dx = 4N(p-1)E(u_0) - \frac{16s_c}{N-2s_c}\int |\nabla u|^2 \leq 4N(p-1)E(u_0).$$

[4] to resolve the loss of compactness of the injection $H^1 \hookrightarrow L^{p+1}$.

a unique global solution $u \in \mathcal{C}([0,+\infty), H^1)$. In the case of NLS, this criterion is sharp as a consequence of the pseudo-conformal transformation (only in the L^2 critical case): if $u(t,x)$ is a solution to NLS, then so is

$$v(t,x) = \frac{1}{|t|^{\frac{N}{2}}} u(-\frac{1}{t}, \frac{x}{t}) e^{i\frac{|x|^2}{4t}}.$$

Applied to the solitary wave solution $u(t,x) = e^{it}Q(x)$, one gets the explicit minimal mass blow-up solution

$$S_{NLS}(t,x) = \frac{1}{|t|^{\frac{N}{2}}} Q\left(\frac{x}{t}\right) e^{i\frac{|x|^2}{4t} - \frac{i}{t}}, \quad \|S_{NLS}(t)\|_{L^2} = \|Q\|_{L^2}.$$

The dynamics generated by the smooth data $S_{NLS}(1)$ is: $S_{NLS}(t)$ scatters as $t \to +\infty$, and blows up as $t \downarrow 0$ with

$$\|\nabla S_{NLS}(t)\|_{L^2} \sim \frac{1}{|t|},$$

$$|S_{NLS}(t)|^2 \rightharpoonup \|Q\|_{L^2}^2 \delta_{x=0} \quad \text{as} \quad t \downarrow 0.$$

Furthermore, this minimal mass blow-up solution is unique:

Theorem 1.2 (Classification of minimal element [9]). *Let $u_0 \in H^1$ such that $\|u_0\|_{L^2} = \|Q\|_{L^2}$, assume that the corresponding solution to NLS blows up in finite time $0 < T < +\infty$, then*

$$u(t) = S_{NLS}(t)$$

up to the symmetries of NLS.

1.3. **Critical gKdV equation and existence of minimal element.** The generalized Korteweg-de Vries equation (gKdV), which is a model of waves on shallow water surface, is a canonical problem similar to (focusing) NLS. Especially, the one-dimensional L^2 critical gKdV is

$$\text{(gKdV)} \quad \begin{cases} u_t + (u_{xx} + u^5)_x = 0, & (t,x) \in [0,T) \times \mathbb{R}, \\ u(0,x) = u_0(x), & x \in \mathbb{R}. \end{cases} \tag{1.9}$$

and the scaling symmetry

$$u_\lambda(t,x) = \lambda^{\frac{1}{2}} u(\lambda^3 t, \lambda x), \quad \lambda > 0$$

leaves the L^2 norm invariant. In the energy space H^1 the local well-posedness for gKdV is similar to NLS, with the mass and energy conservation: $\forall t \in [0,T)$,

$$M(u(t)) = \int u^2(t) = M(u_0), \quad E(u(t)) = \frac{1}{2}\int u_x^2(t) - \frac{1}{6}\int u^6(t) = E(u_0),$$

The solitary wave solution to gKdV is

$$u(t,x) = \frac{1}{\lambda_0^{\frac{1}{2}}} Q\left(\frac{x - \lambda_0^{-2}t - x_0}{\lambda_0}\right), \quad (\lambda_0, x_0) \in \mathbb{R}_+^* \times \mathbb{R},$$

with the ground state as (the same as one-dimensional NLS)

$$Q(x) = \left(\frac{3}{\cosh^2(2x)}\right)^{\frac{1}{4}}, \quad Q'' + Q^5 = Q, \quad E(Q) = 0. \tag{1.10}$$

The existence of blow-up solutions [4] have been obtained for a class of initial data u_0 satisfying

$$\int Q^2 < \int u_0^2 \leq \int Q^2 + \alpha, \quad \text{and} \quad E(u_0) < 0$$

for some small constant $\alpha > 0$[5]. However, the existence of the minimal mass blow-up solution in this setting has been a long standing open problem, due to the fact that the existence of minimal element for NLS relies entirely on the exceptional pseudo-conformal symmetry. In 2012 Martel, Merle and Raphaël [8] proved the existence and uniqueness of minimal element for gKdV:

Theorem 1.3 (Existence and uniqueness of the minimal mass blow-up solution). (i) Existence. *There exists a solution* $S(t) \in \mathcal{C}((0,+\infty), H^1)$ *to* (1.9) *with minimal mass* $\|S(t)\|_{L^2} = \|Q\|_{L^2}$ *which blows up backward at the origin:*

$$S(t,x) - \frac{1}{t^{\frac{1}{2}}} Q\left(\frac{x + \frac{1}{t} + \bar{c}t}{t}\right) \to 0 \quad in \quad L^2 \quad as \quad t \downarrow 0 \tag{1.11}$$

at the speed

$$\|S(t)\|_{H^1} \sim \frac{1}{t} \quad as \quad t \downarrow 0 \tag{1.12}$$

for some universal constant $\bar{c}$.
Moreover, S *is smooth and well localized to the right in space:*

$$\forall x \geq 1, \quad S(1,x) \leq e^{-Cx}. \tag{1.13}$$

(ii) Uniqueness. *Let* $u_0 \in H^1$ *with* $\|u_0\|_{L^2} = \|Q\|_{L^2}$ *and assume that the corresponding solution* $u(t)$ *to* (1.9) *blows up in finite time. Then*

$$u(t) \equiv S(t)$$

up to the symmetries of gKdV.

Remark. (i) Blow-up point. Recall that

$$|S_{NLS}(t)|^2 \rightharpoonup \|Q\|_{L^2}^2 \delta_{x=0} \quad \text{as} \quad t \downarrow 0.$$

In the case of gKdV, now we have

$$S(t,x) - \frac{1}{t^{\frac{1}{2}}} Q\left(\frac{x + \frac{1}{t} + \bar{c}t}{t}\right) \to 0 \quad \text{in} \quad L^2 \quad \text{as} \quad t \downarrow 0,$$

that is to say: $S(t)$ moves to the left as well as focuses, and will blow-up at $-\infty$ in space. The loss of compactness of blow-up point means the loss of spacial symmetry for the blow-up profile of gKdV, hence the left and right tail are of different significance to our analysis.

(ii) The tail of $S(t)$. (1.13) means that $S(t)$ has exponentially decay on the right. However in contrast with $S_{NLS}(t)$, $S(t)$ can not decay too fast on the left. In fact, let $u_0 \in H^1$ with $\|u_0\|_{L^2} = \|Q\|_{L^2}$ and

$$\sup_{x_0>0} x_0^3 \int_{x>x_0} u_0^2(x)dx < +\infty, \tag{1.14}$$

then the corresponding solution $u(t)$ of gKdV is global for $t > 0$ [5]. Since $S(t)$ blows up backward, it can not satisfy (1.14) on the left. In other words, in spite of the convergence (1.11), the left tail of $S(t)$ might still display some oscillations, and thus the convergence is not in H^1.

[5]Note that this condition is not empty due to the variational characterization of the ground state Q.

The construction of S is based on the characterization of solutions with initial data near soliton. More precisely, let $\mathcal{A}$ to be the set of initial data

$$\mathcal{A} = \left\{ u_0 = Q + \varepsilon_0 \text{ with } \|\varepsilon_0\|_{H^1} < \alpha_0 \text{ and } \int_{y>0} y^{10} \varepsilon_0^2 < 1 \right\},$$

and consider the L^2 tube around the manifold of solitary waves

$$\mathcal{T}_{\alpha^*} = \left\{ u \in H^1 \quad \text{with} \quad \inf_{\lambda_0 > 0,\, x_0 \in \mathbb{R}} \left\| u - \frac{1}{\lambda_0^{\frac{1}{2}}} Q\left(\frac{\cdot - x_0}{\lambda_0} \right) \right\|_{L^2} < \alpha^* \right\}. \tag{1.15}$$

Theorem 1.4 (Characterization of the flow in $\mathcal{A}$ [7])**.** *Let $0 < \alpha_0 \ll \alpha^* \ll 1$ and $u_0 \in \mathcal{A}$. Let $u \in \mathcal{C}([0,T), H^1)$ be the corresponding solution of gKdV. Then, one of the following three scenarios occurs:*
(Blow up)*: the solution blows up in finite time $0 < T < +\infty$ with the speed*

$$\|u(t)\|_{H^1} = \frac{\ell(u_0) + o(1)}{T - t} \quad \text{as} \quad t \to T, \quad \ell(u_0) > 0. \tag{1.16}$$

(Soliton)*: the solution is global $T = +\infty$ and converges asymptotically to a solitary wave: there exist $\lambda_\infty > 0$ and $x(t)$ such that*

$$\lambda_\infty^{\frac{1}{2}} u(t, \lambda_\infty \cdot + x(t)) \to Q \quad \text{in } H^1_{\text{loc}} \text{ as } t \to +\infty, \tag{1.17}$$
$$|\lambda_\infty - 1| \leq \delta(\alpha_0), \quad x(t) \sim \frac{t}{\lambda_\infty^2} \quad \text{as} \quad t \to +\infty.$$

(Defocusing and Exit)*: the solution leaves the tube $\mathcal{T}_{\alpha^*}$ at some time $0 < t^* < T \leq +\infty$, and at the exit time the solution is defocused:*

$$\|u_x(t)\|_{L^2} \leq C(\alpha^*)\delta(\alpha_0),$$

where $\delta(\alpha_0)$ is a generic small constant with $\delta(\alpha_0) \to 0$ as $\alpha_0 \to 0$.

Remark. Once the existence and uniqueness of minimal element $S(t)$ is established, it can be proved that in fact, the forward flow of $S(t)$ is an attractor to all the defocused solutions as in the scenario (Defocusing and Exit) [6], thus it is interesting to study the behavior of $S(t)$.

2. Nonlinear profiles and geometrical decomposition near soliton

Notation. We introduce the linearized operator close to Q

$$Lf = -f'' + f - 5Q^4 f, \tag{2.1}$$

and the generator of L^2 scaling

$$\Lambda f(y) = \frac{1}{2} f(y) + y f'(y). \tag{2.2}$$

The self-adjoint operator L is coercive: there exists $\mu_0 > 0$ such that for all $f \in H^1$,

$$(Lf, f) \geq \mu_0 \|f\|_{H^1}^2 - \frac{1}{\mu_0} \left[(f, Q)^2 + (f, y\Lambda Q)^2 + (f, \Lambda Q)^2 \right]. \tag{2.3}$$

[6]Let $u(t,x)$ be a solution of gKdV corresponding to the *(Defocusing and Exit)* scenario in Theorem 1.4 and let $t_u^* \gg 1$ be the corresponding exit time. Then there exist $\tau^* = \tau^*(\alpha^*)$ (independent of u) and (λ_u^*, x_u^*) such that

$$\left\| (\lambda_u^*)^{\frac{1}{2}} u\left(t_u^*, \lambda_u^* x + x_u^*\right) - S(\tau^*, x) \right\|_{L^2} \leq \delta(\alpha_0).$$

For a given small constant $0 < \alpha^* \ll 1$, $\delta(\alpha^*)$ denotes a generic small constant with

$$\delta(\alpha^*) \to 0 \quad \text{as} \quad \alpha^* \to 0.$$

2.1. **Formal derivation of the dynamics of modulation parameters.**
Now focus on the flow of gKdV close to the solitary waves. Formally, we search for a solution of the form

$$u(t,x) = \frac{1}{\lambda^{\frac{1}{2}}(t)} Q_{b(t)}\left(\frac{x - x(t)}{\lambda(t)}\right), \tag{2.4}$$

with the expansion with respect to a small parameter b

$$Q_b = Q + bP + b^2 P_2 + \cdots .$$

Intuitively, $x(t)$ is the position of the main profile (recall that Q is even and exponentially decaying at infinity); $\lambda(t)$ is the scaling of the profile, for example $\lambda(t) \to 0$ is equivalent to blow-up; and $b(t)$ is the small modulation of the profile near the ground state Q, so that the profile is a better approximate to the solution.

Apply it to the equation (1.9), and renormalize as

$$\frac{ds}{dt} = \frac{1}{\lambda^3}, \quad \text{or equivalently } s = \int_0^t \frac{dt'}{\lambda^3(t')},$$

we obtain an equation on the modulation parameters:

$$b_s \frac{\partial Q_b}{\partial b} + b\Lambda Q_b - (-Q_b'' + Q_b' - Q_b^5)' - (\frac{\lambda_s}{\lambda} + b)\Lambda Q_b - (\frac{x_s}{\lambda} - 1)Q_b' = 0. \tag{2.5}$$

Since

$$\Lambda Q_b = \Lambda Q + b\Lambda P + O\left(b^2\right),$$
$$-Q_b'' + Q_b' - Q_b^5 = LQ + bLP + O\left(b^2\right) = bLP + O\left(b^2\right),$$

we let

$$\frac{\lambda_s}{\lambda} + b = 0, \quad \frac{x_s}{\lambda} - 1 = 0, \quad b_s \sim O\left(b^2\right). \tag{2.6}$$

Therefore we get the order b solution of (2.4) is

$$(LP)' = \Lambda Q. \tag{2.7}$$

In this way, the study of the flow is turned into the study of the 3 modulation parameters driven by finite dimensional ODEs (2.6). For example, the study of blow-up corresponds to the form of singularity due to $\lambda \to 0$.

2.2. **Nonlinear profile Q_b.** Define $\mathcal{Y}$ to be the set of functions $f \in C^\infty(\mathbb{R}, \mathbb{R})$ satisfying

$$\forall k \in \mathbb{N},\ \exists C_k,\ r_k > 0,\ \forall y \in \mathbb{R}, \quad |f^{(k)}(y)| \le C_k (1 + |y|)^{r_k} e^{-|y|}.$$

From the property of the operator L, we have the following proposition:

Proposition 2.1. *There exists a unique smooth function P such that $P' \in \mathcal{Y}$ [7] and*

$$(LP)' = \Lambda Q, \quad \lim_{y\to-\infty} P(y) = \frac{1}{2}\int Q, \quad \lim_{y\to+\infty} P(y) = 0, \tag{2.8}$$

[7] $\mathcal{Y}$ is the set of functions $f \in C^\infty(\mathbb{R}, \mathbb{R})$ satisfying

$$\forall k \in \mathbb{N},\ \exists C_k,\ r_k > 0,\ \forall y \in \mathbb{R}, \quad |f^{(k)}(y)| \le C_k (1 + |y|)^{r_k} e^{-|y|}.$$

$$(P,Q)=\frac{1}{16}\left(\int Q\right)^2>0,\quad (P,Q')=0. \tag{2.9}$$

Note that P has a non trivial tail on the left, hence we proceed to a localization: take a cut-off function $\chi\in\mathcal{C}^\infty(\mathbb{R})$ be such that $0\le\chi\le 1$, $\chi'\ge 0$ on $\mathbb{R}$, $\chi\equiv 1$ on $[-1,+\infty)$, $\chi\equiv 0$ on $(-\infty,-2]$. Take $\gamma\in(2/3,1)$ ($\gamma=3/4$ for example), and define the localized profile:

$$Q_b(y)=Q(y)+b\chi_b(y)P(y),\quad\text{with}\quad \chi_b(y)=\chi\left(|b|^\gamma y\right). \tag{2.10}$$

Lemma 2.2. *For $|b|<b^*$ small enough, there hold:*

$$\left|\int Q_b^2-\left(\int Q^2+2b\int PQ\right)\right|\lesssim |b|^{2-\gamma}, \tag{2.11}$$

$$\left|E(Q_b)+b\int PQ\right|\lesssim b^2. \tag{2.12}$$

2.3. **Decomposition and modulation parameters.** Assume $u\in\mathcal{C}^0([0,t_0],H^1)$ is a solution to gKdV close to the manifold of solitary waves, i.e. there exist $(\lambda_1(t),x_1(t))\in\mathbb{R}_+^*\times\mathbb{R}$ and $\varepsilon_1(t)$ such that $\forall t\in[0,t_0]$

$$u(t,x)=\frac{1}{\lambda_1^{\frac12}(t)}(Q+\varepsilon_1)\left(t,\frac{x-x_1(t)}{\lambda_1(t)}\right) \tag{2.13}$$

with

$$\|\varepsilon_1(t)\|_{H^1}\le\alpha^* \tag{2.14}$$

for some small universal constant $\alpha^*>0$. By implicit function theorem (on Banach space), we can refine this decomposition using the Q_b profile in the following way:

Lemma 2.3 (Decomposition and modulation equations [10] [7])**.** *Assume* (2.14). (i) Decomposition*: there exist unique $\mathcal{C}^1$ functions $(\lambda,x,b):[0,t_0]\to\mathbb{R}_+^*\times\mathbb{R}^2$ such that*

$$\forall t\in[0,t_0],\quad \varepsilon(t,y)=\lambda^{\frac12}(t)u(t,\lambda(t)y+x(t))-Q_{b(t)}(y) \tag{2.15}$$

satisfies the orthogonality conditions[8]

$$(\varepsilon(t),y\Lambda Q)=(\varepsilon(t),\Lambda Q)=(\varepsilon(t),Q)=0. \tag{2.16}$$

Moreover, there hold

$$\|\varepsilon(t)\|_{H^1}+|b(t)|+\left|1-\frac{\lambda(t)}{\lambda_1(t)}\right|\lesssim\delta(\alpha^*). \tag{2.17}$$

(ii) Modulation equations*: let*

$$s(t)=\int_0^t\frac{dt'}{\lambda^3(t')}\quad\textit{or equivalently}\quad\frac{ds}{dt}=\frac{1}{\lambda^3},\quad s_0=s(t_0). \tag{2.18}$$

Then the followings hold: on $[0,s_0]$,

$$\left|\frac{\lambda_s}{\lambda}+b\right|+\left|\frac{x_s}{\lambda}-1\right|\lesssim\left(\int\varepsilon^2e^{-\frac{|y|}{10}}\right)^{\frac12}+b^2; \tag{2.19}$$

$$|b_s+2b^2|\lesssim|b|\left(\int\varepsilon^2e^{-\frac{|y|}{10}}\right)^{\frac12}+|b|^3+\int\varepsilon^2e^{-\frac{|y|}{10}}. \tag{2.20}$$

Remark. This lemma is a rigorous version of section 2.1 (compare (2.6) and (2.19) (2.20)), and the orthogonal conditions on the error term $\varepsilon(t)$ are so that the linearized operator L is coercive (see (**??**)).

[8]so that the linearized operator L near Q is coercive, recall (2.3)

Moreover, the decomposition of Lemma 2.3 is stable by H^1-weak limit.

Lemma 2.4 (H^1-weak stability of the decomposition [3]). *Let $u_n(0)$ be a sequence of H^1 initial data such that*

$$u_n(0) \rightharpoonup u(0) \in H^1 \quad \text{as } n \to +\infty.$$

Assume that for some universal time $T_1 > 0$, the corresponding solution u_n of gKdV exists on $[0, T_1]$, belongs to the tube $\mathcal{T}_{\alpha^*}$ (recall its definition in (1.15)) and satisfies for all n,

$$\lambda_n(0) = 1,\ x_n(0) = 0, \quad \text{and} \quad \forall t \in [0, T_1],\ 0 < c \le \lambda_n(t) < C. \tag{2.21}$$

Then the H^1 solution $u(t)$ corresponding to $u(0)$ exists on $[0, T_1]$, belongs to the tube $\mathcal{T}_{\alpha^*}$ and

$$\forall t \in [0, T_1], \quad \lambda_n(t) \to \lambda(t), \quad x_n(t) \to x(t), \quad b_n(t) \to b(t), \quad \varepsilon_n(t) \rightharpoonup \varepsilon(t) \text{ in } H^1.$$

3. Monotonicity formula

Let $\varphi, \psi \in C^\infty(\mathbb{R})$ be such that:

$$\varphi(y) = \begin{cases} e^y & \text{for } \ y < -1, \\ 1 + y & \text{for } \ -\frac{1}{2} < y < \frac{1}{2} \\ y^2 & \text{for for } \ y > 2 \end{cases}, \quad \varphi'(y) > 0, \quad \forall y \in \mathbb{R}, \tag{3.1}$$

$$\psi(y) = \begin{cases} e^{2y} & \text{for } \ y < -1, \\ 1 & \text{for } \ y > -\frac{1}{2} \end{cases}, \quad \psi'(y) \ge 0 \quad \forall y \in \mathbb{R}. \tag{3.2}$$

For $B \ge 100$ a large constant to be fixed, let

$$\psi_B(y) = \psi\left(\frac{y}{B}\right), \quad \varphi_B = \varphi\left(\frac{y}{B}\right),$$

and define

$$\mathcal{N}(s) = \int \varepsilon_y^2(s, y)\psi_B(y)dy + \int \varepsilon^2(s, y)\varphi_B(y)dy. \tag{3.3}$$

The idea of control this $\mathcal{N}(s)$ instead of $\|\varepsilon(s)\|_{H^1}$ comes from the fact that $\varepsilon(s)$ might have very good dacay on the right (for example $\int_{y>0} y^2\varepsilon^2(s, y)dy$ could be small) and bad decay on the left, so that $\varepsilon_y(s, y)$ may not even belong to L^2 on the left). Besides, only the L^2 norm with exponentially-decaying weight is required to bound modulation equations, see (2.19) (2.20).

