

A “modern” approach of the Theorem of the Three Closed Geodesics

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Introduction

All manifolds will be taken smooth and real; sometimes, we will allow boundaries.

Given a riemannian surface, we can (try to) count the number a simple closed geodesics (ie, “circles”) it contains. Of course, the number we get depends on the metric we put but the minimum does not. The case of the genus 0, providing orientability and compactness, has been resolved by Lusternick and Schnirelmann. But the original proof contained a mistake. It has been corrected since of course. Yet it does not seem one can find a proper proof of this theorem in literature.

The approach developed here try (among other things) to remediate this inconvenience. In the first part, we define the curve-shortening flow and give some of its basic –yet useful– properties. The second part try to set the framework of the Lusternick-Schnirelmann theorem. In the third, the two first are used to prove in a “modern”, “softer” way the so-called theorem of the Three Closed Geodesics.

I would like to thank M. Neves for having accepted to be my advisor, for his great advices and ideas and his patience.

1 A sort of example : the case of a torus

Let us consider a torus embedded in \mathbb{R}^3 . Classical Morse theory tells us that x_0, \dots, x_3 are some kind of special points. Of course, they depend on the embedding we took. Therefore, it would be pointless to try characterising them intrinsically. A simple way to find them would be to remark that they are the critical points for the height function h .

Here is another one, more “geometric”.

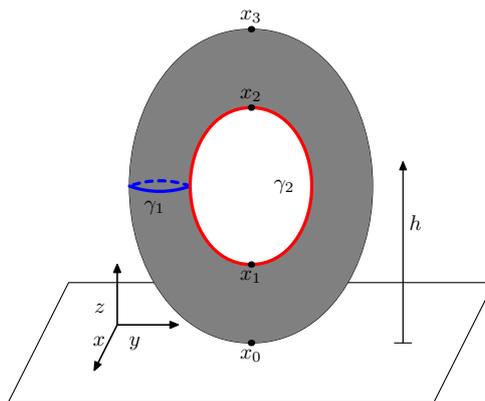


Figure 1: A torus embedded in \mathbb{R}^3

The integral homology of the torus is given by :

$$H_k(\mathbb{T}^2, \mathbb{Z}) = \begin{cases} \mathbb{Z}[\gamma_0] & \text{if } k = 0; \\ \mathbb{Z}[\gamma_1] \oplus \mathbb{Z}[\gamma_2] & \text{if } k = 1; \\ \mathbb{Z}[\gamma_3] & \text{if } k = 2; \\ 0 & \text{else,} \end{cases}$$

where $\gamma_0 = \{pt\}$ and $\gamma_3 = \mathbb{T}^2$. We can now characterise the x_i simply. Indeed, for each $i \in \{0, 1, 2, 3\}$, remark that x_i is solution to the equation :

$$h(x) = \min_{\gamma \in [\gamma_i]} \max_{y \in \gamma} h(y)$$

2 Curve-shortening flow : a digest

Curve-shortening flow is one of the simplest geometric flows¹. It is really helpful only for (real) surfaces; yet it has some strong properties we shall use later in our proof of the Three Closed Geodesics Theorem.

Let M be a (riemannian) surface, ΛM (resp. $\tilde{\Lambda} M$) the space of parametrised (resp. unparametrised) immersed curves in M . Thus, ΛM and $\tilde{\Lambda} M$ are related by $\tilde{\Lambda} M = \Lambda M / \text{Diff}(S^1)$. We equip ΛM with the sup distance $\|\cdot\|_\infty$.

¹This is not really helpful since nobody ever defined what a geometric flow was...

Let $\gamma \in \Lambda M$. A one-parameter family $(\gamma_t)_{0 \leq t \leq \tau}$ of curves in M is a solution to the curve shortening flow with initial data γ if it satisfies the following PDE:

$$\begin{cases} \frac{\partial \gamma_s}{\partial t}(\cdot, t) = \vec{k}(\cdot, t) \\ \gamma_0 = \gamma \end{cases}$$

where $\vec{k}(\cdot, t) = \nabla_T T$ is the vector curvature and T the tangent vector to the curve.

Another way to define Curve-Shortening flow is to define it as the natural gradient flow for the Length functional. Since geodesics are critical points for the it, this might convince us we are doing The Right ThingTM. By writing

It is worth pointing out that the curve shortening flow is invariant by diffeomorphism. So it can be defined not only on ΛM but also on $\tilde{\Lambda M}$. This considerably eases our task. In the original approach, we could not expect such a behaviour and we had to take care all over the proof to make sure we do not distinguish between geodesics (or more generally, loops) that are obtained one from the other by a simple reparametrisation. In the following theorem, uniqueness should be understood as "uniqueness up to time-dependent reparametrisations".

