SINGULAR RATIONALLY CONNECTED SURFACES WITH NON-ZERO PLURI-FORMS

Wenhao OU
supervised by
Stéphane Druel

Abstract

This paper is concerned with projective surfaces X with canonical singularities and having non-zero pluri-forms, *i.e.* $H^0(X, (\Omega_X^1)^{[m]}) \neq \{0\}$ for some $m > 0$, where $(\Omega_X^1)^{[m]}$ is the reflexive hull of $(\Omega_X^1)^{\otimes m}$. For such a surface, we can find a non-constant morphism from X to $\mathbb{P}^1$ which is a fibration. At the same time, we will obtain a method to construct all surfaces of this kind. Moreover, we can find a smooth curve E with positive genus such that $\mathbb{P}^1 = E/G$, where $G = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$. Then X is just the quotient Y/G where Y is the normalization of $X \times_{\mathbb{P}^1} E$ and $H^0(X, (\Omega_X^1)^{[m]}) \cong H^0(E, (\Omega_E^1)^{\otimes m})^G$, the G-invariant part of $H^0(E, (\Omega_E^1)^{\otimes m})$. The proof relies on methods of the minimal model program for surfaces.

Contents

1 Introduction and notation

Throughout this paper, we will work over $\mathbb{C}$, the field of complex numbers. Unless otherwise specified, every variety is an integral $\mathbb{C}$-scheme of finite type. A curve is a variety of dimension 1 and a surface is a variety of dimension 2. For a variety X, we denote the sheaf of Kähler differentials by Ω^1_X. Denote $\bigwedge^p \Omega^1_X$ by Ω^p_X for $p \in \mathbb{N}$.

For a coherent sheaf $\mathscr{F}$ on a variety X, we denote by $\mathscr{F}^{**}$ the reflexive hull of $\mathscr{F}$. There is an important property for reflexive sheaves.

Proposition 1.1. [Har80, Prop.1.6]. Let $\mathscr{F}$ be a coherent sheaf on a normal variety V. Then $\mathscr{F}$ is reflexive if and only if $\mathscr{F}$ is torsion-free and for each open $U \subseteq X$ and each closed subset $Y \subseteq U$ of codimension at least 2, $\mathscr{F}(U) \cong j_*\mathscr{F}(U \setminus Y)$, where $j : U \setminus Y \to U$ is the inclusion map.

If V is a normal variety, let V_{ns} be its smooth locus. We denote a canonical divisor by K_V. Moreover, let $\Omega^{[p]}_V$ (resp. $(\Omega^1_V)^{[p]}$) be the reflexive hull of Ω^p_V (resp. $(\Omega^1_V)^{\otimes p}$). By Proposition 1.1, it's just the push-forward of the locally free sheaf $\Omega^p_{V_{ns}}$ (resp. $(\Omega^1_{V_{ns}})^{\otimes p}$) to V since V is smooth in codimension 1. If V is not normal, $(\Omega^1_V)^{\otimes p}$ is not a locally free sheaf and it may contain some torsion. Thus, it is more interesting to study $H^0(V, (\Omega^1_V)^{[p]})$ than $H^0(V, (\Omega^1_V)^{\otimes p})$.

Definition 1.2. Consider a morphisme $f : X \to Y$. A Cartier divisor C in X is nef if its intersection number with any effective curve is non negative, it's f-nef if its intersection number with any effective curve contracted by f is non negative. A Weil divisor D is a $\mathbb{Q}$-Cartier divisor if there is an positive integer r such that rD is a Cartier divisor, and it's nef (*resp.* f-nef) if rD is a nef (*resp.* f-nef) Cartier divisor.

Definition 1.3. Let S be a normal surface. A morphism $r : \widetilde{S} \to S$ is called the *minimal resolution of singularities* (or *minimal resolution* for short) if $\widetilde{S}$ is smooth and $K_{\widetilde{S}}$ is r-nef.

Remark 1.4. There is a unique minimal resolution of singularities for a normal surface and any resolution of singularities factors through the minimal resolution.

Definition 1.5. Let S be a normal surface and $r : \widetilde{S} \to S$ be the minimal resolution of singularities of S. We say that S has *canonical singularities* if the intersection number $K_{\widetilde{S}} \cdot C$ is zero for every r-exceptional curve C.