Lemma 3.1 (Control of the flow by b). *Under the following assumptions: $\forall s \in [0, s_0]$,*
(H1) smallness*:*

$$\|\varepsilon(s)\|_{L^2} + |b(s)| + \mathcal{N}(s) \le \alpha^*; \tag{3.4}$$

(H2) comparison between b and λ*:*

$$\frac{|b(s)| + \mathcal{N}(s)}{\lambda^2(s)} \le \alpha^*; \tag{3.5}$$

(H3) weighted bound on the right*:*

$$\int_{y>0} y^{10}\varepsilon^2(s, y)dy \le 10\left(1 + \frac{1}{\lambda^{10}(s)}\right), \tag{3.6}$$

for a small universal constant $\alpha^* > 0$. The following inequalities hold
(i) Monotonicity of $\mathcal{N}$*: for all $0 \le s_1 \le s_2 < s_0$,*

$$\mathcal{N}(s_2) \lesssim \mathcal{N}(s_1) + (|b^3(s_2)| + |b^3(s_1)|). \tag{3.7}$$

$$\frac{\mathcal{N}(s_2)}{\lambda^2(s_2)} + \lesssim \frac{\mathcal{N}(s_1)}{\lambda^2(s_1)} + \left[\frac{|b^3(s_1)|}{\lambda^2(s_1)} + \frac{|b^3(s_2)|}{\lambda^2(s_2)}\right]. \tag{3.8}$$

(ii) Monotonicity of b: *for all* $0 \le s_1 \le s_2 < s_0$,

$$\int_{s_1}^{s_2} b^2(s)ds \lesssim \int \left(\varepsilon_y^2 + \varepsilon^2\right)(s_1)\varphi_B' + |b(s_2)| + |b(s_1)|, \tag{3.9}$$

$$\left|\frac{b(s_2)}{\lambda^2(s_2)} - \frac{b(s_1)}{\lambda^2(s_1)}\right| \le \frac{C^*}{10}\left[\frac{b^2(s_1)}{\lambda^2(s_1)} + \frac{b^2(s_2)}{\lambda^2(s_2)} + \frac{1}{\lambda^2(s_1)}\int \left(\varepsilon_y^2 + \varepsilon^2\right)(s_1)\varphi_B'\right]. \tag{3.10}$$

for some universal constant $C^* > 0$.

(iii) Control of the scaling dynamics: *let* $\lambda_0(s) = \lambda(s)(1 - J_1(s))^2$, *then on* $[0, s_0)$,

$$\left|\frac{(\lambda_0)_s}{\lambda_0} + b + c_1 b^2\right| \lesssim \int \varepsilon^2 e^{-\frac{|y|}{10}} + |b|\mathcal{N}^{\frac{1}{2}} + |b|^3, \tag{3.11}$$

$$\left|\frac{\lambda_0}{\lambda} - 1\right| \lesssim |J_1| \lesssim \mathcal{N}^{\frac{1}{2}}. \tag{3.12}$$

Remark. As we have seen in (2.19) and (2.20) of Lemma 2.3, the modulation parameters λ, x, b and the error term ε are coupled, hence it is more complicated than our expectation that a finite dimensional ODE would control the modulation parameters. However, Lemma 3.1 implies the control of the flow by the sole parameter b, and hence gives a chance of decoupling those equations.

The proof of the monotonicity formula is very technical, but the idea is as follows. Assume $\phi = cQ(\frac{x}{K})$ and $\psi(x)$ its distribution function, $u(t)$ is a solution to gKdV, define

$$\mathcal{I}_{x_0,t_0}(t) = \int u^2(t,x)\psi(x - x(t_0) + x_0 - \frac{1}{4}(x(t) - x(t_0)))dx,$$

and note $\tilde{x} = x - x(t_0) + x_0 - \frac{1}{4}(x(t) - x(t_0))$. Thus

$$\frac{d}{dt}\mathcal{I}_{x_0,t_0}(t) \le -3\int u_x^2(t,x)\phi(\tilde{x}) + \left(\frac{1}{K^2} - \frac{x_t}{4}\right)\int u^2(t,x)\phi(\tilde{x}) + \frac{5}{3}\int u^6(t,x)\phi(\tilde{x}). \tag{3.13}$$

The first term is negative, and the second is also negative with carefully chosen K. We decompose the third term so that it is controlled by the first two. Hence[9] $\frac{d}{dt}\mathcal{I}_{x_0,t_0}(t) \le 0$, i.e. $\forall t \ge t_0$,

$$\int u^2(t,x)\psi(x - x(t_0) + x_0 - \frac{1}{4}(x(t) - x(t_0)))dx \le \int u^2(t_0,x)\psi(x - x(t_0) + x_0)dx.$$

When it comes to the monotonicity formula for ε, we will use the equation of $\varepsilon(s)$ to derive an inequality similar to (3.13), and then estimate all the positive terms with respect to the terms with a good sign.

4. Construction of a minimal element

The strategy of the construction of the minimal element is as follows: firstly, take a defocusing sequence (u_n) corresponding to some well prepared data; secondly, at the exit time t_n^*, prove by the control of the dynamical information that $\lambda_n(t_n^*) \sim \sqrt{n}$; thirdly, renormalize u_n by $\lambda_n(t_n^*)$ so that it is more focused, and extract a weak limit; fourthly, study the backward solution of this weak limit and prove it is the minimal element.

[9] Here we cheated a little bit, there is still a small positive term left.

step 1 Well prepared initial data. Let $u_n(0) = Q_{-\frac{1}{n}}$, hence

$$u_n(0) \to Q \quad \text{in } H^1 \text{ as } n \to +\infty, \quad u_n(0) \in \mathcal{A}.$$

By (2.11), one gets $\int u_n^2(0) < \int Q^2$ for n large enough, hence the solution $u_n(t)$ is global. Besides, it cannot converge locally to a solitary wave like in the (soliton) case of Theorem 1.4 due to the strictly subcritical mass and mass conservation. Hence (Defocusing and exit) holds and we define the *exit time* by

$$t_n^* = \sup\{t > 0, \text{ such that } \forall t' \in [0, t],\ u_n(t') \in \mathcal{T}_{\alpha^*}\}.$$

From the continuous dependence of the solution with respect to the initial data, and the fact that $Q(x-t)$ is the corresponding solution with initial data $Q(x)$, one gets that $t_n^* \to +\infty$ as $n \to +\infty$.

On $[0, t_n^*]$, $u_n(t)$ has a decomposition $(\lambda_n, x_n, b_n, \varepsilon_n)$ as in Lemma 2.3. By the definition of t_n^*,

$$\inf_{\lambda_0>0,\ x_0\in\mathbb{R}} \left\| u_n(t_n^*) - \frac{1}{\lambda_0^{\frac{1}{2}}} Q\left(\frac{\cdot - x_0}{\lambda_0}\right) \right\|_{L^2} = \alpha^*, \tag{4.1}$$

so that

$$\left\| u_n(t_n^*) - \frac{1}{\lambda_n^{\frac{1}{2}}(t_n^*)} Q\left(\frac{\cdot - x_n(t_n^*)}{\lambda_n(t_n^*)}\right) \right\|_{L^2} \geq \alpha^*. \tag{4.2}$$

step 2 Description of the modulation parameters of u_n. Combining the mass and energy conservation for u_n, the (almost) monotonicity of $b, \mathcal{N}, \lambda$, and the "boundary" condition , we conclude with the following lemma:

Lemma 4.1.
(i) Estimates at t_n^**:*

$$(\alpha^*)^2 \lesssim \frac{\lambda_n^2(t_n^*)}{n} \lesssim \delta(\alpha^*), \tag{4.3}$$

$$(\alpha^*)^2 \lesssim \int \varepsilon_n^2(t_n^*) \approx -b_n(t_n^*) \lesssim \delta(\alpha^*), \tag{4.4}$$

$$0 < c(\alpha^*) \leq \frac{t_n^*}{\lambda_n^3(t_n^*)} \leq C(\alpha^*). \tag{4.5}$$

(ii) Estimates on $[0, t_n^*]$*:*

$$\|\varepsilon_n(t)\|_{H^1}^2 \lesssim \frac{\lambda_n^2(t)}{n} \lesssim \delta(\alpha^*). \tag{4.6}$$

(iii) Control of the dynamics on $[0, t_n^*]$*:*

$$-(1-\delta(\alpha^*))\frac{b_n(t_n^*)}{\lambda_n^2(t_n^*)} \lesssim (\lambda_{0n})_t(t) \lesssim -(1+\delta(\alpha^*))\frac{b_n(t_n^*)}{\lambda_n^2(t_n^*)}. \tag{4.7}$$

step 3 Renormalization and extraction of the limit. From (4.3), one gets

$$\lambda_n(t_n^*) \sim \sqrt{n},$$

meaning intuitively that the forward flow of u_n is defocused by the scale of $\sqrt{n}$. We renormalize u_n by the scaling $\lambda_n(t_n^*)$ as follows:

$$v_n(\tau, x) = \lambda_n^{\frac{1}{2}}(t_n^*) u_n\left(t_\tau, \lambda_n(t_n^*)x + x(t_n^*)\right) \tag{4.8}$$

$$= \frac{\lambda_n^{\frac{1}{2}}(t_n^*)}{\lambda_n^{\frac{1}{2}}(t_\tau)} (Q_{b_n(t_\tau)} + \varepsilon_n)\left(t_\tau, \frac{\lambda_n(t_n^*)}{\lambda_n(t_\tau)}x + \frac{x(t_n^*) - x(t_\tau)}{\lambda_n(t_\tau)}\right), \tag{4.9}$$

where

$$t_\tau = t_n^* + \tau\lambda_n^3(t_n^*) \quad \text{for} \quad \tau \in [-\frac{t_n^*}{\lambda_n^3(t_n^*)}, 0].$$

By the symmetries of gKdV, v_n is solution of (1.9), and it belongs to the tube $\mathcal{T}_{\alpha^*}$ for $\tau \in [-\frac{t_n^*}{\lambda_n^3(t_n^*)}, 0]$. Moreover, its decomposition $(\lambda_{v_n}, x_{v_n}, b_{v_n}, \varepsilon_{v_n})$ satisfies on $[-\frac{t_n^*}{\lambda_n^3(t_n^*)}, 0]$

$$\lambda_{v_n}(\tau) = \frac{\lambda_n(t_\tau)}{\lambda_n(t_n^*)},\ x_{v_n}(\tau) = \frac{x_n(t_\tau) - x_n(t_n^*)}{\lambda_n(t_n^*)},\ b_{v_n}(\tau) = b_n(t_\tau),\ \varepsilon_{v_n}(\tau) = \varepsilon_n(t_\tau), \tag{4.10}$$

and initially

$$\lambda_{v_n}(0) = 1, \quad x_{v_n}(0) = 0.$$

By (4.6) and (4.4), we have

$$\forall \tau \in [-\frac{t_n^*}{\lambda_n^3(t_n^*)}, 0], \quad \|\varepsilon_{v_n}(\tau)\|_{H^1}^2 = \|\varepsilon_n(t_\tau)\|_{H^1}^2 \lesssim \delta(\alpha^*),$$
$$(\alpha^*)^2 \leq -b_{v_n}(0) = -b_n(t_n^*) \leq \delta(\alpha^*).$$

Therefore

$$\|v_n(0) - Q\|_{H^1} \lesssim \delta(\alpha^*),$$

and we can extract a subsequence of (v_n) (without changing notations) such that they have a weak limit:

$$v_n(0) \rightharpoonup v(0) \text{ in } H^1, \text{ and } \|v(0) - Q\|_{H^1} \lesssim \delta(\alpha^*). \tag{4.11}$$

And by (4.5) and (4.4),

$$\tau_n^* = -\frac{t_n^*}{\lambda_n^3(t_n^*)} \to \tau^* < 0, \quad -b_n(t_n^*) \to b^* > 0.$$

We take $v(\tau)$ to be the backward H^1 solution of gKdV (1.9) with initial data $v(0)$ at $\tau = 0$.

step 4 Minimal mass blow up. we claim that v is a minimal mass blow-up solution with $\|v\|_{L^2} = \|Q\|_{L^2}$ and blows up in finite negative time τ^* with the speed

$$\frac{(1 - \delta(\alpha^*))}{\tau - \tau^*} \leq \|v_x(\tau)\|_{L^2} \leq \frac{(1 + \delta(\alpha^*))}{\tau - \tau^*} \tag{4.12}$$

for τ close enough to τ^*.

By integrating (3.11) on $[0, t]$ (or equivalently on $[\tau_n^*, \tau]$ for λ_{0,v_n}) we get

$$0 < \frac{1}{2}b^*(\tau_0 - \tau^*) \leq \lambda_{v_n}(\tau) \leq -\frac{3}{2}b^*\tau^*$$

for any $\tau_0 \in (\tau^*, 0)$. It follows from the H^1-weak stability Lemma 2.4 that $v(\tau)$ is well defined on $(\tau^*, 0]$ and for all $\tau \in (\tau^*, 0]$,

$$b^*(\tau - \tau^*)(1 - \delta(\alpha^*)) \leq \lambda_v(\tau) \leq b^*(\tau - \tau^*)(1 + \delta(\alpha^*)).$$

On the other hand, by weak convergence (4.11)

$$\int v^2(0) \leq \liminf_{n\to\infty} \int v_n^2(0) = \int Q^2.$$

Since v blows up in finite time, $\|v(0)\|_{L^2} = \|Q\|_{L^2}$ and v is a minimal mass element.

This construction method is clear and elegant. However, since the minimal element is obtained by convergence, there is a drawback that it is difficult to extract more information about its structure. The next section is devoted to the sharp description of the general minimal element, which will in turn be used in the proof of uniqueness.

References

[1] T. Cazenave and P.-L. Lions, Orbital stability of standing waves for some nonlinear Schrödinger equations, Comm. Math. Phys. **85**, (1982), 549–561.

[2] T. Kato, On the Cauchy problem for the (generalized) Korteweg-de Vries equation. Studies in applied mathematics, 93–128, Adv. Math. Suppl. Stud., 8, Academic Press, New York, 1983.

[3] Y. Martel and F. Merle, A Liouville theorem for the critical generalized Korteweg–de Vries equation, J. Math. Pures Appl. **79** (2000), 339–425.

[4] Y. Martel and F. Merle, Blow up in finite time and dynamics of blow up solutions for the L^2-critical generalized KdV equation, J. Amer. Math. Soc. **15** (2002), 617–664.

[5] Y. Martel and F. Merle, Nonexistence of blow-up solution with minimal L^2-mass for the critical gKdV equation. Duke Math. J. **115** (2002), 385–408.

[6] Y. Martel and F. Merle, Stability of blow up profile and lower bounds for blow up rate for the critical generalized KdV equation, Ann. of Math. **155** (2002), 235–280.

[7] Y. Martel, F. Merle and P. Raphaël, Blow up for the critical gKdV equation I: dynamics near the soliton. Preprint.

[8] Y. Martel, F. Merle and P. Raphaël, Blow up for critical gKdV equation II: minimal mass solution. Preprint.

[9] F. Merle, Determination of blow-up solutions with minimal mass for nonlinear Schrödinger equations with critical power. Duke Math. J. **69** (1993), 427–454.

[10] F. Merle, Existence of blow-up solutions in the energy space for the critical generalized KdV equation. J. Amer. Math. Soc. **14** (2001), 555–578.

[11] P. Raphael, On the singularity formation for the non linear Schrödinger equation, Notes of the 2008 Clay Summer School in Zurich.

[12] M.I. Weinstein, Nonlinear Schrödinger equations and sharp interpolation estimates, Comm. Math. Phys. **87** (1983), 567–576.

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Academic Activities

International Conference on Nonlinear PDE and Satellite Workshop *Free Surface & Interface Problems*, University of Oxford, September 10-15, 2012

Fluid Dynamics Summer School, University of Alberta, July 23-27, 2012

Inverse Problems and Partial Differential Equations Summer School, University of Washington, July 2-20, 2012

NSF PIRE Summer School: *New frontiers in multiscale analysis and computing for materials*, University of Minnesota, June 21-29, 2012

Master yearly thesis *Amortissement Landau* (Landau Damping), École Normale Supérieure, 2012

- advised by Prof. Benjamin Texier
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Hadamard Lectures on *Error estimates in stochastic homogenization* by Prof. Felix Otto, University of Paris-Sud XI, March-April, 2012

Undergraduate thesis *Wiener-Hermite Chaos expansion for stochastic Burgers equation driven by Brownian motions*, advised by Prof. Tiejun Li, Peking University, 2011

Summer School on Applied Mathematics, Peking University, July 19-August 20, 2010

- Introduction to molecular simulations
- Mathematical introduction to kinetic theory
- Introduction to statistical physics
- Mathematical introduction to complex fluids
- Numerical methods for multiphase flow

Research Project on Satellite Mission Scheduling, Peking University, September 2010-May 2011

- supervised by Prof. Pingwen Zhang
- using combinatorial techniques to numerically optimize the scheduling of satellite mission-operation based on practical requirements

Multiscale Modeling and Computation, Peking University, March-June 2010

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Advanced Courses

- Elliptic PDEs and calculus of variations
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- Hamiltonian systems in finite and infinite dimensions
- Mathematical theory of fluid dynamics
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Honors & Awards

2011-2013 Sophie Germain International Master scholarship from the Jacques Hadamard Mathematics Foundation (FMJH)

2009-2010 Peking University Academic Innovation Award

✓ Reward to most academically creative students in the university each year

2010 Mathematical Contest in Modeling (MCM) - Meritorious Winner

✓ Top 19% among 2254 teams participated all over the world

2007 Provincial-level Outstanding Student in Henan Province

✓ Out of more than 879,000 senior high school students in Henan Province, only 78 students won this title

MINIMAL MASS BLOW-UP FOR THE CRITICAL GKDV

ZIHUI ZHAO

Contents

1. Introduction

1.1. L^2 **criticality for NLS.** The study of nonlinear dispersive[1] equations has attracted considerable attention recently. One of the most studied and physically significant is the nonlinear Schrödinger equation (NLS), which arises in the mathematical description of electro-magnetic wave propagation through nonlinear media. The focusing[2] nonlinear Schrödinger equation is written as follows

$$(NLS)\quad \begin{cases} iu_t = -\Delta u - |u|^{p-1}u, & (t,x) \in [0,T) \times \mathbb{R}^N, \\ u(0,x) = u_0(x), & u_0(x) : \mathbb{R}^N \to \mathbb{C}. \end{cases} \tag{1.1}$$

where t and x are time and space variable respectively, and u_t is the partial derivative with respect to t. It has the space-time translation symmetry as well as scaling symmetry: if $u(t,x)$ is a solution to NLS, then so is

$$u_\lambda(t,x) = \lambda^{\frac{2}{p-1}} u(\lambda^2 t, \lambda x) \tag{1.2}$$

for any $\lambda > 0$.

[1] Dispersive equation is the equation $\partial_t u + Lu = 0$ which has the wave-type solution $e^{i(x\cdot\xi + h(\xi)t)}$, where $h(\xi)$ depends nonlinearly on ξ. That is to say, different frequencies have different propagation speeds.

[2] Focusing means same sign for diffusion and nonlinearity. If they are of different signs, the energy (1.5) is positive definite and therefore blow-up phenomena would not occur.

Assume the nonlinearity satisfies

$$1 < p < 2^* - 1 \quad \text{with} \quad 2^* = \begin{cases} +\infty & \text{for} \quad N = 1, 2 \\ \frac{2N}{N-2} & \text{for} \quad N \geq 3 \end{cases} \tag{1.3}$$

the Cauchy problem is locally well posed in the energy space: for any $u_0(x) \in H^1(\mathbb{R}^N)$, there exits a unique maximal solution $u(t)$ of (1.1) in $C([0,T), H^1)$, and

$$T < +\infty \text{ implies } \lim_{t \to T} \|\nabla u(t)\|_{L^2} = +\infty.$$

Besides, the mass and energy of NLS are conserved by the flow: $\forall t \in [0, T)$,

$$M(u(t)) = \int |u(t)|^2 = M(u_0), \tag{1.4}$$

$$E(u(t)) = \frac{1}{2} \int |\nabla u(t)|^2 - \frac{1}{p+1} \int |u(t)|^{p+1} = E(u_0). \tag{1.5}$$

Moreover, the Gagliardo-Nirenberg-Sobolev interpolation estimate

$$\forall v \in H^1, \quad \int |v|^{p+1} \leq C(N,p) \left(\int |\nabla v|^2 \right)^{\frac{N(p-1)}{4}} \left(\int |v|^2 \right)^{\frac{N-2}{4}(2^*-1-p)} \tag{1.6}$$

together with the energy conservation implies global existence in the so-called subcritical cases:

Theorem 1.1 (Global existence or blow-up in finite time [11]). *Assume p satisfies* (1.3)*,*
(i)*, if $p < 1 + \frac{4}{N}$ (subcritical), then for every $u_0 \in H^1$, the solution for the Cauchy problem* (1.1) *is global and bounded in H^1;*
(ii)*, if $p = 1 + \frac{4}{N}$ (critical) or $p > 1 + \frac{4}{N}$ (supercritical), then the virial law* [3] *allows for the existence of finite time blow-up solutions, with negative energy.*

Remark. (i) In the linear case, formally the solutions disperse, as indicated by the *dispersive inequality*

$$\|e^{it\Delta} u_0\|_{L^{p'}} \leq C \frac{1}{|t|^{d\left(\frac{1}{p} - \frac{1}{2}\right)}} \|u_0\|_{L^p} \quad \text{for all } 1 \leq p \leq 2.$$

And this theorem states that: when the nonlinearity is sufficiently strong $p \geq 1 + \frac{4}{N}$, the focusing phenomena introduced by the nonlinear media would dominate the linear dispersive effect, and lead to blow-up.
(ii) It is worth mentioning that for $p = 1 + \frac{4}{N}$, the scaling symmetry (1.2) leaves the L^2 norm invariant, that is the so-called L^2 criticality.