Anyway, we regroup in this following theorem all the properties we will need later :

Theorem 2.1. *Suppose M complete and take γ a smooth curve (fixed once for all). Then the following holds:*

Existence *there exists $\tau > 0$ and a one-parameter family $(\gamma_t)_{0 \leq t < \tau}$ solution to the curve shortening flow with initial data γ ;*

Uniqueness *if $(\gamma_t)_{0 \leq t < \tau}$ and $(\gamma'_t)_{0 \leq t < \tau'}$ are two solutions, there exists $0 < \tau'' \leq \min(\tau, \tau')$ such that $\gamma_t = \gamma'_t$ for $0 \leq t < \tau''$.*

Therefore, we can now consider the maximal solution $(\gamma_t)_{0 \leq t < \tau}$ and call it simply the solution.

Length's behaviour *for $0 \leq t \leq s < \tau$, $\text{length}(\gamma_t) \leq \text{length}(\gamma_s)$;*

Long-time behaviour *either the maximal solution shrinks in finite time to a point or it exists for all time and converges strongly to a geodesic ;*

Preservation of embedding *if γ is embedded, so are every γ_t for $t \in]0, \tau[$;*

Proof. We will only prove (iii); it derives from a computation. Indeed, we have:

$$\begin{aligned}
\frac{\partial}{\partial t} L &= \frac{\partial}{\partial t} \int_{S^1} \left\langle \frac{\partial}{\partial u} \gamma_s, \frac{\partial}{\partial u} \gamma_s \right\rangle^{1/2} du \\
&= \int_{S^1} \left\langle \frac{\partial^2}{\partial t \partial u} \gamma_s, \vec{T} \right\rangle du \text{ where } T \text{ is the unit tangent vector} \\
&= \int_{S^1} \left\langle \frac{\partial^2}{\partial u \partial t} \gamma_s, \vec{T} \right\rangle du \\
&= \int_{S^1} \left\langle \frac{\partial}{\partial u} k \vec{N}, \vec{T} \right\rangle du \text{ where } N \text{ is the unit normal vector who points "inside"} \\
&= \int_{S^1} -k^2 ds \leq 0
\end{aligned}$$

The other properties require some quite technical proofs. For example, Hamilton and Gage proved in [1] a very general theorem which implies short-time existence and uniqueness for the curve-shortening flow. The proof of the other properties can also be found in literature. See [4] or [2] for example. \square

3 Preliminary remarks

From now on, M is supposed to be diffeomorphic to a sphere and we take $u : S^2 \rightarrow M$, where S^2 is the so-called canonical sphere in \mathbb{R}^3 , a diffeomorphism. A *circle* on S^2 is the image of the intersection of S^2 with an affine plane. It will be said *strict* if it is not reduced to a point, *great* if the affine plane is actually vectorial. In particular, they are not parametrized !

A circle (resp. strict circle, great circle) on M is the image by u of a circle (resp. strict circle, great circle) on S^2 .

There exists a simple way of parametrising the set of (strict) circles on S^2 (resp. M) (see 2). This parametrisation induces a structure of manifold. It does not depend on the particular u choosen. We will always consider those sets with the structures defined above.

Proposition 3.1. *The following holds :*

- *The set of circles $\tilde{\Sigma}_{S^2}$ (resp. $\tilde{\Sigma}_M$) on S^2 (resp. M) is diffeomorphic to $S^2 \times [0, \frac{\pi}{2}] / (-x, \frac{\pi}{2}) \sim (x, \frac{\pi}{2}) \cong \mathbb{R}P^3$;*
- *The one of strict circles Σ_{S^2} (resp. Σ_M) is diffeomorphic to $\mathbb{R}P^3 \setminus \{\text{ball}\}$;*
- *The space of great circles is diffeomorphic to $\mathbb{R}P^2$.*

Proof. Everything is rather obvious considering : \square

²This is a rather bad definition for it is highly non-intrinsic ; if $M = S^2$ and u is not an isometry for example, we get two definitions of a circle which do not coincide. But it does not really matter for what we need.