Remark 1.6. In [KM98, Def.4.4], Definition 1.5 is the definition for *Du Val singularities*. However, by [KM98, Prop.4.11 and Prop.4.20] if S only has Du Val singularities, it has canonical singularities and it's automatically $\mathbb{Q}$-factorial (*i.e.* for every Weil divisor D on S there is an $m \in \mathbb{N}$ such that mD is a Cartier divisor). Thus these two definitions coincide and we have, in this case, $K_{\widetilde{S}} = r^*K_S$.

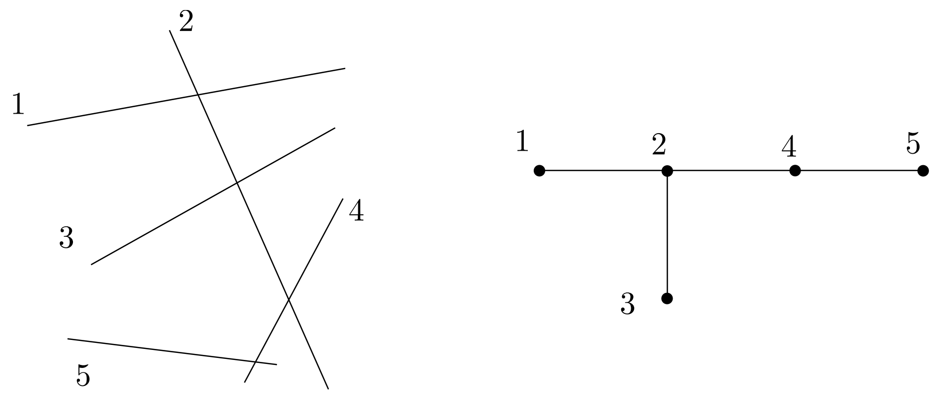
Definition 1.7. Let $p : S \to B$ be a fibration from a normal surface to a smooth curve. If the non-reduced fibers of p are p^*z_1,..., p^*z_r, then the *ramification divisor* R of p is the divisor $p^*z_1 + \cdots + p^*z_r - \mathrm{Supp}(p^*z_1 + \cdots + p^*z_r)$.

Definition 1.8. [KM98, Def.4.6]. Let $C = \bigcup C_i$ be a collection of proper curves on a smooth surface S. The *dual graph* Γ of C is defined as follow:

(1) The vertices of Γ are the curves C_i.

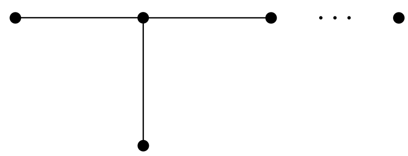
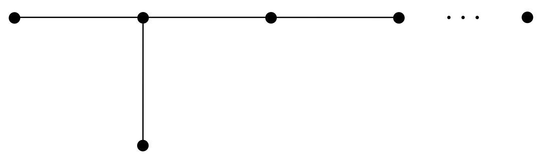
(2) Two vertices $C_i \neq C_j$ are connected with $C_i \cdot C_j$ edges.

For example,



collection of curves | dual graph

Theorem 1.9. Let $(0 \in S)$ be the germ of a Du Val singularity and $r : \widetilde{S} \to S$ be the minimal resolution of S, then we have a table as below (cf. [Reid] for more details)

Type of singularity	Dual graph of the exceptional divisor of r
A_i	
D_i	
E_i	

Definition 1.10. A curve C is a *rational curve* if there exists a non-constant morphism $f : \mathbb{P}^1 \to C$. A variety V is called *rationally connected* if for any two general points x_1, x_2, there exists a rational curve $C \subseteq V$ such that $x_1, x_2 \in C$, cf. [Kol96, Def.3.2 and Prop.3.6]. A variety V is called *rationally chain connected* if any two general points x_1, x_2 are connected by a chain of rational curves.

Remark 1.11. If V is a smooth variety, then V is rationally connected if and only if it's rationally chain connected (cf. [Kol96, IV,Thm.3.10]). In fact, a smooth rationally connected variety is firstly defined by connectedness by chain (cf. [KMM92] or [Cam91]). In [HM07], Hacon and McKernan prove the same equivalence for dlt pairs.

Definition 1.12. [Kol96, IV.Cor.3.8]. If X is a smooth projective rationally connected variety then $H^0(X, (\Omega_X^1)^{\otimes m}) = \{0\}$ for $m > 0$.

There is a conjectured numerical criterion for rational connectivity, which is the converse of the previous theorem. We call it Mumford's conjecture, although it is not clear when and how Mumford formulated it...

Conjecture 1.13. A smooth projective variety X is rationally connected if and only if $H^0(X, (\Omega_X^1)^{\otimes m}) = \{0\}$ for all $m > 0$.