In between the dispersion effect and blow-up, there is a nontrivial solution which propagates without deformation (called the solitary wave), i.e. the solution of the form

$$u(t,x) = e^{it} Q(x) \tag{1.7}$$

where Q satisfies the elliptic equation

$$\Delta Q - Q + |Q|^{p-1} Q = 0, \quad Q \in H^1. \tag{1.8}$$

By elliptic PDE theories, there exists a unique solution of (1.8) which is positive and radially symmetric, called the ground state for NLS. Furthermore, $Q(x)$ is strictly

[3] If $s_c = \frac{N}{2} - \frac{2}{p-1} \geq 0$ and $xu_0 \in L^2$, then

$$\frac{d^2}{dt^2} \int |x|^2 |u(t,x)|^2 dx = 4N(p-1)E(u_0) - \frac{16 s_c}{N - 2s_c} \int |\nabla u|^2 \leq 4N(p-1)E(u_0).$$

positive, smooth and exponentially decay in infinity, and satisfies the following variational characterization:

Theorem 1.2 (Variational characterization of ground state)**.** (i) *If* $1 < p < 1 + \frac{4}{N}$, *assume* $M > 0$, *then the minimisation problem*

$$I(M) = \inf\{E(u) : u \in H^1 \quad \text{with } \|u\|_{L^2}^2 = M\}$$

is attained on the family

$$Q_{\lambda(M)}(x - x_0)e^{i\gamma_0}, \quad x_0 \in \mathbb{R}^N, \quad \gamma_0 \in \mathbb{R},$$

where $\lambda(M)$ *is the unique scaling such that* $\|Q_{\lambda(M)}\|_{L^2} = M$.
(ii) *Any minimizing sequence* v_n *to* $I(M)$ *is relatively compact in* H^1 *up to translation and phase symmetry, i.e. up to a subsequence*

$$v_n(\cdot + x_n)e^{i\gamma_n} \to Q \text{ in } H^1.$$

Based on this characterization and the concentration compactness argument[4], T.Cazenave and P.-L.Lions [1] proved in the subcritical case, the solitary wave is orbital stable for NLS:

Theorem 1.3 (Orbital stability of solitary wave (1.7) for subcritical NLS)**.** *If* $1 < p < 1 + \frac{4}{N}$. *For all* $\varepsilon > 0$, *there exists* $\delta(\varepsilon)$ *such that the following holds true. Let* $u_0 \in H^1$ *with*

$$\|u_0 - Q\|_{H^1} < \delta(\varepsilon),$$

then there exist a translation shift $x(t) \in \mathcal{C}^0(\mathbb{R}, \mathbb{R}^N)$ *and a phase shift* $\gamma(t) \in \mathcal{C}^0(\mathbb{R}, \mathbb{R})$ *such that*

$$\forall t \in \mathbb{R}, \quad \|u(t,x) - Q(x - x(t))e^{i\gamma(t)}\|_{H^1} < \varepsilon.$$

It might not be the case for $p \geq 1 + \frac{4}{N}$.

1.2. **Minimal mass blow-up for critical NLS problem.** Now we focus on the L^2 critical NLS problem, i.e. when $p = 1 + \frac{4}{N}$. In this case, the ground state $Q(x)$ satisfies $E(Q) = 0$ and that the Gagliardo-Nirenberg-Sobolev inequality (1.6) is an equality, hence gives the constant $C(N,p)$ for this sharp (or, optimal) inequality:

$$\forall v \in H^1, \quad \frac{1}{p+1}\int |v|^{p+1} \leq \left(\frac{\int |v|^2}{\int |Q|^2}\right)^{\frac{2}{N}} \left(\frac{1}{2}\int |\nabla v|^2\right).$$

This led to the first derivation of a criterion of global existence for critical NLS by Weinstein [12]. In particular, H^1 initial data with $\|u_0\|_{L^2} < \|Q\|_{L^2}$ generate a unique global solution $u \in \mathcal{C}([0, +\infty), H^1)$. In the case of NLS, this criterion is sharp as a consequence of the pseudo-conformal transformation (only in the L^2 critical case): if $u(t,x)$ is a solution to NLS, then so is

$$v(t,x) = \frac{1}{|t|^{\frac{N}{2}}} u(-\frac{1}{t}, \frac{x}{t})e^{i\frac{|x|^2}{4t}}.$$

Applied to the solitary wave solution $u(t,x) = e^{it}Q(x)$, one gets the explicit minimal mass blow-up solution

$$S_{NLS}(t,x) = \frac{1}{|t|^{\frac{N}{2}}} Q\left(\frac{x}{t}\right) e^{i\frac{|x|^2}{4t} - \frac{i}{t}}, \quad \|S_{NLS}(t)\|_{L^2} = \|Q\|_{L^2}.$$

[4] compensate for the loss of compactness of the injection $H^1 \hookrightarrow L^{p+1}$.

The dynamics generated by the smooth data $S_{NLS}(1)$ is: $S_{NLS}(t)$ scatters as $t \to +\infty$, and blows up as $t \downarrow 0$ with

$$\|\nabla S_{NLS}(t)\|_{L^2} \sim \frac{1}{|t|},$$

$$|S_{NLS}(t)|^2 \rightharpoonup \|Q\|_{L^2}^2 \delta_{x=0} \quad \text{as} \quad t \downarrow 0.$$

Furthermore, this minimal mass blow-up solution is unique:

Theorem 1.4 (Classification of the minimal element [9])**.** *Let $u_0 \in H^1$ such that $\|u_0\|_{L^2} = \|Q\|_{L^2}$, assume that the corresponding solution to NLS blows up in finite time $0 < T < +\infty$, then*

$$u(t) = S_{NLS}(t)$$

up to the symmetries of NLS.

1.3. **Critical gKdV equation and existence of minimal element.** The generalized Korteweg-de Vries equation (gKdV), which is a model of waves on shallow water surface, is a canonical problem similar to (focusing) NLS. Especially, the one-dimensional L^2 critical gKdV is

$$\text{(gKdV)} \quad \begin{cases} u_t + (u_{xx} + u^5)_x = 0, & (t,x) \in [0,T) \times \mathbb{R}, \\ u(0,x) = u_0(x), & x \in \mathbb{R}. \end{cases} \tag{1.9}$$

and the scaling symmetry

$$u_\lambda(t,x) = \lambda^{\frac{1}{2}} u(\lambda^3 t, \lambda x), \quad \lambda > 0$$

leaves the L^2 norm invariant. In the energy space H^1 the local well-posedness for gKdV is similar to NLS, with the mass and energy conservation: $\forall t \in [0,T)$,

$$M(u(t)) = \int u^2(t) = M(u_0), \quad E(u(t)) = \frac{1}{2}\int u_x^2(t) - \frac{1}{6}\int u^6(t) = E(u_0),$$

The solitary wave solution to gKdV is

$$u(t,x) = \frac{1}{\lambda_0^{\frac{1}{2}}} Q\left(\frac{x - \lambda_0^{-2} t - x_0}{\lambda_0}\right), \quad (\lambda_0, x_0) \in \mathbb{R}_+^* \times \mathbb{R},$$

with the ground state as (the same as one-dimensional NLS)

$$Q(x) = \left(\frac{3}{\cosh^2(2x)}\right)^{\frac{1}{4}}, \quad Q'' + Q^5 = Q, \quad E(Q) = 0. \tag{1.10}$$

The existence of blow-up solutions [4] have been obtained for a class of initial data u_0 satisfying

$$\int Q^2 < \int u_0^2 \leq \int Q^2 + \alpha, \quad \text{and} \quad E(u_0) < 0$$

for some small constant $\alpha > 0$[5]. However, the existence of the minimal mass blow-up solution in this setting has been a long standing open problem, due to the fact that the existence of minimal element for NLS relies entirely on the exceptional pseudo-conformal symmetry. In 2012 Martel, Merle and Raphaël [8] proved the existence and uniqueness of minimal element for gKdV:

[5]Note that this condition is not empty due to the variational characterization of the ground state Q.

Theorem 1.5 (Existence and uniqueness of the minimal mass blow-up solution).
(i) Existence. *There exists a solution* $S(t) \in \mathcal{C}((0,+\infty), H^1)$ *to* (1.9) *with minimal mass* $\|S(t)\|_{L^2} = \|Q\|_{L^2}$ *which blows up backward at the origin:*

$$S(t,x) - \frac{1}{t^{\frac{1}{2}}} Q\left(\frac{x + \frac{1}{t} + \bar{c}t}{t}\right) \to 0 \quad \text{in} \quad L^2 \quad \text{as} \quad t \downarrow 0 \tag{1.11}$$

at the speed

$$\|S(t)\|_{H^1} \sim \frac{1}{t} \quad \text{as} \quad t \downarrow 0 \tag{1.12}$$

for some universal constant $\bar{c}$.
Moreover, S is smooth and well localized to the right in space:

$$\forall x \geq 1, \quad S(1,x) \leq e^{-Cx}. \tag{1.13}$$

(ii) Uniqueness. *Let* $u_0 \in H^1$ *with* $\|u_0\|_{L^2} = \|Q\|_{L^2}$ *and assume that the corresponding solution* $u(t)$ *to* (1.9) *blows up in finite time. Then*

$$u(t) \equiv S(t)$$

up to the symmetries of gKdV.

Remark. (i) Blow-up point. Recall that for NLS

$$|S_{NLS}(t)|^2 \rightharpoonup \|Q\|_{L^2}^2 \delta_{x=0} \quad \text{as} \quad t \downarrow 0.$$

In the case of gKdV, now we have

$$S(t,x) - \frac{1}{t^{\frac{1}{2}}} Q\left(\frac{x + \frac{1}{t} + \bar{c}t}{t}\right) \to 0 \quad \text{in} \quad L^2 \quad \text{as} \quad t \downarrow 0,$$

that is to say: $S(t)$ moves to the left as well as focuses, and will blow-up at $-\infty$ in space. The loss of compactness of blow-up point means the loss of spacial symmetry for the blow-up profile of gKdV, hence the left and right tail are of different significance to our analysis.

(ii) The tail of $S(t)$. (1.13) means that $S(t)$ has exponentially decay on the right. However in contrast with $S_{NLS}(t)$, $S(t)$ does not decay too fast on the left. In fact, let $u_0 \in H^1$ with $\|u_0\|_{L^2} = \|Q\|_{L^2}$ and

$$\sup_{x_0 > 0} x_0^3 \int_{x > x_0} u_0^2(x)dx < +\infty, \tag{1.14}$$

then the corresponding solution $u(t)$ of gKdV is global for $t > 0$ [5]. Since $S(t)$ blows up backward, it can not satisfy (1.14) on the left. In fact we expect

$$\limsup_{x_0 \to +\infty} x_0^2 \int_{x < -x_0} S^2(1,x)dx \geq C > 0$$

to be true.

The construction of S is based on the characterization of solutions with initial data near soliton. More precisely, let $\mathcal{A}$ to be the set of initial data

$$\mathcal{A} = \left\{ u_0 = Q + \varepsilon_0 \text{ with } \|\varepsilon_0\|_{H^1} < \alpha_0 \text{ and } \int_{y>0} y^{10} \varepsilon_0^2 < 1 \right\},$$

and consider the L^2 tube around the manifold of solitary waves

$$\mathcal{T}_{\alpha^*} = \left\{ u \in H^1 \quad \text{with} \inf_{\lambda_0 > 0,\, x_0 \in \mathbb{R}} \left\| u - \frac{1}{\lambda_0^{\frac{1}{2}}} Q\left(\frac{\cdot - x_0}{\lambda_0}\right) \right\|_{L^2} < \alpha^* \right\}. \tag{1.15}$$

Theorem 1.6 (Characterization of the flow in $\mathcal{A}$ [7]). *Let $0<\alpha_0\ll\alpha^*\ll 1$ and $u_0\in\mathcal{A}$. Let $u\in\mathcal{C}([0,T),H^1)$ be the corresponding solution of gKdV. Then, one of the following three scenarios occurs:*
(Blow up)*: the solution blows up in finite time $0<T<+\infty$ with the speed*

$$\|u(t)\|_{H^1}=\frac{\ell(u_0)+o(1)}{T-t}\quad as\quad t\to T,\quad \ell(u_0)>0. \tag{1.16}$$

(Soliton)*: the solution is global $T=+\infty$ and converges asymptotically to a solitary wave: there exist $\lambda_\infty>0$ and $x(t)$ such that*

$$\lambda_\infty^{\frac{1}{2}}u(t,\lambda_\infty\cdot+x(t))\to Q\quad in\ H^1_{\text{loc}}\ as\ t\to+\infty, \tag{1.17}$$

$$|\lambda_\infty-1|\le\delta(\alpha_0),\quad x(t)\sim\frac{t}{\lambda_\infty^2}\quad as\quad t\to+\infty.$$

(Defocusing and Exit)*: the solution leaves the tube $\mathcal{T}_{\alpha^*}$ at some time $0<t^*<T\le+\infty$, and the solution is defocused at the exit time:*

$$\|u_x(t)\|_{L^2}\le C(\alpha^*)\delta(\alpha_0),$$

where $\delta(\alpha_0)$ is a generic small constant with $\delta(\alpha_0)\to0$ as $\alpha_0\to0$.

Remark. Once the existence and uniqueness of minimal element $S(t)$ is established, it can be proved that in fact, the forward flow of $S(t)$ is an attractor to all the defocused solutions as in the scenario (Defocusing and Exit) [6], thus it is interesting to study the behavior of $S(t)$.

2. Nonlinear profiles and geometrical decomposition near soliton

Notation. We introduce the linearized operator close to Q

$$Lf=-f''+f-5Q^4f, \tag{2.1}$$

and the generator of L^2 scaling

$$\Lambda f(y)=\frac{1}{2}f(y)+yf'(y). \tag{2.2}$$

The self-adjoint operator L is coercive: there exists $\mu_0>0$ such that for all $f\in H^1$,

$$(Lf,f)\ge\mu_0\|f\|^2_{H^1}-\frac{1}{\mu_0}\left[(f,Q)^2+(f,y\Lambda Q)^2+(f,\Lambda Q)^2\right]. \tag{2.3}$$

For a given small constant $0<\alpha^*\ll1$, $\delta(\alpha^*)$ denotes a generic small constant with

$$\delta(\alpha^*)\to0\quad\text{as}\quad\alpha^*\to0.$$

[6]Let $u(t,x)$ be a solution of gKdV corresponding to the *(Defocusing and Exit)* scenario in Theorem 1.6 and let $t_u^*\gg1$ be the corresponding exit time. Then there exist $\tau^*=\tau^*(\alpha^*)$ (independent of u) and (λ_u^*,x_u^*) such that

$$\left\|(\lambda_u^*)^{\frac{1}{2}}u\left(t_u^*,\lambda_u^*x+x_u^*\right)-S(\tau^*,x)\right\|_{L^2}\le\delta(\alpha_0).$$

2.1. **Formal derivation of the dynamics of modulation parameters.**
Now focus on the flow of gKdV close to the solitary waves. Formally, we search for a solution of the form

$$u(t,x) = \frac{1}{\lambda^{\frac{1}{2}}(t)} Q_{b(t)}\left(\frac{x-x(t)}{\lambda(t)}\right), \tag{2.4}$$

with the expansion with respect to a small parameter b

$$Q_b = Q + bP + b^2 P_2 + \cdots .$$

Intuitively, $x(t)$ means the position of the main profile; $\lambda(t)$ the scaling of the profile, for example $\lambda(t) \to 0$ means blow up; and $b(t)$ the small modulation of the profile near the ground state Q, so that the profile is a better approximate to the solution.

Apply it to the equation (1.9), and renormalize as

$$\frac{ds}{dt} = \frac{1}{\lambda^3}, \quad \text{or equivalently } s = \int_0^t \frac{dt'}{\lambda^3(t')},$$

we obtain an equation on the modulation parameters $\lambda(s), x(s), b(s)$:

$$b_s \frac{\partial Q_b}{\partial b} + b\Lambda Q_b - (-Q_b'' + Q_b' - Q_b^5)' - (\frac{\lambda_s}{\lambda} + b)\Lambda Q_b - (\frac{x_s}{\lambda} - 1)Q_b' = 0. \tag{2.5}$$

Since

$$\Lambda Q_b = \Lambda Q + b\Lambda P + O\left(b^2\right),$$

$$-Q_b'' + Q_b' - Q_b^5 = LQ + bLP + O\left(b^2\right) = bLP + O\left(b^2\right),$$

we let

$$\frac{\lambda_s}{\lambda} + b = 0, \quad \frac{x_s}{\lambda} - 1 = 0, \quad b_s \sim O\left(b^2\right). \tag{2.6}$$

Therefore we get the order b solution of (2.4) is

$$(LP)' = \Lambda Q. \tag{2.7}$$

In this way, the study of the flow is turned into the study of the 3 modulation parameters driven by finite dimensional ODEs (2.6). For example, the study of blow-up corresponds to the form of singularity due to $\lambda \to 0$.

2.2. **Nonlinear profile Q_b.** Define $\mathcal{Y}$ to be the set of functions $f \in C^\infty(\mathbb{R},\mathbb{R})$ satisfying

$$\forall k \in \mathbb{N},\ \exists C_k,\ r_k > 0,\ \forall y \in \mathbb{R}, \quad |f^{(k)}(y)| \le C_k(1+|y|)^{r_k} e^{-|y|}.$$

From the property of the operator L, we have the following proposition:

Proposition 2.1. *There exists a unique smooth function P such that $P' \in \mathcal{Y}$ and*

$$(LP)' = \Lambda Q, \quad \lim_{y\to-\infty} P(y) = \frac{1}{2}\int Q, \quad \lim_{y\to+\infty} P(y) = 0, \tag{2.8}$$

$$(P,Q) = \frac{1}{16}\left(\int Q\right)^2 > 0, \quad (P,Q') = 0. \tag{2.9}$$

Moreover,

$$Q_b = Q + bP$$

is an $O(b)$ approximate solution to (2.5) in the sense that:

$$\left\| \left(Q_b'' - Q_b + Q_b^5\right)' + b\Lambda Q_b \right\|_{L^\infty} \lesssim b^2, \tag{2.10}$$

where $\lesssim$ means $\le$ up to a universal constant.

Note that P has a non trivial tail on the left, hence we proceed to a localization: take a cut-off function $\chi \in \mathcal{C}^\infty(\mathbb{R})$ be such that $0 \leq \chi \leq 1$, $\chi' \geq 0$ on $\mathbb{R}$, $\chi \equiv 1$ on $[-1, +\infty)$, $\chi \equiv 0$ on $(-\infty, -2]$. Take $\gamma \in (2/3, 1)$ ($\gamma = 3/4$ for example), and redefine the localized profile:

$$Q_b(y) = Q(y) + b\chi_b(y)P(y), \quad \text{with} \quad \chi_b(y) = \chi\left(|b|^\gamma y\right). \tag{2.11}$$

Note that

$$\int \chi_b^2 P^2 \sim C|b|^{-\gamma}. \tag{2.12}$$

Lemma 2.2. *For $|b| < b^*$ small enough, there hold:*
(i) Estimates on Q_b: *For all $y \in \mathbb{R}$,*

$$|Q_b(y)| \lesssim e^{-|y|} + |b|\left(\mathbf{1}_{[-2,0]}(|b|^\gamma y) + e^{-\frac{|y|}{2}}\right), \tag{2.13}$$

$$|Q_b^{(k)}(y)| \lesssim e^{-|y|} + |b|e^{-\frac{|y|}{2}} + |b|^{1+k\gamma}\mathbf{1}_{[-2,-1]}(|b|^\gamma y), \quad \textit{for } k \geq 1. \tag{2.14}$$

where $\mathbf{1}_I$ denotes the characteristic function of the interval I.
(ii) Equation of Q_b: *Q_b is still an $O(b)$ approximate solution to* (2.5) *in the sense that:*

$$\|\left(Q_b'' - Q_b + Q_b^5\right)' + b\Lambda Q_b\|_{L^\infty} \lesssim |b|^{1+\gamma}. \tag{2.15}$$

(iii) Mass and energy properties of Q_b:

$$\left|\int Q_b^2 - \left(\int Q^2 + 2b\int PQ\right)\right| \lesssim |b|^{2-\gamma}, \tag{2.16}$$

$$\left|E(Q_b) + b\int PQ\right| \lesssim b^2. \tag{2.17}$$

2.3. **Decomposition and modulation parameters.** Assume $u \in \mathcal{C}^0([0, t_0], H^1)$ is a solution to gKdV close to the manifold of solitary waves, i.e. there exist $(\lambda_1(t), x_1(t)) \in \mathbb{R}_+^* \times \mathbb{R}$ and $\varepsilon_1(t)$ such that $\forall t \in [0, t_0]$

$$u(t,x) = \frac{1}{\lambda_1^{\frac{1}{2}}(t)}(Q + \varepsilon_1)\left(t, \frac{x - x_1(t)}{\lambda_1(t)}\right) \tag{2.18}$$

with[7]

$$\|\varepsilon_1(t)\|_{L^2} + \left(\int (\partial_y \varepsilon_1)^2 e^{-\frac{|y|}{2}} dy\right)^{\frac{1}{2}} \leq \alpha^* \tag{2.19}$$

for some small universal constant $\alpha^* > 0$. By implicit function theorem (on Banach space), we can refine this decomposition using the Q_b profile in the following way:

Lemma 2.3 (Decomposition and modulation equations [10] [7])**.** *Assume* (2.19).
(i) Decomposition*: There exist unique $\mathcal{C}^1$ functions $(\lambda, x, b) : [0, t_0] \to \mathbb{R}_+^* \times \mathbb{R}^2$ such that*

$$\forall t \in [0, t_0], \quad \varepsilon(t, y) = \lambda^{\frac{1}{2}}(t) u(t, \lambda(t) y + x(t)) - Q_{b(t)}(y) \tag{2.20}$$

satisfies the orthogonality conditions

$$(\varepsilon(t), y\Lambda Q) = (\varepsilon(t), \Lambda Q) = (\varepsilon(t), Q) = 0. \tag{2.21}$$

Moreover, there hold

$$\|\varepsilon(t)\|_{L^2} + |b(t)| + \left|1 - \frac{\lambda(t)}{\lambda_1(t)}\right| \lesssim \delta(\alpha^*), \quad \|\varepsilon(t)\|_{H^1} \lesssim \delta(\|\varepsilon_1(t)\|_{H^1}). \tag{2.22}$$

[7]Note that $\|\varepsilon_1(t)\|_{H^1} \leq \alpha^*$ implies this condition.