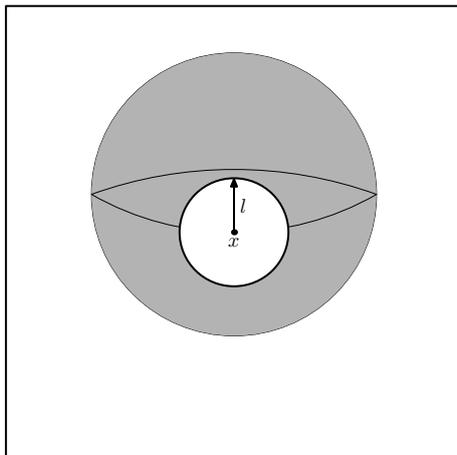


Figure 2: Parametrisation of the set of circles

We define a map $\pi_M : \Sigma_M \rightarrow P^2(\mathbb{R})$ by $\pi_M := p \circ u^{(-1)}$, where $u^{(-1)} : \Sigma_M \rightarrow \Sigma_{S^2}$ is the reverse image function induced by u and $p : \Sigma_{S^2} \rightarrow P^2(\mathbb{R})$ the one which maps a circle on S^2 to its axe. Thus, π_M is a fibration of D^1 fiber.

Take $\omega := (\omega_0, \omega_1, \omega_2)$ a triple of simplex such that ω_i generates the i degree (\mathbb{Z}_2)-homology of $P^2(\mathbb{R})$ and suppose that $\omega_0 \subset \omega_1 \subset \omega_2$. For any such ω , we consider the following subspaces of Σ :

- $\Omega_1(\omega) := \pi_M^{-1}(\omega_0)$ (homeomorphic to an open interval);
- $\Omega_2(\omega) := \pi_M^{-1}(\omega_1)$ (homeomorphic to a mobius band);
- $\Omega_3(\omega) := \pi_M^{-1}(\omega_2)$,

and for each $i \in \{1, 2, 3\}$ define the critical length-value $x_i(\omega)$ of $\Omega_i(\omega)$ by

$$x_i(\omega) = \inf_{t \rightarrow \infty} \sup_{\gamma \in \Omega_i(\omega)} \text{length}(\gamma_t)$$

where we used the notations defined in section 1.

From the very definition of the x_i ³, the following holds:

Proposition 3.2. *We have $0 \leq x_1 \leq x_2 \leq x_3$.*

Our goal is to prove that for each i there exists a closed geodesics of length x_i and to study what happens whenever two critical values happen to be equal (ie, there exists $x_i = x_j$ with $i \neq j$).

³We dropped the ω . Better get used to...

4 The Lusternick-Schnirelmann theorem

We first start by stating the aforesaid theorem :

Theorem 4.1. *Let M be a riemannian manifold whose underlying topological space is homeomorphic to S^2 . There exists a least three distinct closed geodesics on M .*

More precisely, the critical values satisfy $0 < x_1 \leq x_2 \leq x_3$ and for each i , there exists a simple closed geodesic with length x_i . If two of them are equal for any ω , then there exists infinitely many such geodesics.

The rest of this *mémoire* is devoted to the proof of this theorem. It consists on a succession of (finitely) many steps.

Proof. Step 1: We show that x_1 is actually positive.

Let γ be a simple curve on M . The isoperimetric inequality states that there exists a constant $c > 0$ (not depending on γ) such that $length(\gamma)^2 \geq c \times area(\gamma)(area(M) - area(\gamma))$, where $area(\gamma)$ is the area of one of the two domains bounded by γ (the expression does not depend on the one we have choosen). For $t = 0$, there exists a circle γ such that $area(\gamma) = area(M) - area(\gamma)$. More generally, for any time t , there exists a circle γ such that $area(\gamma_t) = area(M) - area(\gamma_t)$. For such a γ_t , we have

$$length(\gamma_t)^2 \geq c \left(\frac{area(M)}{2} \right)^2$$

From the very definition of x_1 , it follows that $x_1 > 0$.

Step 2: For each i , there exists a simple closed geodesics of length x_i .

Take $\delta \ll x_i$. Curve-shortening flow is length decreasing. Therefore, we have that

$$\inf_{t \rightarrow \infty} \sup_{\gamma \in \Omega_i} length(\gamma_t) = \inf_{t \rightarrow \infty} \sup_{\gamma \in \Omega_i, length(\gamma) \geq x_i - \delta} length(\gamma_t)$$

But $\Omega'_i := \{\gamma \in \Omega_i | length(\gamma) \geq x_i - \delta\}$ is compact and the length fonctionnal is continuous on it. Thus, there exists $\gamma \in \Omega'_i$ such that $x_i = \inf_{t \rightarrow \infty} length(\gamma_t) = \lim_{t \rightarrow \infty} length(\gamma_t)$. Since $x_i > 0$, (γ_t) does not shrink in finite time to a point and so (2.1) converge strongly to a geodesic. Again using 2.1, we deduce there exists a simple closed geodesics of length x_i

Step 3: Case where $0 < x_1 < x_2 < x_3$.