There are some version of Theorem 1.12 for singular rationnally connected varieties. In [GKKP11, Thm.5.1], it's shown that if a pair (X, D) is klt and X is rationally connected, then $H^0(X, \Omega_X^{[m]}) = \{0\}$ for $m > 0$, where $\Omega_X^{[m]}$ is the reflexive hull of Ω_X^m. On the other hand, by [GKP12, Thm.3.3], if X is factorial, rationally connected and with canonical singularities, then $H^0(X, (\Omega_X^1)^{[m]}) = \{0\}$ for $m > 0$, where $(\Omega_X^1)^{[m]}$ is the reflexive hull of $(\Omega_X^1)^{\otimes m}$. However, this will not be true without the assumption of being factorial. There is an example given in [GKP12, example 3.7]. In this paper, our aim is to classify rationally connected surfaces with canonical singularities which have non-zero reflexive pluri-forms.

The following example is the one given in [GKP12, example 3.7].

Example 1.14. Let $\pi' : X' \to \mathbb{P}^1$ be any smooth ruled surface. Choose four distinct points q_1, q_2, q_3, q_4 in $\mathbb{P}^1$. For each point q_i, perform the following sequence of birational transformations of the ruled surface:

(i) Blow up a point x_i in the fiber over q_i. Then we get two (-1)-curves which meet transversely at x_i'.

(ii) Blow up the point x_i'. Over q_i, we get two disjoint (-2)-curves and one (-1)-curve. The (-1)-curve appears in the fiber with multiplicity two.

(iii) Blow down the two (-2)-curves. We get two singular points on the fiber, each of them is of type A_1.

In the end, we get a rationally connected surface $\pi : X \to \mathbb{P}^1$ with canonical singularities such that $H^0(X, (\Omega_X^1)^{[2]}) \neq \{0\}$.

In fact, we will prove that every projective rationally connected surface X with canonical singularities and having non-zero pluri-forms can be constructed by similar methods from a smooth ruled surface over $\mathbb{P}^1$. We have several steps:

(i) Take a smooth ruled surface $\pi_0 : X_0 \to \mathbb{P}^1$ and choose distinct points $q_1, ..., q_r$ in $\mathbb{P}^1$ with $r \geq 4$.

(ii) For each q_i, perform the same sequence of birational transformations as in Example 1.14. We get a fibered surface $\pi_1 : X_1 \to \mathbb{P}^1$. The non-reduced fibers of π_1 are $\pi_1^* q_1, ..., \pi_1^* q_r$

(iii) Perform finitely many times this birational transformation: blow up a smooth point on a non-reduced fiber and then blow down the strict transform of the initial fiber. We obtain another fibered surface $p : X_f \to \mathbb{P}^1$.

(iv) Starting from X_f, perform a sequence of blow-ups of smooth points, we get a surface X_{aux}.

(v) Blow down some exceptional (-2)-curves for $X_f \to X_{aux}$, we get the wanted surface X.

We have a theorem:

Theorem 1.15. If X is a projective rationally connected surface with canonical singularities such that $H^0(X, (\Omega^1_X)^{[m]}) \neq \{0\}$ for some $m > 0$, then X can be constructed by the method described above.

Note that we may produce some non-reduced fibers over $\mathbb{P}^1$ during the process above. In fact, they are the source of non-zero forms. We will prove the theorem below.

Theorem 1.16. Let X be a projective rationally connected surface with canonical singularities and having non-zero reflexive pluri-forms. Let X_f be a result of the MMP, then X_f is a Mori fiber space over $\mathbb{P}^1$. Let $p : X_f \to \mathbb{P}^1$ be the fibration. If r is the number of points over which p has non-reduced fibers, we have $r \geq 4$ and

$$H^0(X, (\Omega^1_X)^{[m]}) \cong H^0(X_f, (\Omega^1_{X_f})^{[m]}) \cong H^0(\mathbb{P}^1, \mathscr{O}_{\mathbb{P}^1}(-2m + [\frac{m}{2}]r))$$

for $m > 0$.

We note that both in Theorem 1.16 and in the construction of Theorem 1.15, we meet a surface named X_f. In fact, these two surfaces can be identical. On the other hand, there is a projective surface Y and a $4:1$ cover $\Gamma : Y \to X$. More precisely, we will prove the theorem below.