(ii) Equation of ε: *Let*

$$s(t)=\int_0^t \frac{dt'}{\lambda^3(t')} \quad \textit{or equivalently} \quad \frac{ds}{dt}=\frac{1}{\lambda^3}, \quad s_0=s(t_0). \tag{2.23}$$

Then for all $s\in[0,s_0]$, ε *satisfies the following equation*

$$\begin{aligned}\varepsilon_s-(L\varepsilon)_y+b\Lambda\varepsilon=&\left(\frac{\lambda_s}{\lambda}+b\right)(\Lambda Q_b+\Lambda\varepsilon)+\left(\frac{x_s}{\lambda}-1\right)(Q_b+\varepsilon)_y\\ &+\Phi_b+\Psi_b-(R_b(\varepsilon))_y-(R_{\rm NL}(\varepsilon))_y,\end{aligned} \tag{2.24}$$

where the residual terms are defined:

$$\Phi_b=-b_s\left(\chi_b+\gamma y(\chi_b)_y\right)P, \quad \Psi_b=-(Q_b''-Q_b+Q_b^5)'-b\Lambda Q_b, \tag{2.25}$$

$$R_b(\varepsilon)=5\left(Q_b^4-Q^4\right)\varepsilon, \quad R_{\rm NL}(\varepsilon)=(\varepsilon+Q_b)^5-5Q_b^4\varepsilon-Q_b^5. \tag{2.26}$$

(iii) Rough modulation equations: *on* $[0,s_0]$,

$$\left|\frac{\lambda_s}{\lambda}+b\right|+\left|\frac{x_s}{\lambda}-1\right|\lesssim\left(\int\varepsilon^2e^{-\frac{|y|}{10}}\right)^{\frac12}+|b|^2, \tag{2.27}$$

$$|b_s+2b^2|\lesssim|b|\left(\int\varepsilon^2e^{-\frac{|y|}{10}}\right)^{\frac12}+|b|^3+\int\varepsilon^2e^{-\frac{|y|}{10}}. \tag{2.28}$$

(iv) Refined modulation equations: *Assuming in addition the following uniform* L^1 *control on the right*

$$\forall t\in[0,t_0), \quad \int_{y>0}|\varepsilon(t)|\lesssim\delta(\kappa_0), \tag{2.29}$$

then

- *Law of* λ:

$$J_1(s)=(\varepsilon(s),\rho_1) \quad \textit{with} \quad \rho_1(y)=\frac{4}{\left(\int Q\right)^2}\int_{-\infty}^y\Lambda Q \tag{2.30}$$

  is well defined, and for some constant c_1,

$$\left|\frac{\lambda_s}{\lambda}+b+c_1b^2-2\left((J_1)_s+\frac12\frac{\lambda_s}{\lambda}J_1\right)\right|\lesssim\int\varepsilon^2e^{-\frac{|y|}{10}}+|b|\left(\int\varepsilon^2e^{-\frac{|y|}{10}}\right)^{\frac12}+|b|^3. \tag{2.31}$$

- *Law of* $\frac{b}{\lambda^2}$: *let*

$$\rho_2=\frac{16}{\left(\int Q\right)^2}\left(\frac{(\Lambda P,Q)}{\|\Lambda Q\|_{L^2}^2}\Lambda Q+P-\frac12\int Q\right)-8\rho_1, \quad J_2(s)=(\varepsilon(s),\rho_2), \tag{2.32}$$

  then $\rho:=4\rho_1+\rho_2\in\mathcal{Y}$, *so*

$$J=(\varepsilon,\rho) \textit{ is well-defined, and } |J|\lesssim\left(\int\varepsilon^2e^{-\frac{|y|}{10}}\right)^{\frac12},$$

  and for some constant c_0

$$\left|\frac{d}{ds}\left(\frac{b}{\lambda^2}\right)+\frac{b}{\lambda^2}\left(J_s+\frac12\frac{\lambda_s}{\lambda}J\right)+c_0\frac{b^3}{\lambda^2}\right|\lesssim\frac{1}{\lambda^2}\left(\int\varepsilon^2e^{-\frac{|y|}{10}}+|b|^4\right). \tag{2.33}$$

Remark. This lemma is a rigorous version of section 2.1 (compare (2.6) and (2.27) (2.28)), and the orthogonal conditions on the error term $\varepsilon(t)$ are so that the linearized operator L is coercive (see (4.11)). The refined modulation equations in *(iv)* might be intimating, but we will prove later that $J_1(s)$ and $J(s)$ are in fact very small, therefore they are nothing more than a slightly better estimate than the rough modulation equations in *(iii)*.

Moreover, the decomposition of Lemma 2.3 is stable by H^1-weak limit.

Lemma 2.4 (H^1-weak stability of the decomposition [3]). *Let $u_n(0)$ be a sequence of H^1 initial data such that*

$$u_n(0) \rightharpoonup u(0) \in H^1 \quad \text{as } n \to +\infty.$$

Assume that for some universal time $T_1 > 0$, the corresponding solution u_n of gKdV exists on $[0, T_1]$, belongs to the L^2 tube $\mathcal{T}_{\alpha^*}$ (recall its definition in (1.15)) and satisfies for all n,

$$\lambda_n(0) = 1, \; x_n(0) = 0, \quad \text{and} \quad \forall t \in [0, T_1], \; 0 < c \leq \lambda_n(t) < C. \tag{2.34}$$

Then the H^1 solution $u(t)$ corresponding to $u(0)$ exists on $[0, T_1]$, belongs to the tube $\mathcal{T}_{\alpha^*}$ and

$$\forall t \in [0, T_1], \quad \lambda_n(t) \to \lambda(t), \quad x_n(t) \to x(t), \quad b_n(t) \to b(t), \quad \varepsilon_n(t) \rightharpoonup \varepsilon(t) \text{ in } H^1.$$

3. Monotonicity formula

3.1. **Monotonicity formula and control of the flow.** Let $\varphi, \psi \in C^\infty(\mathbb{R})$ be such that:

$$\varphi(y) = \begin{cases} e^y & \text{for } \; y < -1, \\ 1 + y & \text{for } \; -\frac{1}{2} < y < \frac{1}{2} \\ y^2 & \text{for for } \; y > 2 \end{cases}, \quad \varphi'(y) > 0, \quad \forall y \in \mathbb{R}, \tag{3.1}$$

$$\psi(y) = \begin{cases} e^{2y} & \text{for } \; y < -1, \\ 1 & \text{for } \; y > -\frac{1}{2} \end{cases}, \quad \psi'(y) \geq 0 \quad \forall y \in \mathbb{R}. \tag{3.2}$$

For $B \geq 100$ a large constant to be fixed, let

$$\psi_B(y) = \psi\left(\frac{y}{B}\right), \quad \varphi_B = \varphi\left(\frac{y}{B}\right),$$

and define

$$\mathcal{N}(s) = \int \varepsilon_y^2(s, y)\psi_B(y)dy + \int \varepsilon^2(s, y)\varphi_B(y)dy. \tag{3.3}$$

The idea of control this $\mathcal{N}(s)$ instead of $\|\varepsilon(s)\|_{H^1}$ comes from the fact that $\varepsilon(s)$ might have very good dacay on the right (for example $\int_{y>0} y^2 \varepsilon^2(s, y)dy$ could be small) and bad decay on the left, so that $\varepsilon_y(s, y)$ may not even belong to L^2 on the left). Besides, only the L^2 norm with exponentially-decaying weight is required to bound modulation equations, see (2.27) (2.28).

Proposition 3.1 (Lyapunov function and monotonicity [7]). *There exist $\mu > 0$ and $0 < \kappa^* < \kappa_0$ such that the following holds for $B > 100$ large enough. Assume that $u(t)$ is a solution of gKdV which satisfies (2.19) on $[0, t_0]$ and thus admits a decomposition (2.20) as in Lemma 2.3.Assume the following a priori bounds: $\forall s \in [0, s_0]$,*
(H1) smallness*:*

$$\|\varepsilon(s)\|_{L^2} + |b(s)| + \mathcal{N}(s) \leq \kappa^*; \tag{3.4}$$

(H2) comparison between b and λ*:*

$$\frac{|b(s)| + \mathcal{N}(s)}{\lambda^2(s)} \leq \kappa^*; \tag{3.5}$$

(H3) weighted bound on the right*:*

$$\int_{y>0} y^{10}\varepsilon^2(s,y)dy \leq 10\left(1+\frac{1}{\lambda^{10}(s)}\right). \tag{3.6}$$

For $j\in\{1,2\}$, define the Lyapunov function

$$\mathcal{F}_j = \int\left[\varepsilon_y^2\psi_B + \varepsilon^2(1+\mathcal{J}_j)\varphi_B - \frac{1}{3}\left((\varepsilon+Q_b)^6 - Q_b^6 - 6\varepsilon Q_b^5\right)\psi_B\right], \tag{3.7}$$

with

$$\mathcal{J}_j = (1-J_1)^{-4j} - 1. \tag{3.8}$$

Then the following bounds hold on $[0,s_0]$:
(i) Lyapounov inequality*:*

$$\frac{d\mathcal{F}_1}{ds} + \mu\int\left(\varepsilon_y^2+\varepsilon^2\right)\varphi_B' \lesssim |b|^4. \tag{3.9}$$

$$\frac{d}{ds}\left\{\frac{\mathcal{F}_2}{\lambda^2}\right\} + \frac{\mu}{\lambda^2}\int\left(\varepsilon_y^2+\varepsilon^2\right)\varphi_B' \lesssim \frac{|b|^4}{\lambda^2}. \tag{3.10}$$

(ii) Coercivity of $\mathcal{F}_j$ and pointwise bounds*:*

$$\mathcal{N} \lesssim \mathcal{F}_j \lesssim \mathcal{N}, \quad j=1,2. \tag{3.11}$$

$$|J_1| \lesssim \mathcal{N}^{\frac{1}{2}}, \tag{3.12}$$

The consequence of Proposition 3.1 is:

Lemma 3.2 (Control of the flow by b)**.** *Under the assumptions of Proposition 3.1, the following hold*
(i) Monotonicity of $\mathcal{N}$*: for all $0\leq s_1\leq s_2<s_0$,*

$$\mathcal{N}(s_2)+\int_{s_1}^{s_2}\left[\int\left(\varepsilon_y^2+\varepsilon^2\right)(s)\varphi_B' + |b|^4(s)\right]ds \lesssim \mathcal{N}(s_1)+(|b^3(s_2)|+|b^3(s_1)|), \tag{3.13}$$

$$\frac{\mathcal{N}(s_2)}{\lambda^2(s_2)}+\int_{s_1}^{s_2}\left[\int\left(\varepsilon_y^2+\varepsilon^2\right)(s)\varphi_B' + |b|^4(s)\right]\frac{ds}{\lambda^2(s)} \lesssim \frac{\mathcal{N}(s_1)}{\lambda^2(s_1)}+\left[\frac{|b^3(s_1)|}{\lambda^2(s_1)}+\frac{|b^3(s_2)|}{\lambda^2(s_2)}\right]. \tag{3.14}$$

(ii) Monotonicity of b*: for all $0\leq s_1\leq s_2<s_0$,*

$$\int_{s_1}^{s_2} b^2(s)ds \lesssim \int\left(\varepsilon_y^2+\varepsilon^2\right)(s_1)\varphi_B' + |b(s_2)| + |b(s_1)|, \tag{3.15}$$

$$\left|\frac{b(s_2)}{\lambda^2(s_2)} - \frac{b(s_1)}{\lambda^2(s_1)}\right| \leq \frac{C^*}{10}\left[\frac{b^2(s_1)}{\lambda^2(s_1)}+\frac{b^2(s_2)}{\lambda^2(s_2)}+\frac{1}{\lambda^2(s_1)}\int\left(\varepsilon_y^2+\varepsilon^2\right)(s_1)\varphi_B'\right]. \tag{3.16}$$

for some universal constant $C^*>0$.
(iii) Control of the scaling dynamics*: let $\lambda_0(s)=\lambda(s)(1-J_1(s))^2$, then on $[0,s_0)$,*

$$\left|\frac{(\lambda_0)_s}{\lambda_0}+b+c_1b^2\right| \lesssim \int\varepsilon^2 e^{-\frac{|y|}{10}} + |b|\mathcal{N}^{\frac{1}{2}} + |b|^3, \tag{3.17}$$

$$\left|\frac{\lambda_0}{\lambda}-1\right| \lesssim |J_1| \lesssim \mathcal{N}^{\frac{1}{2}}. \tag{3.18}$$

Remark. As we have seen in (2.24), (2.27) and (2.28) of Lemma 2.3, the modulation parameters λ, x, b and the error term ε are coupled, hence it is more complicated than our expectation that a finite dimensional ODE would control the modulation parameters. However, Lemma 3.2 implies the control of the flow by the sole parameter b, and hence gives a chance of decoupling those equations.

The proof of the monotonicity formula (3.9) (3.10) is very technical and will not be shown here. Instead, I will present a simple version and give the proof, so that the reader is able to taste the flavour of this kind of monotonicity result.

3.2. **A simple version of monotonicity formula and its proof.** We claim that the mass close to the soliton (in some sense) of $u(t)$ is almost a decreasing function of time. Define for $K > 0$,

$$\phi(x) = cQ(\frac{x}{K}),\ \psi(x) = \int_{-\infty}^{x} \phi(y)dy,\ \text{where } c = \frac{1}{K\int_{-\infty}^{+\infty} Q}, \tag{3.19}$$

hence the smooth function ψ is a distribution function

$$\forall x \in \mathbb{R},\ 0 \le \psi(x) \le 1,\ \lim_{x\to-\infty} \psi(x) = 0,\ \lim_{x\to+\infty} \psi(x) = 1.$$

Lemma 3.3 (Almost monotonicity of the mass [10]). *There exist $\alpha^* > 0$ and $R > 0$ such that the following holds. Assume the decomposition* (2.18) *exists with $\|\varepsilon_1(t)\|_{H^1} \le \alpha^*$ on $[0,T]$* [8], *and $0 < \lambda(t) < 1.1$ for all $t_0 \le t \le T$, then there exists a constant $C > 0$ such that $\forall x_0 \ge R$, $\forall t \ge t_0$,*

$$\int u^2(t,x)\psi(x-x(t_0)+x_0-\frac{1}{4}(x(t)-x(t_0)))dx \le \int u^2(t_0,x)\psi(x-x(t_0)+x_0)+Ce^{-\frac{x_0}{3}}, \tag{3.20}$$

where ψ is defined as in (3.19) *with $K = 3\sqrt{2}$.*

Define

$$\mathcal{I}_{x_0,t_0}(t) = \int u^2(t,x)\psi(x - x(t_0) + x_0 - \frac{1}{4}(x(t) - x(t_0)))dx,$$

and note $\tilde{x} = x - x(t_0) + x_0 - \frac{1}{4}(x(t) - x(t_0))$. First we have the following lemma:

Lemma 3.4 (Kato identity). *Let $g(x)$ be any $\mathcal{C}^3$ function and $v(t,x)$ be a solution of gKdV, then*

$$\frac{d}{dt}\int v^2 g dx = -3\int v_x^2 g' dx + \int v^2 g''' dx + \frac{5}{3}\int v^6 g' dx. \tag{3.21}$$

Apply Kato identity, one gets

$$\frac{d}{dt}\mathcal{I}_{x_0,t_0}(t) = -3\int u_x^2(t,x)\psi'(\tilde{x})+\int u^2(t,x)\psi'''(\tilde{x})+\frac{5}{3}\int u^6(t,x)\psi'(\tilde{x})-\frac{x_t}{4}\int u^2(t,x)\psi'(\tilde{x}). \tag{3.22}$$

Note that $\psi' = \phi$, and since $Q'' = Q - Q^5 \le Q$, we have

$$\forall x \in \mathbb{R},\ \phi''(x) \le \frac{c}{K^2}Q(\frac{x}{K}) = \frac{1}{K^2}\phi(x).$$

[8] This condition is not empty. In fact, as proved in [10], it is satisfied by the solution with initial data $u_0 \in H^1$ such that

$$\int Q^2 < \int u_0^2 \le \int Q^2 + \alpha \text{ and } E(u_0) < 0$$

where α is a small enough constant.

Thus

$$\frac{d}{dt}\mathcal{I}_{x_0,t_0}(t) \le -3\int u_x^2(t,x)\phi(\tilde{x}) + \left(\frac{1}{K^2} - \frac{x_t}{4}\right)\int u^2(t,x)\phi(\tilde{x}) + \frac{5}{3}\int u^6(t,x)\phi(\tilde{x}). \tag{3.23}$$

The first term is negative. From (2.27) of Lemma 2.3, one gets

$$0 < \frac{9}{10\lambda^2} \le x_t \le \frac{11}{10\lambda^2} \tag{3.24}$$

for α^* small enough. Since $0 < \lambda(t) < 1.1$, we have $\frac{1}{K^2} = \frac{1}{18} < \frac{x_t}{8}$, so that the second term is also negative. Now we try to resolve the third nonlinear term so that it can be controlled by the first two.

Lemma 3.5. *There is a constant C such that for all $u \in H^1(\mathbb{R})$ and $R \in \mathbb{R}_+^*$,*

$$\|u^2\phi^{\frac{1}{2}}\|^2_{L^\infty(|x|>R)} \le C\left(\int_{|x|>R} u^2\right)\left(\int u_x^2\phi + \int u^2\phi\right).$$

This implies

$$\begin{aligned}(I) = \int_{|x-x(t)|>R} u^6\phi(\tilde{x}) &\le \|u^2\phi^{\frac{1}{2}}(\tilde{x})\|^2_{L^\infty(|x-x(t)|>R)}\left(\int_{|x-x(t)|>a} u^2\right)\\ &\le \left(\int_{|x-x(t)|>a} u^2\right)^2\left(\int u_x^2\phi(\tilde{x}) + \int u^2\phi(\tilde{x})\right). \end{aligned}\tag{3.25}$$

Since

$$\int_{|x-x(t)|>R} u^2 \le \int_{|y|>\frac{R}{1.1}} (Q_b+\varepsilon)^2 \lesssim e^{-\frac{R}{1.1}} + |b|^{2-\gamma} + \int \varepsilon^2,$$

we can choose R large enough so that

$$(I) \le \frac{1}{36}\left(\int u_x^2\phi(\tilde{x}) + \int u^2\phi(\tilde{x})\right).$$

On the other hand, we have

$$\|\varepsilon(t)\|_{L^\infty} \le \|\varepsilon(t)\|_{H^1} \le \delta(\alpha^*),$$

$$\|u(t)\|^4_{L^\infty} = \left\|\frac{1}{\lambda^{\frac{1}{2}}(t)}\left(Q_{b(t)}+\varepsilon\right)\left(t, \frac{x-x(t)}{\lambda(t)}\right)\right\|^4_{L^\infty} \le \frac{C}{\lambda^2(t)} \le Cx_t.$$

Therefore

$$\begin{aligned}(II) = \int_{|x-x(t)|<R} u^6\phi(\tilde{x}) &\le \|u(t)\|^2_{L^2}\|\phi(\tilde{x})\|_{L^\infty(|x-x(t)|<R)}\|u(t)\|^4_{L^\infty}\\ &\le Cx_t \max_{|x-x(t)|<R} e^{-\frac{1}{K}|x-x(t_0)+x_0-\frac{1}{4}(x(t)-x(t_0))|}. \end{aligned}\tag{3.26}$$

Note the fact that $x(t)$ increases in time. If $x_0 \le R$, then $|x-x(t)| < R$ implies $x - x(t_0) + x_0 - \frac{1}{4}(x(t)-x(t_0)) > \frac{3}{4}(x(t)-x(t_0)) \ge 0$. Thus

$$(II) \le Cx_t e^{\frac{3}{4K}(x(t)-x(t_0))}e^{-\frac{x_0}{K}}.$$

To conclude, we have

$$\forall t \ge t_0,\ \frac{d}{dt}\mathcal{I}_{x_0,t_0}(t) \le Cx_t e^{\frac{3}{4K}(x(t)-x(t_0))}e^{-\frac{x_0}{K}}.$$

By integration in time, it follows that

$$\mathcal{I}_{x_0,t_0}(t) - \mathcal{I}_{x_0,t_0}(t_0) \le Ce^{-\frac{x_0}{K}} \le Ce^{-\frac{x_0}{3}},$$

and Lemma 3.3 is proved.

When it comes to the monotonicity formula for ε, we will use the equation of $\varepsilon(s)$ (2.24) to derive a Kato-type equation similar to (3.21), and then estimate all the positive terms with respect to the terms with a good sign.

4. **Construction of minimal element**

The strategy of the construction of the minimal element is as follows: firstly, take a defocusing sequence (u_n) corresponding to some well prepared data; secondly, at the exit time t_n^*, prove by the control of the dynamical information that $\lambda_n(t_n^*) \sim \sqrt{n}$; thirdly, renormalize u_n by $\lambda_n(t_n^*)$ so that it is more focused, and extract a weak limit; fourthly, study the backward solution of this weak limit and prove it is the minimal element.

step 1 Well prepared initial data. Let $u_n(0) = Q_{-\frac{1}{n}}$, hence

$$u_n(0) \to Q \quad \text{in } H^1 \text{ as } n \to +\infty, \quad u_n(0) \in \mathcal{A}.$$

By (2.16), one gets $\int u_n^2(0) < \int Q^2$ for n large enough, hence the solution $u_n(t)$ is global. Besides, it cannot converge locally to a solitary wave like in the (Soliton) case of Theorem 1.6 due to the strictly subcritical mass and mass conservation. Hence (Defocusing and Exit) holds and we define the *exit time* by

$$t_n^* = \sup\{t > 0,\ \text{such that } \forall t' \in [0,t],\ u_n(t') \in \mathcal{T}_{\alpha^*}\}.$$

From the continuous dependence of the solution with respect to the initial data, and the fact that $Q(x-t)$ is the corresponding solution with initial data $Q(x)$, one gets that $t_n^* \to +\infty$ as $n \to +\infty$.