Obvious from what precedes. Indeed, the only problem we could have is that among the critical values, some are (or all) of them are just multiple coverings of the same "drawing". But curve-shortening flow respect embedness (2.1), meaning that if γ_0 is embedded, so are γ_t for every $0 \leq t \leq \tau$. In particular, in our case, γ_t cannot have self-intersection and therefore one of the geodesics we obtained cannot be a covering of another one.

Step 4: Case where $x_1 = x_2 = x > 0$.

Suppose that for *some* ω , $x_1 = x_2$ and that the number of simple closed geodesics with length $x_1(\omega) = x_2(\omega)$ ($= x$ briefly) is finite. Let's take one of them (say $\omega = (\omega_0, \omega_1, \omega_2)$)

once for all and drop its notation again. Up to reparametrisation, there is therefore only finitely many simple closed geodesics c_1, \dots, c_m with length x . For each $i \in \{1, \dots, m\}$, let U_i be an open neighbourhood "small enough" of c_i in ΛM such that for $i \neq j$, $U_i \cap U_j = \emptyset$. We define also U to be the union of the U_i .

For t large enough (but fixed), the following alternative is true : for $x \in \Omega_2$, either $x_t \in U$ or $length(x_t) < x$ ⁴. (The "or" is certainly not exclusive.) Let us put $O := \{x \in \Omega_2 | x_t \in U\}$. Note that O is simply connected (ie, each loop is homotopic to a point) and that $O \subset\subset \Omega_2$. Therefore, O is a (finite) union of disjoint open balls. So there exist $z_1, \dots, z_l \in O$ (one in each connected component of O) such that $H_*(\Omega_2 \setminus O) \rightarrow H_*(\Omega_2 \setminus \{z_1, \dots, z_l\})$ is an isomorphism.

Take V to be a disjoint union a ball $\Omega_2 \supset\supset V_i \ni z_i$. The long exact sequence of relative homology, where $Y = \Omega_2 \setminus O$ writes as:

$$\dots \rightarrow H_1(Y) \rightarrow H_1(\Omega_2) \rightarrow H_1(\Omega_2, Y) \rightarrow \dots$$

But by excision property

$$H_1(\Omega_2, Y) = H_1(\Omega_2 \setminus \overset{\circ}{Y}, Y \setminus \overset{\circ}{Y}) = H_1(\bigsqcup V_i, \bigsqcup V_i \setminus \{z_i\}) = \prod H_1(V_i, V_i \setminus \{z_i\}) = 0$$

So $H_1(Y) \rightarrow H_1(\Omega_2)$ is surjective. Id est, there exists a cycle κ which represents the one degree homology class of Ω_2 with no points in O . By drawing the same kind of argument as in Step 2, we would find a 0-cycle ω'_0 in κ (i.e. point in Ω_2 , i.e. a circle on M) such that

$$\inf_{t \rightarrow \infty} \sup_{\gamma \in \kappa} length(\gamma_t) = \lim_{t \rightarrow \infty} length(\omega'_{0_t}).$$

Now, consider $\omega' = (\omega'_0, \omega_1, \omega_2)$. By construction, $x_1(\omega') < x_2(\omega')$ so we can apply step 3.

Step 5: Case where $x_2 = x_3 = x' > 0$.

Same as Step 4. □

Remark. This result is optimal. Meaning there exist riemannian spheres with no more than three distinct simple closed geodesics. An ellipsoid with almost equal yet different axes for example. For the non-orientable case of genus 0, the result is the same : there exists at least three simple closed geodesics on any projective plane. For non-orientable case of genus 1 (the klein bottle), the answer is five. And for every other surface whether orientable or not, there always exists infinitely many simple closed geodesics.

References

- [1] M. Gage and R. S. Hamilton. The heat equation shrinking convex plane curves. *Journal of Differential Geometry*, 23:pp. 69–96, 1986.

⁴Indeed, if not, for any time t there would exist $x^t \in \Omega_2$ such that $x^t \notin U$ and $length(x^t) \geq x$. Proceeding just like in step 2, we would find a geodesic c with length x which is not in U ; this is absurd by definition of U .

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