On $[0, t_n^*]$, $u_n(t)$ has a decomposition $(\lambda_n, x_n, b_n, \varepsilon_n)$ as in Lemma 2.3, with initially

$$\lambda_n(0) = 1, \quad x_n(0) = 0, \quad b_n(0) = -\frac{1}{n}, \quad \varepsilon_n(0) = 0. \tag{4.1}$$

By the definition of t_n^*,

$$\inf_{\lambda_0>0,\ x_0\in\mathbb{R}} \left\| u_n(t_n^*) - \frac{1}{\lambda_0^{\frac12}} Q\left(\frac{\cdot - x_0}{\lambda_0}\right) \right\|_{L^2} = \alpha^*, \tag{4.2}$$

so that

$$\left\| u_n(t_n^*) - \frac{1}{\lambda_n^{\frac12}(t_n^*)} Q\left(\frac{\cdot - x_n(t_n^*)}{\lambda_n(t_n^*)}\right) \right\|_{L^2} \geq \alpha^*. \tag{4.3}$$

step 2 Description of the modulation parameters of u_n. Combining the mass and energy conservation for u_n, the (almost) monotonicity of $b, \mathcal{N}, \lambda$, and the "boundary" condition , we conclude with the following lemma:

Lemma 4.1.
(i) Estimates at t_n^*:

$$(\alpha^*)^2 \lesssim \frac{\lambda_n^2(t_n^*)}{n} \lesssim \delta(\alpha^*), \tag{4.4}$$

$$(\alpha^*)^2 \lesssim \int \varepsilon_n^2(t_n^*) \approx -b_n(t_n^*) \lesssim \delta(\alpha^*), \tag{4.5}$$

$$0 < c(\alpha^*) \leq \frac{t_n^*}{\lambda_n^3(t_n^*)} \leq C(\alpha^*). \tag{4.6}$$

(ii) Estimates on $[0,t_n^*]$:

$$\|\varepsilon_n(t)\|_{H^1}^2 \lesssim \frac{\lambda_n^2(t)}{n} \lesssim \delta(\alpha^*). \tag{4.7}$$

(iii) Control of the dynamics on $[0,t_n^*]$:

$$-(1-\delta(\alpha^*))\frac{b_n(t_n^*)}{\lambda_n^2(t_n^*)} \lesssim (\lambda_{0n})_t(t) \lesssim -(1+\delta(\alpha^*))\frac{b_n(t_n^*)}{\lambda_n^2(t_n^*)}. \tag{4.8}$$

Proof. By the mass conservation

$$\int u_n^2(0) = \int u_n^2(t), \quad \text{for all } t\in[0,t_n^*].$$

Since

$$\int u_n^2(0) = \int Q_{-\frac{1}{n}}^2 = \int Q^2 - O\left(\frac{1}{n}\right),$$

$$\int u_n^2(t) = \int Q_{b_n(t)}^2 + \int \varepsilon_n^2(t) + 2(\varepsilon_n(t), Q_{b_n(t)}),$$

we have

$$\int Q_{b_n(t)}^2 - \left(\int Q^2 + 2b_n(t)(P,Q)\right) + 2b_n(t)(P,Q) + \int \varepsilon_n^2(t) + 2(\varepsilon_n(t), Q_{b_n(t)}) = O\left(\frac{1}{n}\right). \tag{4.9}$$

By the orthogonal condition and (2.12), one gets

$$|(\varepsilon_n(t), Q_{b_n(t)})| = |(\varepsilon_n(t), b_n(t)\chi_{b_n(t)}P)| \lesssim |b_n(t)|^{\frac{1}{8}} \int \varepsilon_n^2(t) + |b_n(t)|^{1+\frac{9}{8}}. \tag{4.10}$$

Now combine (4.9),(2.16),(2.22) and (4.10), we conclude that for $t\in[0,t_n^*]$,

$$\int \varepsilon_n^2(t) \approx -b_n(t) \lesssim \delta(\alpha^*).$$

Moreover, by (4)

$$\alpha^* \le \left\| u_n(t) - \frac{1}{\lambda_n^{\frac{1}{2}}(t)} Q\left(\frac{\cdot - x_n(t)}{\lambda_n(t)}\right)\right\|_{L^2} = \|b_n(t_n^*)\chi_{b_n(t_n^*)}P + \varepsilon_n(t_n^*)\|_{L^2} \le \delta(\alpha^*).$$

Therefore for n large, we obtain

$$(\alpha^*)^2(1+\delta(\alpha^*)) \le \int \varepsilon_n^2(t_n^*) \approx -b_n(t_n^*).$$

By the energy conservation and coercivity of the operator L

$$\begin{aligned} \lambda_n^2(t)E(u_n(0)) = \lambda_n^2(t)E(u_n(t)) &= E(Q_{b_n(t)} + \varepsilon_n(t)) \\ &\ge \frac{1}{2}(L\varepsilon_n(t), \varepsilon_n(t)) + C|b_n(t)| - \delta(\alpha^*)\|\varepsilon_n(t)\|_{H^1}^2 \\ &\gtrsim \|\varepsilon_n(t)\|_{H^1}^2 + C|b_n(t)|, \end{aligned} \tag{4.11}$$

hence we conclude from (2.17) that

$$\|\varepsilon_n(t)\|_{H^1}^2 \lesssim \lambda_n^2(t)E(u_n(0)) \approx \frac{\lambda_n^2(t)}{n}. \tag{4.12}$$

From the dynamical information on $\frac{b}{\lambda^2}$ of Lemma 3.2 and the initial information (4.1), one gets: for all $t\in[0,t_n^*]$,

$$\frac{1-\delta(\alpha^*)}{n} \lesssim -\frac{b_n(t)}{\lambda_n^2(t)} \lesssim \frac{1+\delta(\alpha^*)}{n}, \tag{4.13}$$

combined with the boundedness of $-b_n(t_n^*)$ (4.5) one gets

$$(\alpha^*)^2 \lesssim \frac{\lambda_n^2(t_n^*)}{n} \lesssim \delta(\alpha^*). \tag{4.14}$$

(3.17) divided by λ^2 gives

$$\left|(\lambda_{0n})_t\frac{\lambda_n}{\lambda_{0n}} + \frac{b_n}{\lambda_n^2}\right| \lesssim \frac{\int \varepsilon_n^2 e^{-\frac{|y|}{10}}}{\lambda_n^2} + \frac{|b_n|}{\lambda_n^2}(\mathcal{N}_n^{\frac{1}{2}} + |b_n|^2) \lesssim \frac{1}{\lambda_n^2}(\mathcal{N}_n + |b_n|^2). \tag{4.15}$$

From the dynamical information on $\frac{b}{\lambda^2}$ and $\frac{\mathcal{N}}{\lambda^2}$ of Lemma 3.2, one gets

$$(1-\delta(\alpha^*))\frac{|b_n(t_n^*)|}{\lambda_n^2(t_n^*)} \lesssim \frac{|b_n(t)|}{\lambda_n^2(t)} \lesssim (1+\delta(\alpha^*))\frac{|b_n(t_n^*)|}{\lambda_n^2(t_n^*)}, \tag{4.16}$$

and for n large enough

$$\frac{\mathcal{N}_n(t)}{\lambda_n^2(t)} \lesssim \frac{|b_n(t)|^3}{\lambda_n^2(t)} + \frac{|b_n(0)|^3}{\lambda_n^2(0)} \lesssim \delta(\alpha^*)\frac{|b_n(t_n^*)|}{\lambda_n^2(t_n^*)}. \tag{4.17}$$

Combine (4.15), (4.16) and (4.17), one gets

$$(1-\delta(\alpha^*))\frac{|b_n(t_n^*)|}{\lambda_n^2(t_n^*)} \lesssim (\lambda_{0n})_t(t) \lesssim (1+\delta(\alpha^*))\frac{|b_n(t_n^*)|}{\lambda_n^2(t_n^*)}.$$

We integrate on $[0, t_n^*]$ and then divide by $\lambda_{0,n}(t_n^*)$ to obtain

$$-t_n^*\frac{b_n(t_n^*)}{\lambda_n^3(t_n^*)}(1-\delta(\alpha^*)) \le 1 - \frac{1}{\lambda_{0,n}(t_n^*)} \le -t_n^*\frac{b_n(t_n^*)}{\lambda_n^3(t_n^*)}(1+\delta(\alpha^*)), \tag{4.18}$$

together with the boundedness of $-b_n(t_n^*)$ (4.5) and $\frac{\lambda_n^2(t_n^*)}{n}$ (4.4), this implies

$$0 < c(\alpha^*) \le \frac{t_n^*}{\lambda_n^3(t_n^*)} \le C(\alpha^*).$$

This finishes the proof of the Lemma. □

step 3 Renormalization and extraction of the limit. From (4.4), one gets

$$\lambda_n(t_n^*) \sim \sqrt{n},$$

meaning intuitively that the forward flow of u_n is defocused by the scale of $\sqrt{n}$. We renormalize u_n by the scaling $\lambda_n(t_n^*)$ as follows:

$$\begin{aligned} v_n(\tau, x) &= \lambda_n^{\frac{1}{2}}(t_n^*)u_n\left(t_\tau, \lambda_n(t_n^*)x + x(t_n^*)\right) &(4.19)\\ &= \frac{\lambda_n^{\frac{1}{2}}(t_n^*)}{\lambda_n^{\frac{1}{2}}(t_\tau)}(Q_{b_n(t_\tau)} + \varepsilon_n)\left(t_\tau, \frac{\lambda_n(t_n^*)}{\lambda_n(t_\tau)}x + \frac{x(t_n^*) - x(t_\tau)}{\lambda_n(t_\tau)}\right), &(4.20)\end{aligned}$$

where

$$t_\tau = t_n^* + \tau\lambda_n^3(t_n^*) \quad \text{for} \quad \tau \in [-\frac{t_n^*}{\lambda_n^3(t_n^*)}, 0].$$

By the symmetries of gKdV, v_n is solution of (1.9), and it belongs to the tube $\mathcal{T}_{\alpha^*}$ for $\tau \in [-\frac{t_n^*}{\lambda_n^3(t_n^*)}, 0]$. Moreover, its decomposition $(\lambda_{v_n}, x_{v_n}, b_{v_n}, \varepsilon_{v_n})$ satisfies on $[-\frac{t_n^*}{\lambda_n^3(t_n^*)}, 0]$

$$\lambda_{v_n}(\tau) = \frac{\lambda_n(t_\tau)}{\lambda_n(t_n^*)},\ x_{v_n}(\tau) = \frac{x_n(t_\tau) - x_n(t_n^*)}{\lambda_n(t_n^*)},\ b_{v_n}(\tau) = b_n(t_\tau),\ \varepsilon_{v_n}(\tau) = \varepsilon_n(t_\tau), \tag{4.21}$$

and initially

$$\lambda_{v_n}(0) = 1, \quad x_{v_n}(0) = 0.$$

By (4.7) and (4.5), we have

$$\forall \tau \in [-\frac{t_n^*}{\lambda_n^3(t_n^*)}, 0], \quad \|\varepsilon_{v_n}(\tau)\|_{H^1}^2 = \|\varepsilon_n(t_\tau)\|_{H^1}^2 \lesssim \delta(\alpha^*),$$
$$(\alpha^*)^2 \le -b_{v_n}(0) = -b_n(t_n^*) \le \delta(\alpha^*).$$

Therefore

$$\|v_n(0) - Q\|_{H^1} \lesssim \delta(\alpha^*),$$

and we can extract a subsequence of (v_n) (without changing notations) such that they have a weak limit:

$$v_n(0) \rightharpoonup v(0) \text{ in } H^1, \text{ and } \|v(0) - Q\|_{H^1} \lesssim \delta(\alpha^*). \tag{4.22}$$

And by (4.6) and (4.5),

$$\tau_n^* = -\frac{t_n^*}{\lambda_n^3(t_n^*)} \to \tau^* < 0, \quad -b_n(t_n^*) \to b^* > 0.$$

We take $v(\tau)$ to be the backward H^1 solution of gKdV (1.9) with initial data $v(0)$ at $\tau = 0$.

step 4 Minimal mass blow up. We claim that v is a minimal mass blow-up solution with $\|v\|_{L^2} = \|Q\|_{L^2}$ and blows up in finite negative time τ^* with the speed

$$\frac{(1 - \delta(\alpha^*))}{\tau - \tau^*} \lesssim \|v_x(\tau)\|_{L^2} \lesssim \frac{(1 + \delta(\alpha^*))}{\tau - \tau^*} \tag{4.23}$$

for τ close enough to τ^*.

Indeed, we integrate (4.8) and obtain for $t \in [0, t_n^*]$ and n large enough,

$$t\frac{b^*}{\lambda^2(t_n^*)}(1 - \delta(\alpha^*)) \le \lambda_{0,n}(t) - \lambda_{0,n}(0) \le t\frac{b^*}{\lambda^2(t_n^*)}(1 + \delta(\alpha^*)).$$

We rewrite this for λ_{v_n} using (4.21) and obtain for $\tau \in [\tau_n^*, 0]$,

$$b^*(\tau - \tau_n^*)(1 - \delta(\alpha^*)) \le \lambda_{0,v_n}(\tau) - \lambda_{0,v_n}(\tau_n^*) \le b^*(\tau - \tau_n^*)(1 + \delta(\alpha^*)). \tag{4.24}$$

Take $\tau_0 \in (\tau^*, 0)$. Since from (4.4)

$$\lambda_{0,v_n}(\tau_n^*) = \lambda_{v_n}(\tau_n^*)(1 - J_1(\tau_n^*))^2 = \frac{\lambda_n(0)}{\lambda_n(t_n^*)}(1 - J_1(\tau_n^*))^2 \to 0 \quad \text{as} \quad n \to +\infty$$

and

$$1 - \delta(\alpha^*) \le \frac{\lambda_{0,v_n}}{\lambda_{v_n}} \le 1 + \delta(\alpha^*),$$

we conclude from (4.24) that for n large enough (depending on τ_0),

$$\forall \tau \in [\tau_0, 0], \quad b^*(\tau - \tau^*)(1 - \delta(\alpha^*)) \le \lambda_{v_n}(\tau) \le b^*(\tau - \tau^*)(1 + \delta(\alpha^*)),$$

and hence

$$0 < \frac{1}{2}b^*(\tau_0 - \tau^*) \le \lambda_{v_n}(\tau) \le -\frac{3}{2}b^*\tau^*.$$

It follows from the H^1-weak stability Lemma 2.4 that $v(\tau)$ is well-defined and $\lambda_{v_n}(\tau) \to \lambda_v(\tau)$ on $[\tau_0, 0]$. Therefore $v(\tau)$ exists on $(\tau^*, 0]$ and for all $\tau \in (\tau^*, 0]$,

$$b^*(\tau - \tau^*)(1 - \delta(\alpha^*)) \le \lambda_v(\tau) \le b^*(\tau - \tau^*)(1 + \delta(\alpha^*)).$$

In addition,

$$\|Q_y\|_{L^2} - \| (Q_{b_v} - Q)_y \|_{L^2} - \|(\varepsilon_v)_y\|_{L^2} \le \| (Q_{b_v} + \varepsilon_v)_y \|_{L^2} \le \| (Q_{b_v})_y \|_{L^2} + \|(\varepsilon_v)_y\|_{L^2},$$

as a consequence $\| (Q_{b_v} + \varepsilon_v)_y \|_{L^2}$ is bounded from upper and below. Therefore we obtain the estimate of $\|v_x(\tau)\|_{L^2}$ as in (4.23).

On the other hand, by weak convergence (4.22)

$$\int v^2(0) \leq \liminf_{n\to\infty} \int v_n^2(0) = \int Q^2.$$

Since v blows up in finite time, $\|v(0)\|_{L^2} = \|Q\|_{L^2}$ and v is a minimal mass element.

This construction method is clear and elegant. However, since the minimal element is obtained by convergence, the drawback is that it is difficult to extract more information about its structure. The next section is devoted to the sharp description of the general minimal element, which will in turn be used in the proof of uniqueness.

5. **Sharp description of the minimal element**

Now assume $u(t) \in \mathcal{C}((T,t_0],H^1)$ is any backward minimal mass blow-up solution which blows up in finite or infinite time: $-\infty \leq T < t_0$. The aim of this section is to derive sharp qualitative bounds on the modulation parameters, hence give a sharp description of the solution $u(t)$ near blow-up time. In particular we prove the blow-up time is necessarily finite.

5.1. **Sharp bounds of modulation parameters.** From standard concentration compactness argument (see for example [11]) one gets $u(t)$ satisfies the decomposition (2.18) with in addition

$$\|\varepsilon_1(t)\|_{H^1} \to 0 \quad t \to T.$$

Therefore choose a t_0 close enough to T, $u(t)$ admits on $(T,t_0]$ a decomposition

$$u(t,x) = \frac{1}{\lambda^{\frac{1}{2}}(t)}(Q_{b(t)} + \varepsilon)\left(t, \frac{x-x(t)}{\lambda(t)}\right)$$

as in Lemma 2.3, and

$$\forall t \in (T,t_0], \quad |b(t)| + \|\varepsilon(t)\|_{H^1} \leq \delta(\alpha^*). \tag{5.1}$$

As in Lemma 2.3 we define the rescaled time

$$s(t) = s_0 - \int_t^{t_0} \frac{ds}{\lambda^3}.$$

Since $u(t)$ blows up at time T, it is easy to get that $(T,t_0]$ corresponds to $(-\infty,s_0]$ for the s variable. We have the following proposition:

Proposition 5.1. *Assume $u(t) \in \mathcal{C}((T,t_0],H^1)$ is a backward blow-up solution with critical mass $\|u_0\|_{L^2} = \|Q\|_{L^2}$ and which blows up in finite or infinite time: $-\infty \leq T < t_0$.* (i) Finite time blow up*: there holds*

$$T > -\infty.$$

(ii) Sharp controls near blow-up time*: there exist universal constants c_λ, c_x, c_b and $\ell^* = \ell^*(u) > 0$, $x^* = x^*(u) \in \mathbb{R}$ such that, for t close to T,*

$$\|\varepsilon(t)\|_{L^\infty} \leq \|\varepsilon(t)\|_{H^1} \lesssim (t-T), \tag{5.2}$$

$$\lambda(t) = \ell^*(t-T) + O\left[(t-T)^3\right], \tag{5.3}$$

$$x(t) = -\frac{1}{(\ell^*)^2(t-T)} + x^* + O\left[(t-T)^2\right], \tag{5.4}$$

$$\frac{b(t)}{\lambda^2(t)} = -\ell^* + O\left[(t-T)^2\right], \tag{5.5}$$

$$\mathcal{N}(t) \lesssim (t-T)^6. \tag{5.6}$$

(iii) Global forward behavior*: the solution is globally defined for $t > T$, $u \in \mathcal{C}^\infty((T,+\infty)\times\mathbb{R})$ and for some $C(t)$, $\gamma(t) > 0$,*

$$\forall t > T,\ \forall x > 0, \quad |u(t,x)| \le C(t)e^{-\gamma(t)x}. \tag{5.7}$$

Once this proposition is proved, using the symmetries of gKdV it is easy to get a sharp description of the minimal element $S(t)$ obtained in section 4:

Corollary 5.2. *There exists a solution $S \in \mathcal{C}((0,+\infty),H^1)\cap\mathcal{C}^\infty((0,+\infty)\times\mathbb{R})$ to gKdV with critical mass $\|S(t)\|_{L^2} = \|Q\|_{L^2}$ and blows up backward at the origin such that:*

$$\|\partial_x S(t)\|_{L^2} \sim \frac{\|\partial_x Q\|_{L^2}}{t} \quad \text{as } t\downarrow 0, \tag{5.8}$$

$$S(t,x) - \frac{1}{t^{\frac12}}Q\left(\frac{x+\frac1t+\bar{c}t}{t}\right) \to 0 \quad \text{in} \quad L^2 \quad \text{as} \quad t\downarrow 0, \tag{5.9}$$

$$\forall x > 0, \quad |S(1,x)| \lesssim e^{-\gamma x} \tag{5.10}$$

for some universal constants $(\bar{c},\gamma)\in\mathbb{R}\times\mathbb{R}_+^*$.

5.2. **Strategy for the proof.** The proof of Proposition 5.1 is based on the following lemma:

Lemma 5.3 (Decay on the right for blow-up solution [6])**.** *Let $u(t)$ be a solution of gKdV defined on $(T,0]$, which blows up in finite or infinite time $-\infty \le T < 0$. Assume*

$$\int u^2(0) \le \int Q^2 + \alpha_0, \tag{5.11}$$

for $\alpha_0 > 0$ small enough and consider $(\lambda(t),x(t),b(t),\varepsilon(t))$ the decomposition of $u(t)$ for t close enough to T. Assume further that

$$u^2(t,x+x(t)) \rightharpoonup \left(\int u_0^2\right)\delta_{x=0} \quad \text{as } t\downarrow T. \tag{5.12}$$

If for some $\bar{t} > T$ close enough to T, there holds

$$\forall t\in(T,\bar{t}), \quad \lambda(t) \le 1.1\lambda(\bar{t}), \tag{5.13}$$

then

$$\forall y > 0, \quad |\varepsilon(\bar{t},y)| \lesssim e^{-\frac{y}{20}}. \tag{5.14}$$

step 1 Verify that the hypotheses (H1), (H2), (H3) hold on $(-\infty, s_0]$.

Recall by the proof of Lemma 4.1, especially (4.11), that

$$b(s) < 0, \quad 0 < E(u_0) \lesssim |b(s)| \le \delta(\alpha^*), \tag{5.15}$$

$$|b(s)| + \|\varepsilon(s)\|_{H^1}^2 \lesssim \lambda^2(s)E(u_0) \lesssim \lambda^2(s)\delta(\alpha^*). \tag{5.16}$$

And the blow-up assumption (finite time or infinite time) is equivalent to

$$\lim_{s\to-\infty}\lambda(s) = 0. \tag{5.17}$$

It therefore remains to prove (H3) i.e. decay on the right of the data.
From (5.17) and (5.16), we have the following blow-up profile

$$u^2(t,x+x(t)) \rightharpoonup \left(\int Q^2\right)\delta_{x=0} \quad \text{as } t\to T. \tag{5.18}$$

Since $\lambda(s)\to 0$, we can extract a sequence $s_n\to-\infty$ such that

$$\lambda(s_n) = 2^{-n} \quad \text{and} \quad \lambda(s) < 2^{-n}\ \forall s \le s_n. \tag{5.19}$$

Now we can apply Lemma 5.3 and obtain

$$\forall n \geq 1, \quad \forall y > 0, \qquad |\varepsilon(s_n, y)| \lesssim e^{-\frac{y}{20}}.$$

Therefore for a large constant A

$$\int_{y>0} y^{10}\varepsilon^2(s_n) \leq A^{10}\int \varepsilon^2(s_n) + \int_{y>A} y^{10}e^{-\frac{y}{10}} \leq \delta(\alpha^*),$$

for n large enough. Hence (H3) holds for the sequence (s_n). And the proof for $\frac{\mathcal{N}}{\lambda^2}$ is very similar.

By a continuity argument, there exists a small time s_0 such that (H3) holds on $(-\infty, s_0]$.

step 2 Prove that

$$\mathcal{N} \lesssim \lambda^6, \qquad \left|\frac{b}{\lambda^2} + \ell^* - \frac{c_0}{2}\frac{b^2}{\lambda^2}\right| \lesssim \lambda^3 \tag{5.20}$$

for some constant $\ell^* > 0$.
Inspired by the idea of using the relative sizes of $b(s)$ and $\mathcal{N}(s)$ to characterize different scenarios in [7], we first claim that there exists a sequence $s_m \to -\infty$ such that

$$\forall m \geq 1, \quad b(s_m) \leq -C^*\int \left(\varepsilon_y^2 + \varepsilon^2\right)(s_m)\varphi_B', \tag{5.21}$$

where $C^* > 0$ is the constant in (3.16). Assume by contradiction that there exists a time $s^* \leq s_0$ such that

$$\forall s < s^*, \quad |b(s)| \leq C^*\int \left(\varepsilon_y^2 + \varepsilon^2\right)(s)\varphi_B'. \tag{5.22}$$

Integrating (3.17) one gets $\forall s < s^*$,

$$\left|\log\left(\frac{\lambda_0(s)}{\lambda_0(s^*)}\right)\right| \lesssim \int_s^{s^*}\int \left(\varepsilon_y^2 + \varepsilon^2\right)(s)\varphi_B' ds \tag{5.23}$$

$$\lesssim \mathcal{N}(s) + |b^3(s^*)| + |b^3(s)| \tag{5.24}$$

$$\lesssim 1, \tag{5.25}$$

which contradicts the blow-up assumption $\lambda(s) \to 0$ as $s \to -\infty$.

By (H2) and the blow-up assumption (5.17), we have

$$\lim_{s\to-\infty} \mathcal{N}(s) = \lim_{s\to-\infty} b(s) = 0.$$

Thus passing (3.13) to the limit $s_1 \to -\infty$ we get

$$\mathcal{N}(s) + \int_{-\infty}^{s}\int (\varepsilon_y^2 + \varepsilon^2)\varphi_B' ds \lesssim |b^3(s)| \lesssim \lambda^6(s), \quad \forall s \leq s_0. \tag{5.26}$$

Take (5.21) in (3.16) yields

$$\frac{b(s)}{\lambda^2(s)} \sim \frac{b(s_m)}{\lambda^2(s_m)} \quad \forall s \in [s_m, s_0],$$

hence we conclude

$$\frac{b(s)}{\lambda^2(s)} \sim \frac{b(s_0)}{\lambda^2(s_0)} < 0,$$

Therefore there exists $\ell^* > 0$ such that

$$\lim_{s\to-\infty} \frac{b(s)}{\lambda^2(s)} = -\ell^* < 0. \tag{5.27}$$

Moreover, we can derive a precise bound on the limit. In fact, recall the refined modulation equations on $\frac{b}{\lambda^2}$ from (2.33):

$$\left|\frac{d}{ds}\left(\frac{b}{\lambda^2}e^J\right)+c_0\frac{b^3}{\lambda^2}\right| \le \frac{1}{\lambda^2}\left(\int \varepsilon^2 e^{-\frac{|y|}{10}}+|b|^4\right), \tag{5.28}$$

where

$$|e^J-1|\lesssim |J|\lesssim \mathcal{N}^{\frac{1}{2}}\lesssim \lambda^3.$$

Passing (3.14) to the limit $s_1\to-\infty$ yields that

$$\int_{-\infty}^{s}\frac{1}{\lambda^2}\left(\int \varepsilon^2 e^{-\frac{|y|}{10}}+|b|^4\right)ds\lesssim \frac{b^3}{\lambda^2}(s)\lesssim \lambda^4. \tag{5.29}$$

Combine this estimate with the rough modulation equations on b and λ

$$\left|\frac{\lambda_s}{\lambda}+b\right|+\left|\frac{x_s}{\lambda}-1\right|\lesssim \left(\int \varepsilon^2 e^{-\frac{|y|}{10}}\right)^{\frac{1}{2}}+b^2; \tag{5.30}$$

$$|b_s+2b^2|\lesssim |b|\left(\int \varepsilon^2 e^{-\frac{|y|}{10}}\right)^{\frac{1}{2}}+|b|^3+\int \varepsilon^2 e^{-\frac{|y|}{10}}, \tag{5.31}$$

we conclude from integration by part that

$$\int_{-\infty}^{s}\frac{b^3}{\lambda^2}=-\frac{1}{2}\frac{b^2(s)}{\lambda^2(s)}+O(\lambda^4).$$

Now integrating (5.28) on $(-\infty,s]$ and it follows that

$$\left|\frac{b}{\lambda^2}+\ell^*-\frac{c_0}{2}\frac{b^2}{\lambda^2}\right|\lesssim O(\lambda^4)+\frac{|b|}{\lambda^2}(1-e^J)\lesssim\lambda^3.$$

step 3 Solving modulation equations. Now since the modulation parameters and ε are all bounded by λ, we could decouple the modulation equation for λ.

From the previous step we have

$$\lambda_0=\lambda+O(\lambda J_1)=\lambda+O(\lambda^4) \tag{5.32}$$

and

$$b=-\ell^*\lambda^2+c_0(\ell^*)\lambda^4+O(\lambda^5). \tag{5.33}$$

Therefore

$$(\lambda_0)_t=\frac{\lambda_0}{\lambda^3}\frac{(\lambda_0)_s}{\lambda_0}=-\frac{\lambda_0}{\lambda^3}\left[b-c_1b^2+O(\mathcal{N}+|b|\mathcal{N}^{\frac{1}{2}}+|b|^3)\right] \tag{5.34}$$

$$=\ell^*-c_1(\ell^*)\lambda^2+O(\lambda^3)>\frac{\ell^*}{2}>0. \tag{5.35}$$

$T=-\infty$ is in contradiction with $\lambda_0\to 0$ as $t\downarrow T$, hence $T<-\infty$, i.e. the solution blows up in finite time.

Now integrating the decoupled ODE of λ_0 (5.34) we can easily get

$$\lambda(t)=|\ell^*|(t-T)+c(\ell^*)^4(t-T)^3+O\left[(t-T)^4\right],$$

and (5.5) follows. Besides

$$x_t=\frac{1}{\lambda^2}\frac{x_s}{\lambda}=\frac{1}{\lambda^2}\left[1+O(\mathcal{N}^{\frac{1}{2}}+b^2)\right]=\frac{1}{\lambda^2}+O(\lambda),$$

and inserting (5.3) one gets

$$x_t=\frac{1}{(\ell^*)^2(t-T)^2}-2c\ell^*+O(t-T).$$

Integrating on $(T,t_0]$ implies (5.4).

step 4 Global existence for the forward solution. Consider the forward flow starting from $s = s_0$. Recall that $\forall y > 0, |\varepsilon(s_0, y)| \lesssim e^{-\frac{y}{20}}$, and the exponential decay of Q and P (only on the right). We conclude that $u(t_0)$ has exponential decay on the right. From the Remark before, the critical mass solution $u(t)$ is globally defined for $t \geq t_0$. Finally it is proved by Kato [2] that solution of gKdV with such exponential decay on the right is in fact smooth in space and time. This finishes our proof.

References

[1] T. Cazenave and P.-L. Lions, Orbital stability of standing waves for some nonlinear Schrödinger equations, Comm. Math. Phys. **85**, (1982), 549–561.

[2] T. Kato, On the Cauchy problem for the (generalized) Korteweg-de Vries equation. Studies in applied mathematics, 93–128, Adv. Math. Suppl. Stud., 8, Academic Press, New York, 1983.

[3] Y. Martel and F. Merle, A Liouville theorem for the critical generalized Korteweg–de Vries equation, J. Math. Pures Appl. **79** (2000), 339–425.

[4] Y. Martel and F. Merle, Blow up in finite time and dynamics of blow up solutions for the L^2-critical generalized KdV equation, J. Amer. Math. Soc. **15** (2002), 617–664.

[5] Y. Martel and F. Merle, Nonexistence of blow-up solution with minimal L^2-mass for the critical gKdV equation. Duke Math. J. **115** (2002), 385–408.

[6] Y. Martel and F. Merle, Stability of blow up profile and lower bounds for blow up rate for the critical generalized KdV equation, Ann. of Math. **155** (2002), 235–280.

[7] Y. Martel, F. Merle and P. Raphaël, Blow up for the critical gKdV equation I: dynamics near the soliton. Preprint.

[8] Y. Martel, F. Merle and P. Raphaël, Blow up for critical gKdV equation II: minimal mass solution. Preprint.

[9] F. Merle, Determination of blow-up solutions with minimal mass for nonlinear Schrödinger equations with critical power. Duke Math. J. **69** (1993), 427–454.

[10] F. Merle, Existence of blow-up solutions in the energy space for the critical generalized KdV equation. J. Amer. Math. Soc. **14** (2001), 555–578.

[11] P. Raphael, On the singularity formation for the non linear Schrödinger equation, Notes of the 2008 Clay Summer School in Zurich.

[12] M.I. Weinstein, Nonlinear Schrödinger equations and sharp interpolation estimates, Comm. Math. Phys. **87** (1983), 567–576.

Amortissement Landau

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1 Introduction

1.1 Modélisation

Le problème à plusieurs corps est très difficile, celui à trois corps, le problème de Poincaré, a déjà une énorme complexité. Mais quand le nombre de particules dans le système devient très grand, on pourra passer du discret au continu, ce qui nous offre de nombreuses méthodes de résolution.

Ici nous considérons un système dont la force d'intéraction dérive d'un potentiel W qui est a une certaine intégrabilité, par exemple dans le cas linéaire, $\nabla W \in L^1(\mathbb{T}^d)$ ou dans le cas non-linéaire, sa transformée de Fourier doit satisfaire

$$|\widehat{W}(k)| = O(\frac{1}{|k|^{1+\gamma}}), \text{ pour } \gamma \geqslant 1.$$

En particulier, quand $\gamma = 1$, W est une solution fondamentale de $\pm\Delta$, qu'on appelle *le couplage de Poisson*. C'est un cas limite qui physiquement intéressant, (par exemple quand $d = 3$, cela correspond à l'interaction de Coulomb, la gravitation, etc.)

On considère N particules identiques, notées par leur position et vitesse, $\{(x_1(t), v_1(t)), \dots (x_N(t), v_N(t))\}$. Connaitre l'état du système à temps t est équivalent à connaitre tous les (x_i, v_i), *i.e* connaitre la mesure

$$\hat{\mu}_t^N = \frac{1}{N}\sum_{i=1}^{N} \delta_{(x_i(t), v_i(t))} \tag{1}$$

où $\delta_{(x,v)}$ dénote la masse de Dirac dans l'espace de phase $X^d \times \mathbb{R}^d$, où $X^d = \mathbb{R}^d$ ou $\mathbb{T}^d$. Dans notre exposé, nous ne considérerons que le cas $\mathbb{T}^d$, *i.e* ayant une périodicité spatiale, ce qui simplifie énormément l'étude de comportement asymptotique des solutions et est plutôt réaliste dans un plasma.

Notons que $\mathbb{P}(X^d \times \mathbb{R}^d)$ l'espace de probabilité auquel appartient $\hat{\mu}_t^N$, ne dépend pas du nombre N de particules. On va établir le lien entre les lois de Newton et l'équation des mesures par la proposition suivante.

Proposition 1. *Soit un système sans collisions qui vérifie les lois de Newton :*

$$\forall i,\ \dot{x_i} = v_i, \quad \dot{v_i} = -c \sum_{j \neq i} \nabla W(x_i - x_j) = -cN(\nabla W *_{x,v} \hat{\mu}^N),$$

alors

$$\partial_t \hat{\mu}^N + v \cdot \nabla_x \hat{\mu}^N + F^N[\hat{\mu}^N] \cdot \nabla_v \hat{\mu}^N = 0 \tag{2}$$

où

$$F^N[\hat{\mu}^N](t,x) = -c \sum_j \nabla W(x - x_j) = -cN(\nabla W *_{x,v} \hat{\mu}^N)$$

Démonstration. L'équation (2) est au sens de distribution, prenant une fonction test $\varphi = \varphi(x,v)$ appliquée à la partie gauche de l'équation, on a

$$\frac{d}{dt}\left[\frac{1}{N}\sum_i \varphi(x_i, v_i)\right] - \frac{1}{N}\sum_i (v \cdot \nabla_x \varphi)|_{(x_i,v_i)} - \sum_i (F^N \cdot \nabla_v \varphi)|_{(x_i,v_i)}$$
$$= \frac{1}{N}\sum_i (\nabla_x \varphi \cdot \dot{x_i} + \nabla_v \varphi \cdot \dot{v_i} - \nabla_x \varphi \cdot v_i - \nabla_v \varphi \cdot F^N(x_i)) = 0 \qquad \square$$

Supposons que $\hat{\mu}_t^N$ converge faiblement vers un certain μ_t, que

$$cN \xrightarrow[N \to \infty]{} \lambda, \tag{*}$$

et que W est uniformément continu, alors la mesure dépendant de temps $\mu = \mu_t(dx\,dv)$ est formellement solution de

$$\begin{cases} \partial_t \mu + v \cdot \nabla_x \mu + F[\mu] \cdot \nabla_v \mu = 0 \\ F = -\lambda \nabla W *_{x,v} \mu \end{cases}$$

Les cas où W a une singularité, Poisson par exemple, sont plus délicats à gérer, on admet alors que le problème est donné sous la forme de (2).

Physiquement il est intéressant de ne considérer que de répartition continue de matière, qui correspond à une mesure à densité, *i.e* $\mu_t(dx\,dv) = f(t,x,v)\,dx\,dv$. Le régime asymptotique (*) porte le nom "*limite de champ moyen*". On se ramène de milliers d'équations pour chaque particule suivant la loi de Newton à une seule équation dont la solution $f = f(t,x,v)$ est la densité de la répartition de matière à temps t dans l'espace des phases. On trouve finalement l'équation de Vlasov :

$$\begin{cases} \dfrac{\partial f}{\partial t} + v \cdot \nabla_x f + F[f] \cdot \nabla_v f = 0 \\ F[f] = -\nabla W *_x \rho \quad \rho(t,x) = \displaystyle\int f(t,x,v)dv. \end{cases} \tag{3}$$

L'étude du comportement asympotique de cette équation différentielle partielle non linéaire, en particulier la stabilité des solutions spatialement homogènes, fait l'objet de notre exposé.

1.2 Analogie avec l'équation différentielle en dimension finie

Pour étudier la stabilité d'une équilibre d'une équation différentielle autonome $X : \mathbb{R}^n \to \mathbb{R}^n$, *i.e* une solution $g(t) = g^0$ telle que $X(g^0) = 0$, on fait souvent appel au théorème de Lyapunov, qui ramène l'équation en question non-linéaire en une équation linéaire.

Théorème 2. *Si une équation différentielle autonome $X : \mathbb{R}^n \to \mathbb{R}^n$ est linéaire, alors l'équilibre* 0 *est* stable *si et seulement si :*

$$\mathrm{sp}(X) \subset \{z \in \mathbb{C},\ Re\, z \leqslant 0\} \quad et \quad (z \in \mathrm{sp}(X) \cap i\mathbb{R} \Rightarrow z \ est\ semi\text{-}simple)$$

et 0 *est* asymptotiquement stable *si et seulement si*

$$\mathrm{sp}(X) \subset \{z \in \mathbb{C},\ Re\, z < 0\}$$

Théorème 3 (Lyapunov)**.** *Si f^0 est un équilibre asympototiquement stable pour le système linéarisé $y' = X'(f^0)(y - f^0)$, alors f^0 est asymptotiquement stable pour X.*
Si f^0 est un équilibre de X tel que $X'(f^0)$ a une valeur propre de partie réelle strictement positive, alors f^0 est instable.

Ainsi, la stabilité asympotique linéaire implique la stabilité asymptotique non-linéaire. Et la non-stabilité linéaire due à une valeur propre de partie réelle > 0 implique la non-stabilité non-linéaire.

Le problème de stabilité est alors réduit à une question élémentaire d'algèbre, il suffit d'étudier le spectre de l'opérateur linéaire $X'(f^0)$.

On aimerait appliquer ces résultats à

$$\partial_t f = -(v \cdot \nabla_x f + F[f] \cdot \nabla_v f) = X(f)$$

où $X : C^\infty(\mathbb{T}^d \times \mathbb{R}^d, \mathbb{R}) \to C^\infty(\mathbb{T}^d \times \mathbb{R}^d, \mathbb{R})$. Malheureusement, on n'a pas une critère similiaire à celui de Lyapunov pour l'espace C^∞ qui n'est pas un Banach, sinon la question de la stabilité deviendrait très simple.

Mais on verra quand même une ressemblance entre les conditions suffisantes de stabilité de la Vlasov avec celles de la Vlasov linéarisée qui consistent à imposer une certaine régularité sur la donnée initiale f^i, l'équilibre étudié f^0 et le potentiel W, parmi lesquelles, une condition **(L)** est la même :

(L) Il existe λ, $\kappa > 0$ tels que

$$\forall \xi \in \mathbb{C} \text{ avec } 0 \leqslant Re(\xi) < \lambda|k|, \quad \inf_{k \in \mathbb{Z}^d} |K^L(\xi, k) - 1| \geqslant \kappa$$

avec

$$K^L(\xi,k)=\int_0^t e^{2\pi\xi^* t}K(t,k)\,dt,\ \text{ et, }\ K(t,k)=-4\pi^2\widehat{W}(k)\tilde{f}^0(kt)|k|^2 t.$$

Condition **(L)** est impliquée par exemple par les critères de Penrose, dont l'une est une condition de petitesse de $|\widehat{W}(k)|,|\tilde{f}^0(\eta)|$. Sous la forme de $F(\widehat{W}(k),\tilde{f}^0(\eta))<1$, où F est une fonction croissante.

1.3 Les résultats

La stabilité des solutions de la Vlasov linéarisée a été d'abord étudiée par Landau. On verra déjà une interprétation de la réversibilité de l'amortissement. Le travail de C.Mouhot et C.Villani [MV] étend les résultats de Landau pour l'équation de Vlasov-Poisson non-linéaire sous le régime perturbative (*i.e* la donnée initiale est suffisamment proche de l'équilibre, ce qui n'est pas nécessaire dans le cas linéaire).

Soient $f^0=f^0(v)$ un équilibre homogène analytique satisfaisant les critères de Penrose, et le potentiel d'interaction W qui satisfait

$$|\widehat{W}(k)|\leqslant\frac{C_W}{|k|^{1+\gamma}}\quad\text{pour tout}\quad k\in\mathbb{Z}^d$$

pour un certain $C_W>0$ et $\gamma\geqslant 1$. Alors on a la stabilité non-linéaire et l'amortissement non-linéaire au voisinage de f^0.

Plus précisément, on définit la norme

$$\|f\|_{\lambda,\mu}=\sup_{k\in\mathbb{Z}^d,\eta\in\mathbb{R}^d}e^{2\pi\mu|k|}e^{2\pi\lambda|\eta|}|\tilde{f}(k,\eta)|.$$

On peut alors énoncer le résultat de C.Mouhot et C.Villani :

Théorème 4 ([MV] C.Mouhot & C.Villani)**.** *Soit*

1. *f^0 et W satisfont la condition* ***(L)*** *avec un certain λ, $\kappa>0$;*
2. *avec le même paramètre λ, il existe un constant $C_0>0$ tell que*

$$\sup_{\eta\in\mathbb{R}^d}e^{2\pi\lambda_0|\eta|}|\tilde{f}^0(\eta)|\leqslant C_0,\quad \text{et}\quad \sum_{n\in\mathbb{N}^d}\frac{\lambda_0^n}{n!}\|\nabla_v^n f^0\|_{L^1(dv)}\leqslant C_0.$$

Alors pour tout $0<\lambda'<\lambda$, $\beta>0$ et $0<\mu'<\mu$, il existe

$$\epsilon=\epsilon(d,C_W,C_0,\kappa,\lambda,\lambda',\mu,\mu',\beta,\gamma)$$

tell que : si la condition initiale f^i est une perturbation de f^0 au sens où

$$\|f^i-f^0\|_{\lambda,\mu}+\int_{\mathbb{T}^d}\int_{\mathbb{R}^d}|f^i(x,v)-f^0(v)|e^{2\pi\beta|v|}dvdx\leqslant\epsilon,$$

alors il existe une solution globale, (ce qui n'est pas du tout évident pour une équation différentielle non-linéaire). Et il existe un équilibre homogène $f_{\pm\infty}(v)$ (correspond à $t \to \pm\infty$), tel que

$$\|f(t, x + vt, v) - f_{\pm\infty}(v)\|_{\lambda',\mu'} = O(e^{-2\pi\lambda'|t|}),$$
$$\|F(t,\cdot)\|_{C^r(\mathbb{T}^d)} = O(e^{-2\pi\lambda'|t|}), \quad \textit{pour tout} \quad r \in \mathbb{N}.$$

Nous donnerons le fil principal de la démonstration dans la partie 3, et y développerons l'étude de l'équation de la réaction dans un milieu oscillant où se produit un effet d'écho. Avant de se lancer dans la présentation des ingrédients principaux de leur preuve, on va d'abord voir une conséquence impressionnante de leurs résultats : C'est un amortissement réversible.

1.4 Quantité invariante

Par la méthode des caractéristiques, $f(t, S_{0,t}(x,v)) = f^i(x,v)$, où $S_{0,t}(x,v)$ est le flot du système :

$$\begin{cases} \dot{X}_t = V_t, \quad \dot{V}_t = F(t, X_t), \quad F = -\nabla W * \rho, \\ (X_0, V_0) = (x, v) \end{cases} \tag{4}$$

Proposition 5. *L'équation différentielle en A : $A'(t) = B(t)A(t)$, où $A(t), B(t) \in M_n(\mathbb{R})$ vérifient : $det(A(t)) = e^{\int_0^t tr(B(t))\,dt} det(A_0)$.*

Démonstration. Comme $det(A(t))$ vérifie l'équation différentielle : si $A(t_0)$ est inversible, alors

$$(det(A(t)))'|_{t=t_0} = d_{A(t_0)} det(A'(t_0))$$
$$= tr(A(t_0)^{-1}A'(t_0)) det(A(t_0)) = tr(B(t_0)) det(A(t_0)),$$

cette égalité est aussi vérifiée si $A(t_0)$ non inversible par la densité de $GL_n(\mathbb{R})$ et le régularité au moins C^2 de det. On a ainsi l'égalité cherchée par intégration en t. □

Proposition 6. *Le flot $S_{0,t}$ de (4) préserve la mesure $dx\,dv$,* i.e *le jacobien de $S_{0,t}$ est égal à 1 en tout (x,v).*

Démonstration. Un système d'équation de flot $S_{0,t}$, pour (x,v) fixé,

$$S'_{0,t}(x,v) = (\dot{x}_t, \dot{v}_t) = B_t(S_{0,t}(x,v))$$

ici $B_t(x,v) = (\nabla_v H, -\nabla_x H)$, en posant $H(X,V) = \frac{V^2}{2} + W * \rho$

$$\text{Jac}(S_{0,t})(x,v)'|_{t=t_0} = \text{Jac}(B_{t_0})(x(t_0), v(t_0)) \text{Jac}(S_{0,t_0})(x,v).$$

Si $tr(\mathrm{Jac}(B_t)(x(t), v(t))) = 0$, $\forall t$, alors en appliquant la proposition précédente, $det(\mathrm{Jac}(S_{0,t})(x, v)) = 1$ comme $S_{0,0} = Id$.
Ici c'est le cas car, le système (4) a une structure hamiltonienne,

$$\begin{cases} \dot{X}_t = \nabla_V H \\ \dot{V}_t = -\nabla_X H. \end{cases}$$

le théorème de Schwartz assure que la trace est constamment nulle. □

Remarque. Le résultat ci-dessus n'est pas vrai dans une variété quelconque.

Cette structure hamiltonienne assure l'invariance de mesure par $S_{0,t}$. Il en résulte que pour toute application $A : \mathbb{R} \to \mathbb{R}$ continue,

$$\iint A(f(t, x, v))\, dx\, dv$$

est invariante dans le temps. En particulier **l'entropie**

$$-\iint f \log(f)\, dx\, dv$$

est constante.

On obtient ainsi qu'autour d'une solution homogène $f^0(v)$ asymptotiquement stable pour une certaine topologie, $f(t, x, v)$ converge *sans augmentation d'entropie* vers $f^0(v)$ quand t tend vers $+\infty$, alors que la force associée $F(t, x)$ tend vers 0, d'où le nom «l'amortissement», et très contre-intuitivement, un amortissement réversible.

2 Approximation linéaire

2.1 Équation de transport libre

On va d'abord considérer l'équation de transport libre :

$$\begin{cases} \partial_t f + v \cdot \nabla_x f = 0 \\ f|_{t=0} = f^i(x, v) \end{cases} \tag{5}$$

On trouve rapidement, par la méthode des caractéristiques, l'égalité

$$f(t, x, v) = f^i(x - vt, v).$$

En appliquant la transformée de Fourier par rapport à $(x, v) \in \mathbb{T}^d \times \mathbb{R}^d$, on trouve

$$\tilde{f}(t, k, \eta) = \iint f^i(x - vt, v) e^{-2\pi i x \cdot k} e^{-2\pi i v \cdot \eta} dv dx = \tilde{f}^i(k, \eta + kt).$$

– Quand $k = 0$, $\tilde{f}(t,0,\eta) = \tilde{f}^i(0,\eta)$, signifie que $\langle f \rangle = \int f dx$ est conservé.
– Si $\tilde{f}^i(k,\eta) \leqslant C(e^{-2\pi\lambda|\eta|})$, pour η fixé et $k \neq 0$, on a alors $\tilde{f}(t,k,\eta) = \tilde{f}^i(k,\eta+kt) = O(e^{-2\pi\lambda|k|t})$, on a d'après [MV]

$$f(t,x,v) \xrightarrow[t\to\infty]{faiblement} \langle f^i \rangle$$

La notion de convergence faible n'est pas très claire dans leur papier. On peut l'interpréter au sens suivant, on pose

$$g(t,x,v) = f(t,x,v) - \langle f \rangle_x(v)$$

si pour tout t, , $g \in L^2(\mathbb{T}^d \times \mathbb{R}^d)$, alors pour toute fonction test $\varphi \in C_c^\infty(\mathbb{T}^d \times \mathbb{R}^d)$,

$$< g, \varphi >_{L^2(\mathbb{T}^d\times\mathbb{R}^d)} = < \tilde{g}, \tilde{\varphi} >_{L^2(\mathbb{Z}^d\times\mathbb{R}^d)} \xrightarrow[t\to\infty]{} 0$$

où la première égalité résulte du théorème de Plancherel, et la convergence s'obtient par le théorème de convergence dominée.

2.2 Équation de Vlasov linéarisée

Comme $F = -\nabla W * \rho$ est un gradient, $f(t,x,v) = f^0(v)$ est bien une solution homogène de (3). Nous allons étudier la stabilité de ces équilibres homogènes.

Soit f^0 un de ces équilibre, on cherche une solution de la forme

$$f(t,x,v) = f^0(v) + h(t,x,v) \quad \text{avec} \quad \|h\| = O(\epsilon).$$

On le remplace dans (3), et on espère que le terme $\|F[h]\cdot\nabla_v h\|$ est négligeable devant $\|F[h]\cdot\nabla_v f^0\|$. On obtient ainsi l'équation linéaire

$$\begin{cases} \partial_t h + v\cdot\nabla_x h + F[h]\cdot\nabla_v f^0 = 0 \\ h|_{t=0} = h^i(x,v) \end{cases} \tag{6}$$

Physiquement on peut penser se retrouver dans un milieu où interagissent deux sortes de particules, et seulement entre eux, dont l'une a une répartition invariante dans le temps.

Si l'on écrit $S = F[h]\cdot\nabla_v f^0$, alors $\partial_t h + v\cdot\nabla_x h = -S$. Par la méthode de caractéristiques et le principe de Duhamel, on a

$$h(t,x,v) = h^i(x-vt,v) - \int_0^t S(\tau, x - v(t-\tau), v)d\tau$$

et sa transformée de Fourier

$$\tilde{h}(t,k,\eta) = \tilde{h}^i(k,\eta+kt) - \int_0^t \tilde{S}(\tau,k,\eta+k(t-\tau))d\tau.$$

Une remarque très importante est que $S = F[h] \cdot \nabla_v f^0$ est le produit d'une fonction de (t, x) avec une fonction de v, cela implique que pour la transformée de Fourier de S, on a l'égalité :

$$\begin{aligned}
\tilde{S}(\tau, k, \eta) &= \widehat{F[h]}(t, k) \cdot \widetilde{\nabla_v f^0}(\eta) \\
&= -\widehat{\nabla W}(k)\hat{\rho}(t, k) \cdot \widetilde{\nabla_v f^0}(\eta) \\
&= 4\pi^2 k \cdot \eta\, \widehat{W}(k)\tilde{f}^0(\eta)
\end{aligned}$$

- si $k = 0,\ \tilde{S}(\tau, 0, \eta) = 0, \tilde{h}(t, 0, \eta) = \tilde{h}^i(0, \eta)$
- si $k \neq 0,$ soit $\rho^1(t, x) = \int h\, dv,$ alors $\hat{\rho}(t, k) = \widehat{\int f^0(v)\, dv + \int h(t, x, v)\, dv} = \hat{\rho}^1(t, k).$

d'où

$$\tilde{h}(t, k, \eta) = \tilde{h}^i(k, \eta + kt) - 4\pi^2 \widehat{W}(k) \int_0^t \hat{\rho}^1(\tau, k)\tilde{f}^0(\eta + k(t - \tau))k \cdot [\eta + k(t - \tau)]\, d\tau \tag{7}$$

on prend $\eta = 0$, on trouve

$$\hat{\rho}^1(t, k) = \tilde{h}^i(k, \eta + kt) + \int_0^t K(t - \tau, k)\hat{\rho}^1(\tau, k)\, d\tau \tag{8}$$

avec $K(t, k) = -4\pi^2 \widehat{W}(k)\tilde{f}^0(kt)|k|^2 t.$

La séparation des variables de S fait que les différentes modes de ρ^1 ne se couplent pas. On peut donc résoudre (8) mode par mode.

2.3 Équation de Volterra

On cherche à résoudre les équations (8) qui sont de forme

$$\phi(t) = a(t) + \int_0^t K(t - \tau)\phi(\tau)\, d\tau \tag{9}$$

Soit $\Phi(t) = e^{2\pi\lambda' t}\phi(t)$, $A(t) = e^{2\pi\lambda' t}a(t)$ dont on déterminera la valeur de λ' plus tard, (9) devient

$$\Phi(t) = A(t) + \int_0^t K(t - \tau)e^{2\pi\lambda'(t-\tau)}\Phi(\tau)\, d\tau$$

on prolonge Φ, K, A par 0 pour $t \leqslant 0$, et on prend la transformée par rapport à t, alors

$$\forall \omega \in \mathbb{R}, \quad \widehat{\Phi}(\omega) = \widehat{A}(\omega) + K^L(\lambda' + i\omega)\widehat{\Phi}(\omega)$$

où K^L désigne la transformée de Laplace complexe

$$K^L(\xi) = \int_0^{+\infty} e^{2\pi\xi^* t}K(t)\, dt,\ \xi \in \mathbb{C}$$

soit finalement

$$\widehat{\Phi}(\omega) = \frac{\widehat{A}(\omega)}{1 - K^L(\lambda' + i\omega)}$$

On suppose qu'il existe des constantes C_0, C_i, λ_0, λ, Λ, $\kappa > 0$, telles que

1. $|K(t)| \leqslant C_0 e^{-2\pi\lambda_0 t} \qquad |a(t)| \leqslant C_i e^{-2\pi\lambda_0 t}$
2. $|K^L(\xi) - 1| \geqslant \kappa > 0 \qquad$ pour $0 \leqslant Re\xi \leqslant \Lambda$

alors si $\lambda' < min\{\Lambda, \lambda, \lambda_0\}$, $K^L(\lambda' + i\omega)$ sont bien définis, et

$$|1 - K^L(\lambda' + i\omega)| \geqslant \kappa.$$

D'après le théorème de Plancherel,

$$\|\Phi\|_{L^2(dt)} = \|\widehat{\Phi}\|_{L^2(d\omega)} \leqslant \frac{\|\widehat{A}\|_{L^2(d\omega)}}{\kappa} = \frac{\|A\|_{L^2(dt)}}{\kappa} \leqslant \frac{C_i}{\kappa\sqrt{4\pi(\lambda - \lambda')}}$$

on peut donc borner Φ. On voit donc que la condition de type **(L)** est cruciale.

$$\begin{aligned}
\|\Phi\|_{L^\infty(dt)} &\leqslant \|A\|_{L^\infty(dt)} + \|(K e^{2\pi\lambda' t}) * \Phi\|_{L^\infty(dt)} \\
&\leqslant \|A\|_{L^\infty(dt)} + \|(K e^{2\pi\lambda' t})\|_{L^2(dt)} \|\Phi\|_{L^2(dt)} \\
&\leqslant C_i + \frac{C_0\, C_i}{4\pi\kappa\sqrt{(\lambda_0 - \lambda')(\lambda - \lambda')}}
\end{aligned}$$

d'où

$$\forall \lambda' < min\{\lambda, \lambda_0, \Lambda\},\ |\phi(t)| \leqslant C C_i e^{-2\pi\lambda' t} \text{ avec } C = C(C_0, \lambda_0, \lambda, \Lambda, \kappa, \lambda')$$

2.4 Conclusion du cas linéaire

Revenons à nos moutons, $K(t,k) = -4\pi^2 \widehat{W}(k) \tilde{f}^0(kt)|k|^2 t$.

Proposition 7. *Si on a*

1. Condition sur le potentiel $\|\nabla W\|_{L^1} \leqslant C_W < \infty$,

2. Condition sur la donnée initiale et le équilibre f^0
Il existe des constantes $\lambda > 0$, $C > 0$, tel que $\forall k \in \mathbb{Z}^d, \eta \in \mathbb{R}^d$

$$|\hat{f}^0(\eta)| + |\tilde{h}^i(k,\eta)| \leqslant C e^{-2\pi\lambda|\eta|}$$

3. (**L**) *est vérifié avec les constantes λ, $\kappa > 0$.*

Alors, la solution h de l'équation (6) vérifie $\forall \lambda' < \lambda$,

- *on a $\forall k \neq 0$, $|\hat{\rho}^1(t,k)| = O(e^{-2\pi\lambda'|k|t})$ donc uniformément en $k \in \mathbb{Z}^d_*$,*
- *$\|\rho(t,\cdot) - \langle\rho\rangle\|_{C^r} = O(e^{-2\pi\lambda'|t|})$ pour tout $r \in \mathbb{N}$,*

– $|\tilde{h}(t,k,\eta) - \tilde{h}^i(k,\eta+kt)| = O(e^{-2\pi\lambda'|kt|})$ *pour tout* $(k,\eta) \in \mathbb{Z}^d \times \mathbb{R}^d$.

Démonstration. – Le noyau K vérifie les conditions de l'équation de Volterra, on obtient : $\forall k \neq 0$, $|\hat{\rho}^1(t,k)| \leqslant C(\lambda,\lambda',\kappa)e^{-2\pi\lambda'|k|t}$.

– De plus $\|\rho^1 - \langle\rho^1\rangle\|_{H^s(\mathbb{T}^d)}(t) = O(e^{-2\pi\lambda' t})$ pour tout $s \in \mathbb{N}$. L'injection de Sobolev montre que la convergence est forte dans C^r.

– On peut contrôler l'intégrale dans l'équation (7), en prenant en compte $|\hat{\rho}(\tau,k)| = O(e^{-2\pi\lambda'|k|\tau})$ et $|\tilde{f}^0(\eta + k(t-\tau))| = O(e^{-2\pi\lambda'|\eta+k(t-\tau)|})$ et en cédant un peu sur λ', on a pour tout $\lambda'' < \lambda'$,
$|4\pi^2\widehat{W}(k)\int_0^t \hat{\rho}^1(\tau,k)\tilde{f}^0(\eta+k(t-\tau))k\cdot[\eta+k(t-\tau)]\,d\tau| = O(e^{-2\pi\lambda''|\eta+kt|})$.
D'où $|\tilde{h}(t,k,\eta) - \tilde{h}^i(k,\eta+kt)| = O(e^{-2\pi\lambda'|kt|})$. □

Remarque. 1. $h(t,\cdot) \xrightarrow[t\to\infty]{faiblement} \langle h^i\rangle$ par même argument dans la partie 2.1.

2. Comme $\tilde{h}(t,k,-kt) = O(1)$, la décroissance de $\tilde{h}$ n'est pas uniforme, mais chaque mode a une décroissance exponentielle pour $k \neq 0$, et pour $k = 0$ on sait déjà que $\tilde{h}(t,0,\eta) = \tilde{h}^i(0,\eta)$.

3. Au cours du temps, h voit une oscillation forte en mode $(k,-kt)$ qui correspond à une perte de régularité en vitesse.

Même si la densité spatiale converge fortement vers sa moyenne, et la force associée $F[h]$ converge exponentiellement vite vers 0, (dû au fait que $F[h] = -\nabla W * \rho$, $\widehat{F[h]}(t,k) = -\widehat{\nabla W}\cdot\hat{\rho}(t,k)$ qui a une même décroissance que ρ), l'information n'est pas vraiment perdue, elle est stockée dans l'oscillation des vitesses.

3 Équation de Vlasov non-linéaire

3.1 Méthode de Newton

La preuve de C. Mouhot et C. Villani utilise de manière essentielle la méthode de Newton, qui est une méthode itérative pour chercher un zéro x_∞ d'une fonction $\Phi(x)$. En faisant une bonne estimation, on fixe x_0 proche de x_∞. Au rang n, on résout l'équation

$$\Phi(x_n) + \nabla\Phi(x_n)\cdot(x_{n+1} - x_n) = 0.$$

Alors x_n converge vers le zéro x_∞ cherché avec l'estimation

$$|x_n - x_\infty| \leqslant C\delta^{2^n}, \quad \text{où } \delta = |x_0 - x_\infty|.$$

Pour une EDP abstraite

$$\begin{cases} \dfrac{\partial f}{\partial t} = Q(f) \\ f|_{t=0} = f^i(x,v) \end{cases},$$

la solution de cette EDP proche d'un équilibre homogène $f^0(v)$ est equivalente à un zéro au voisinage de $f^0(v)$ de la équation

$$Q(f) = \left(\frac{\partial f}{\partial t} - Q(f), f(0,\cdot) - f^i\right).$$

À partir de $f_0 = f^0(v)$, la méthode de Newton est de résoudre

$$\Phi(f_n) + \nabla\Phi(f_n) \cdot (f_{n+1} - f_n) = 0$$

Il faut faire attention que ici la dérivée ∇ est par rapport à la fonction au lieu d'une variable de dimension finie. C'est à dire, soit $h^n = f^n - f^{n-1}$, on veut résoudre l'EDP

$$\begin{cases} \partial_t h^1 = Q'(f^0) \cdot h^1 \\ h^1(0,\cdot) = f^i - f^0, \end{cases}$$

et pour $n \geqslant 1$,

$$\begin{cases} \partial_t h^{n+1} = Q'(f^n) \cdot h^{n+1} - [\partial_t f^n - Q(f^n)] \\ h^{n+1}(0,\cdot) = 0 \end{cases} \tag{10}$$

Par induction, pour $n \geqslant 1$, on a

$$\begin{cases} \partial_t h^{n+1} = Q'(f^n) \cdot h^{n+1} + \Big[Q(f^{n-1} + h^n) - Q(f^{n-1}) - Q'(f^{n-1}) \cdot h^n\Big] \\ h^{n+1}(0,\cdot) = 0 \end{cases}$$

Dans le cas de l'équation de Vlasov,

$$Q(f) = -v \cdot \nabla_x f - F[f] \cdot \nabla_v f$$

où la non-linéarité provient du terme quadratique, via la force $F[f] = \nabla W * \int f dv$. Donc on peut simplifier (10) notre équation avec

$$Q(f^{n-1} + h^n) - Q(f^{n-1}) - Q'(f^{n-1}) \cdot h^n = -F[h^n] \cdot \nabla_v h^n.$$

Donc, on définit la méthode de Newton pour la équation de Vlasov non-linéaire par :

$$f^0 = f^0(v) \quad \text{un équilibre homogène donné,}$$

$$f^n = f^0 + h^1 + \cdots + h^n,$$

où

$$\begin{cases} \partial_t h^1 + v \cdot \nabla_x h^1 + F[h^1] \cdot \nabla_v f^0 = 0 \\ h^1(0,\cdot) = f^i - f^0 \end{cases} \tag{11}$$

et pour $n \geqslant 1$,

$$\begin{cases} \partial_t h^{n+1} + v \cdot \nabla_x h^{n+1} + \underbrace{F[f^n] \cdot \nabla_v h^{n+1}}_{(a)} + \underbrace{F[h^{n+1}] \cdot \nabla_v f^n}_{(b)} = \underbrace{-F[h^n] \cdot \nabla_v h^n}_{(c)} \\ h^{n+1}(0, \cdot) = 0 \end{cases} \tag{12}$$

L'équation (11) est le même comme l'équation de Vlasov linéaire que on a étudiée dans la section 2. Mais l'équation (12) est assez compliquée. L'idée c'est de la diviser en trois parties.

Sans les termes (a), (b) et (c), (12) est l'équation de transport libre. Avec seulement (a), c'est une perturbation du transport libre avec une force $F[f^n]$ assez petite. Avec seulement (b), et l'équation est similaire à l'équation de Vlasov linéaire sauf que f^n dépend de (t, x, v) donc il n'y a pas de séperation des variables comme on a remarqué dans la partie 2. Et on peut montrer que la norme de (c) est petite dans un régime perturbatif, (mais bien entendu, il faut prouver que si f^i est proche de f^0, alors la solution reste toujours proche de f^0). L'étude de l'effet de (a) sera développée dans la section 3.3, l'effet de (b) dans 3.4.

3.2 La norme analytique

Le choix des normes est important dans l'analyse de l'équation de Vlasov, donc on présente ici quelques propriétés importantes de ces normes sans preuve. A priori, parmi les normes pour les fonctions analytiques, il y en a deux qui interviennent souvent dans la preuve.

Définition 1. Soit f une fonction analytique d'un variable, $p \in [1, \infty]$ et $\lambda \geqslant 0$. On définit

$$\|f\|_{\mathcal{C}^{\lambda;p}} := \sum_{n \in \mathbb{N}_0^d} \frac{\lambda^n}{n!} \|f^{(n)}\|_{L^p}, \quad \text{et} \quad \|f\|_{\mathcal{F}^{\lambda;p}} := \left(\sum_{k \in \mathbb{Z}^d} e^{2\pi\lambda p|k|} |\hat{f}(k)|^p \right)^{1/p},$$

Si $p = \infty$, ce dernier sera $\sup_{k \in \mathbb{Z}^d} |\hat{f}(k)| e^{2\pi\lambda|k|}$. On écrit

$$\mathcal{C}^\lambda := \mathcal{C}^{\lambda;\infty}, \quad \text{et} \quad \mathcal{F}^\lambda := \mathcal{F}^{\lambda;1}.$$

Les normes $\mathcal{C}^\lambda$ et $\mathcal{F}^\lambda$ nous intéressent parce qu'ils sont des normes d'algebre :

$$\|fg\|_\lambda \leqslant \|f\|_\lambda \|g\|_\lambda.$$

On utilise de nouveau la méthode des caractéristiques :

$$f(t, S_{0,t}(x, v)) = f^i(x, v),$$

ou l'inverse :

$$f(t,x,v) = f^i(S_{t,0}(x,v)) = f^i(S_{0,t}^{-1}(x,v)),$$

où $S_{0,t}$ est la caractéristique définite dans la section (1.4). Donc on voudrait considérer la propriété de ces normes quand on fait la composition d'une fonction analytique et la caractéristique. Mouhot et Villani ont prouvé la proposition suivante [MV] :

Proposition 8. *1. Pour tout $\lambda > 0$,*

$$\|f \circ (Id + G)\|_{\mathcal{F}^\lambda} \leqslant \|f\|_{\mathcal{F}^{\lambda+\nu}}, \quad \nu = \|G\|_{\dot{\mathcal{F}}^\lambda};$$

2. Pour tout $\lambda > 0$ et tout $a > 0$,

$$\|f \circ (aId + G)\|_{\mathcal{C}^\lambda} \leqslant \|f\|_{\mathcal{F}^{a\lambda+\nu}}, \quad \nu = \|G\|_{\dot{\mathcal{C}}^\lambda};$$

où

$$\|G\|_{\dot{\mathcal{F}}^\lambda} := \sum_{k\in\mathbb{Z}^d\setminus\{0\}} e^{2\pi\lambda|k|}|\hat{f}(k)|, \quad et \quad \|G\|_{\dot{\mathcal{C}}^\lambda} := \sum_{n\in\mathbb{N}_0^d\setminus\{0\}} \frac{\lambda^n}{n!}\|f^{(n)}\|_{L^\infty}$$

sont les normes de $\mathcal{F}^\lambda$ et $\mathcal{C}^\lambda$ privées de 0.

Quand on choisit une norme pour notre question (un variable périodique $x \in \mathbb{T}^d$, et l'autre variable $v \in \mathbb{R}^b$), on considère : d'abord, la norme $\mathcal{F}^\lambda$ applique au cas périodique $x \in \mathbb{T}^d$, et elle s'adapte bien à l'amortissement de Landau grâce à sa correspondance avec la décroissance exponentielle de la transformée de Fourier ; mais on n'a pas de transformée de Fourier pour $S_{0,t}(x,v)$ par rapport à v (note que l'application affine $x \mapsto a{\cdot}x+b$ appartient à $\dot{\mathcal{C}}^\lambda$ mais pas à $\dot{\mathcal{F}}^\lambda$), donc on choisit la norme $\mathcal{C}^\lambda$ pour la variable v. Donc on introduit la norme analytique hybride de deux variables :

Définition 2 (La norme hybride)**.** Pour λ, $\mu > 0$ et $p \in [1,\infty]$, on définit

$$\|f\|_{\mathcal{Z}^{\lambda,\mu;p}} := \sum_{k\in\mathbb{Z}^d}\sum_{n\in\mathbb{N}_0^d} \frac{\lambda^n}{n!} e^{2\pi\mu|l|}\|\widehat{\nabla_v^n f}(k,v)\|_{L^p(\mathbb{R}_v^d)}.$$

Il reste à traiter l'oscillation rapide de la variable v. Comme on a noté dans le cas du transport libre,

$$\tilde{f}(t,k,\eta) = \tilde{f}^i(k,\eta+kt).$$

Ça nous emene à introduire un parametre τ et appliquer le semi-groupe du transport libre $S_{0,\tau}^0(x,v) = (x+v\tau, v)$ à la fonction de répartition, ou précisément,

Définition 3. Pour λ, $\mu > 0$ et $p \in [1, \infty]$, on définit

$$\|f\|_{\mathcal{Z}_\tau^{\lambda,\mu;p}} := \|f \circ S_{0,\tau}^0\|_{\mathcal{Z}^{\lambda,\mu;p}} \sum_{k \in \mathbb{Z}^d} \sum_{n \in \mathbb{N}_0^d} \frac{\lambda^n}{n!} e^{2\pi\mu|l|} \left\| (\nabla_v + 2\pi i \tau k)^n \widehat{f}(k,v) \right\|_{L^p(\mathbb{R}_v^d)}.$$

L'idée c'est que on choisit τ égal ou au moins asymptotiquement égal à t, et on adapte l'ampleur de la régularité à l'oscillation de v, et alors on se concentre aux modes de Fourier importants qui glissent au cours de temps.

On peut montrer rapidement que :

Proposition 9. *1. Soit $f = f(x)$, alors*

$$\|f\|_{\mathcal{Z}_\tau^{\lambda,\mu}} = \|f\|_{\mathcal{F}^{\lambda|\tau|+\mu}};$$

2. soit $f = f(v)$, alors

$$\|f\|_{\mathcal{Z}_\tau^{\lambda,\mu;p}} = \|f\|_{\mathcal{C}^{\lambda;p}}.$$

Remarque. Note que la force $F[f]$ est une fonction de (t, x), donc pour tout τ

$$\|F[f](t,\cdot)\|_{\mathcal{Z}_\tau^{\lambda,\mu}} = \|F[f](t,\cdot)\|_{\mathcal{F}^{\lambda|\tau|+\mu}}.$$

En gros ça corresponde à la décroissance exponentielle de la transformée de Fourier de $F[f]$.

De plus,

Proposition 10. *1. Soit $\lambda \leqslant \lambda'$ et $\mu \leqslant \mu'$, alors*

$$\|f\|_{\mathcal{Z}_\tau^{\lambda,\mu}} \leqslant \|f\|_{\mathcal{Z}_\tau^{\lambda',\mu'}};$$

2. soit $\tau, \tau' \in \mathbb{R}$, alors

$$\|f\|_{\mathcal{Z}_{\tau'}^{\lambda,\mu}} \leqslant \|f\|_{\mathcal{Z}_\tau^{\lambda,\mu+\lambda|\tau'-\tau|}}.$$

Remarque. La première est facile à comprendre : on peut considérer λ et μ comme le largeur de convergence, donc on a plus de regularité avec l'augmentation de λ et μ. La dernière nous dit que les familles des espaces $\mathcal{Z}_\tau^{\lambda,\mu}$ ne sont pas monotone par rapport au parametre τ, donc on ne peut pas les comparer.

La norme (3) semble trop compliquée, mais on peut prouver l'injections entre elle et une norme plus standard

$$\|f\|_{X_\tau^{\lambda,\mu}} := \sup_{k \in \mathbb{Z}^d, \eta \in \mathbb{R}^d} |\tilde{f}(k,\eta)| e^{2\pi\lambda|\eta+kt|} e^{2\pi\mu|k|}$$

(avec une perte de la régularité suffisament petite). Finalement, même si on travaille et estimate la norme dans les espaces compliquées $\mathcal{Z}_\tau^{\lambda,\mu}$, on peut obtenir les résultats plus clairs sur la norme $X_\tau^{\lambda,\mu}$.

3.3 Caractéristiques avec l'amortissement de la force

Dans cette section on va étudier la première partie de la méthode de Newton, en gardant que le terme (a), on a l'équation :

$$\partial_t h^{n+1} + v \cdot \nabla_x h^{n+1} + F[f^n] \cdot \nabla_v h^{n+1} = 0$$

avec une force $F[f^n]$ assez petite. Pour comparer la dynamique perturbée avec le transport libre, on définit alors l'opérateur de deflection :
pour $0 \leqslant \tau \leqslant t$,

$$\Omega_{t,\tau}(x, v) := S_{t,\tau} \circ S^0_{\tau,t}(x, v),$$

Ici $S^0_{\tau,t}(x, v) = (x + (t - \tau)v, v)$ est la caractéristique du transport libre, $S_{\tau,t}(x, v) = (X_{\tau,t}, V_{\tau,t})$ est la caractéristique avec une perturbation $F(t, x)$, dont l'équation des caractéristiques s'écrit comme

$$\begin{cases} \dfrac{dX_{\tau,t}}{dt} = V_{\tau,t}, \quad \dfrac{dV_{\tau,t}}{dt} = F(t, X_{\tau,t}) \\ (X_{\tau,t}, V_{\tau,t})|_{t=\tau} = (x, v) \end{cases}$$

Comme c'est un processus réversible, $S_{t,\tau}$ est l'inverse de $S_{\tau,t}$. C'est à dire, on part au temps τ et évolue par la dynamique libre jusqu'au temps t, et après on revient en arrière en évoluant par la dynamique avec la perturbation. L'idée est de comparer $\Omega_{t,\tau}$ avec l'identité pour étudier l'influence de la perturbation. On a la proposition suivante :

Proposition 11. *Soit* $0 < \lambda' < \lambda$, $0 < \mu' < \mu$ *et*

$$\|F\| := \sup_{t \geqslant 0} \|F(t, \cdot)\|_{\lambda t + \mu} \leqslant \frac{(\mu - \mu')(\lambda - \lambda')^2}{C}$$

avec un constant C *assez grand, alors*

$$\|\Omega_{t,\tau} - Id\|_{\mathcal{Z}^{\lambda',\mu'}_{\tau}} \leqslant C\|F\| e^{-2\pi(\lambda - \lambda')\tau} \min(t - \tau, \frac{1}{\lambda - \lambda'}).$$

Remarque. Note que cette estimation est

1. de décroissance exponentielle quand $\tau \to \infty$,
2. uniforme quand $t \to \infty$,
3. petite quand $\tau \to t$.

3.4 Echos - Réaction contre un milieu oscillant

Dans cette partie, on va développer un peu plus sur l'effet donné par le terme (b) dans l'équation (12), $F[h^{n+1}] \cdot \nabla_v f^n$, *i.e* dans la méthode de Newton, on néglige :

$$\begin{cases} \partial_t h^{n+1} + v \cdot \nabla_x h^{n+1} + \cancel{F[f^n] \cdot \nabla_v h^{n+1}} + F[h^{n+1}] \cdot \nabla_v f^n = -\cancel{F[h^n] \cdot \nabla_v h^n} \\ h^{n+1}(0, \cdot) = 0 \end{cases}$$

L'équation étudiée est donc

$$\partial_t f + v \cdot \nabla_x f + F[f](t,x) \cdot \nabla_v \bar{f}(t,x,v) = 0 \tag{13}$$

où la fonction $\bar{f}$ est "le milieu oscillant forcé" supposé donné, qui diffère de la linéarisation discutée dans la partie 2 par sa dépendance en x. Il en résulte que dans la transformée de Fourier, la source devient convolution des coefficients de Fourier au lieu d'un simple produit.

On pose $\bar{f} = f^0 + \bar{h}$, f^0 est l'équilibre étudié. Comme dans la partie 2, on applique le principe de Duhamel, puis la transformation de Fourier, en intégrant par rapport à v, on a

$$\begin{aligned}
&\hat{\rho}(t,k) = \tilde{f}^i(k,kt) \\
&+ \int_0^t \iint (\nabla W * \rho)(\tau, x - v(t-\tau)) \cdot (\nabla_v f^0)(\tau, x - v(t-\tau), v) e^{-2i\pi k\cdot x} \, dx \, dv \, d\tau \\
&+ \int_0^t \iint (\nabla W * \rho)(\tau, x - v(t-\tau)) \cdot (\nabla_v \bar{h})(\tau, x - v(t-\tau), v) e^{-2i\pi k\cdot x} \, dx \, dv \, d\tau
\end{aligned}$$

dont le dernier intégrale égale à

$$\begin{aligned}
&\int_0^t \iint (\nabla W * \rho) \cdot (\nabla_v \bar{h})(\tau, x, v) e^{-2i\pi k\cdot x} e^{-2i\pi k\cdot v(t-\tau)} \, dx \, dv \, d\tau \\
&= \int_0^t \int \sum_{l\in\mathbb{Z}^d} \widehat{\nabla W}(k-l) \hat{\rho}(\tau, k-l) (\widehat{\nabla_v \bar{h}})(\tau, l, v) e^{-2i\pi k\cdot v(t-\tau)} \, dv \, d\tau \\
&= \int_0^t \sum_{l\in\mathbb{Z}^d} \widehat{\nabla W}(k-l) \hat{\rho}(\tau, k-l) (\widetilde{\nabla_v \bar{h}})(\tau, l, k(t-\tau)) \, d\tau
\end{aligned} \tag{14}$$

Remarque. Maintenant les modes de ρ sont couplés entre eux.

Pour voir l'effet de cette nouvelle source, on va supposer que $f^0 = 0$.

On rappelle la norme d'analyticité $\mathcal{F}$:

$$\|f\|_{\mathcal{F}^\lambda} = \sum_{k\in\mathbb{Z}} e^{2\pi\lambda|k|} |\hat{f}(k)|$$

Dans la partie 2, nous avons une borne des coefficients de Fourier de ρ, $|\hat{\rho}^1(t,k)| = O(e^{-2\pi\lambda'|k|t})$, il peut paraître naturelle d'étudier $\|\rho(t)\|_{\mathcal{F}^{\lambda t+\mu}}$, pour certains λ, μ déterminés par la petitesse de f^i et $\bar{h}$. En supposant

$$\widehat{\nabla W}(k-l) = O\left(\frac{1}{|k-l|^\gamma}\right), \text{ pour } \gamma \geqslant 1,$$

(14) permet d'obtenir une estimation de $\|\rho(t,\cdot)\|_{\mathcal{F}^{\lambda t+\mu}} = \varphi(t)$, avec $\bar{\mu} > \mu$ et $\bar{\lambda} > \lambda$ arbitraire fixés,

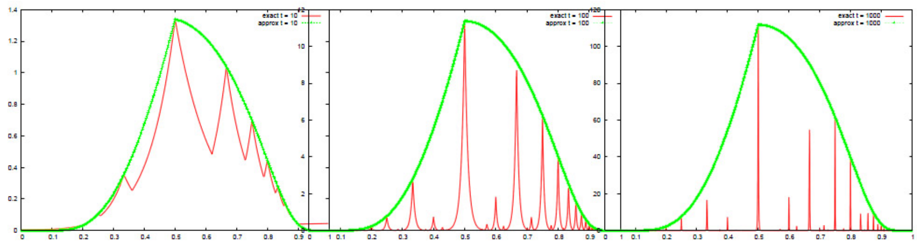
$$\|\rho(t,\cdot)\|_{\mathcal{F}^{\lambda t+\mu}} \leqslant A(t) + C \int_0^t K(t,\tau) \|\rho(\tau,\cdot)\|_{\mathcal{F}^{\lambda\tau+\mu}} \, d\tau \tag{15}$$

dans laquelle $A(t)$ est la contribution de la donnée initiale, et

$$K(t,\tau)=\sup_{k,l}\left(\frac{|k|(t-\tau)e^{-2\pi(\bar{\lambda}-\lambda)|k(t-\tau)+l\tau|}e^{-2\pi(\bar{\mu}-\mu)|l|}}{1+|k-l|^{\gamma}}\right). \tag{16}$$

Remarque. $K(t,\tau)$ dépend implicitement $\lambda,\mu,\bar{\lambda},\bar{\mu},\gamma$.

En gardant les termes dominants $l=-1$ et $k>0$ dans (16), on trace les courbes $K(t,\tau)$ en τ pour les valeurs $t=10$, 100, 1000, qui sont sous l'enveloppe dans la figure ci-dessous.



La masse de K se concentre sur certaines valeurs de τ telles qu'il existe $k,l\in\mathbb{Z}$

$$k(t-\tau)+l\tau=0.$$

Ainsi on pourra considérer K comme une somme de masse de Dirac à constante près. Il en produit un effet de resonance pour ces valeurs discrètes. Si la masse de K se concentre sur τ qui est loin de t, par exemple $\tau/2$, $A(t)$ borné par A une constante, alors (15) peut s'écrire comme :

$$\varphi(t)\leqslant A+c\frac{t}{2}\varphi\left(\frac{t}{2}\right).$$

Par simple itération, on obtient

$$\varphi(t)\leqslant A\sum_{n}\frac{c^n t^n}{2^{n(n-1)/2}} \tag{17}$$

qui a une croissance comme $Ae^{c'(\log t)^2}$, qui est plus lent que tous les $e^{\epsilon t}$, $\forall\epsilon>0$. Donc $\|\rho(t)\|_{\mathcal{F}^{\lambda' t+\mu'}}$ ainsi que pour $F(t)=\nabla W*\rho(t)$, en supposant $\hat{F}(t,0)=0$, $\forall\lambda'<\lambda$, on a toujours

$$\|F(t)\|_{\mathcal{F}^{\lambda' t+\mu'}}\leqslant CAe^{-2\pi(\lambda-\lambda')t}e^{\epsilon t}.$$

Pour ϵ assez petit, on a la décroissance exponentielle de F.

Des études plus poussées du noyau K permettent d'obtenir l'énoncé suivant :

Proposition 12. *Soit f^0 tel que $|\tilde{f}^0(\eta)| = O(e^{-2\pi\lambda_0|\eta|})$ et satisfait les critères de Penrose avec le largeur de stabilité λ_L, et $\widehat{\nabla W}(k) = O\left(\frac{1}{|k|^\gamma}\right)$, avec $\gamma \geqslant$. Et f^i et $\bar{h}$ vérifient une certaine petitesse associée à λ, μ, avec $0 < \lambda < min(\lambda_0, \lambda_L)$, et $\mu > 0$.*

Alors $\forall \mu' < \mu$, et $\lambda/2 < \lambda' < \lambda$,

$$\sup_{t\geqslant 0} \|F(t)\|_{\mathcal{F}^{\lambda' t+\mu'}} < C(\lambda', \mu'),$$

où $C(\lambda', \mu')$ est une fonction qu'on peut expliciter.

Remarque. L'intérêt d'avoir une régularité glissante de F bornée, c'est qu'on peut la transformer en tout moment en une *décroissance exponentielle dans le temps*, pourvu que F a une moyenne nulle, ici c'est le cas, car F est un gradient.
Plus précisément,

$$|||F||| := \sup_{t\geqslant 0} \|F(t, \cdot)\|_{\mathcal{F}^{\lambda t+\mu}} \tag{18}$$

alors $\forall\, \lambda' < \lambda$, $\mu' < \mu$, $s > \tau$,

$$\|F(s, \cdot)\|_{\mathcal{F}^{\lambda'\tau+\mu'}} \leqslant e^{-2\pi(\lambda s-\lambda'\tau)} \|F(s, \cdot)\|_{\mathcal{F}^{\lambda s+\mu}} \leqslant e^{-2\pi(\lambda-\lambda')s} |||F||| \tag{19}$$

De belles expériences physiques ont été réalisées par Malmberg et ses collegues, qui ont enregistrées des vrais effets de la résonance dans un plasma. *cf* [MA].

Références

[GL] *Gauss's law* from en.wikipedia.org.

[LCE] Lawrence C. Evans, *Partial differential equations*, American Mathematical Society, 1997.

[LS] Z. Lin, W. Strauss, *Nonlinear stability and instablity of relativistic Vlasov-Maxwell systems*, Communications on Pure and Applied Mathematics, 2007.

[MA] B. Malmberg, C. Wharton, R. Gould, T. O'Neil *Plasma wave echo experiment*, Pys. Rev. Letters 20, 3 (1968), 95 - 97.

[MV] C. Mouhot, C. Villani, *On Landau damping*, Acta Mathematica 207(2011), 29-201.

[SS] Elias M. Stein, Rami Shakarchi, *Princeton lectures in analysis : Complex analysis*, Princeton University Press, 2007.

[TB] B. Texier, *Équations différentielles ordinaires, Feuille de TD 1*, FIMFA, premier semestre 2011-2012.

[Vi] C. Villani, *Landau damping*, notes de cours, 2010, http ://math.univ-lyon1.fr/homes-www/villani/surveys.html

[VIA] V. I. Arnold, *Mathematical methods of classical mechanics*, Springer, 1989.