Theorem 1.17. *Let X be a projective rationally connected surface with canonical singularities and having non-zero pluri-forms, then we have a fibration $X \to \mathbb{P}^1$ given by Theorem 1.16. There is a smooth curve E with positive genus, a normal projective surface Y with canonical singularities and an action on E of the group $G := \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ such that Y is the normalization of $X \times_{\mathbb{P}^1} E$ and $X \cong Y/G$, $\mathbb{P}^1 \cong E/G$.*

Remark 1.18. With the notation in Theorem 1.17, we note that Y is not rationally connected. Moreover, if $r_Y : \widetilde{Y} \to Y$ is the minimal resolution of Y, then $H^0(\widetilde{Y}, (\Omega^1_{\widetilde{Y}})^{\otimes m}) \cong H^0(Y, (\Omega^1_Y)^{[m]})$, and the G-invariant part is isomorphic to $H^0(X, (\Omega^1_X)^{[m]})$. We have a diagram:

$$\begin{array}{ccc}
\widetilde{Y} & & \\
\Big\downarrow r_Y & & \\
Y & \xrightarrow[4:1\ \mathit{cover}]{\Gamma} & X \\
\Big\downarrow \pi' & & \Big\downarrow \pi \\
E & \xrightarrow[4:1\ \mathit{cover}]{\gamma} & \mathbb{P}^1
\end{array}$$

2 Preparation for the proof of Theorem 1.16

Minimal model program In algebraic geometry, the minimal model program is part of the birational classification of algebraic varieties. Its goal is to construct a birational model of any complex projective variety which is as simple as possible. Form a variety X, perform a sequence of birational applications $X = X_0 \dashrightarrow X_1 \dashrightarrow \cdots$, and the aim is to obtain a variety X_f (if we can) in the end, such that either K_{X_f} is nef, or we have a fibration $X_f \to Y$ such that $-K_{X_f}$ is ample on general fibers (for more details of MMP, cf. [KM98, §1.4 and §3.7]).

Let S be a projective rationally connected surface with canonical singularities, then we can run the minimal model program for S . We obtain a sequence of extremal contractions

$$S = S_0 \to S_1 \cdots \to S_n.$$

Proposition 2.1. With the notations above, S_n is a Mori fiber space, *i.e.* K_{S_n} is not nef.

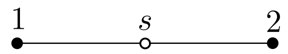
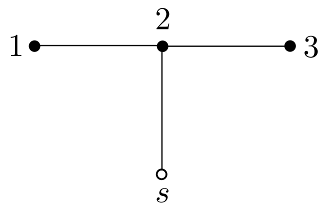
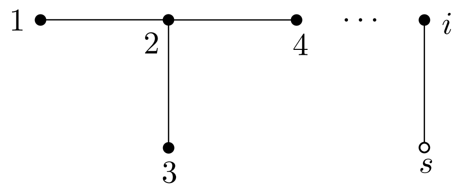
Proof. Since S is rationally connected and has canonical singularities, so is S_n. Assume that K_{S_n} is nef. Let $r : Y \to S_n$ be its minimal resolution of singularities, then $K_Y = r^* K_{S_n}$ is nef and Y is also rationally connected. But this contradicts [Kol96, IV. Cor.3.8]. □

By Proposition 2.1, we have a Mori fibration $p : S_n \to B$. Therefore we have two possibilities: either $\dim B = 0$ or $\dim B = 1$. Since S_n is rationally connected, so is B. Thus, if $\dim B = 1$, then $B \cong \mathbb{P}^1$.

A Fano surface S is a normal projective surface such that $-K_S$ is an ample $\mathbb{Q}$-Cartier divisor. Our aim is to prove the proposition below.

Proposition 2.2. Let S be a Fano surface with canonical singularities and with Picard number 1, then $H^0(S, (\Omega_S^1)^{[m]}) = \{0\}$ for any $m > 0$.

Theorem 2.3. Let S be a quasi-projective surface with canonical singularities and B be a smooth curve such that there is a Mori fibration $p : S \to B$ which has non-reduced fiber over $0 \in B$. Let $r : \widetilde{S} \to S$ be the minimal resolution and $\tilde{p}$ be $p \circ r$, then we have a table

Type of fiber	Dual graph
$(A_1 + A_1)$	
(D_3)	
(D_i)	

where the dual graph is the one of the support of $\tilde{p}^*0 \subseteq \widetilde{S}$ and s corresponds to $\widetilde{C}$.

3 Proof of Theorem 1.16

We will first prove Theorem 1.16. Let X be a rationally connected projective normal surface such that X has canonical singularities and $H^0(X, \Omega_X^{[m]}) \neq \{0\}$ for some $m > 0$. Run a MMP for X and we will get a sequence of divisorial contractions

$$X = X_0 \to X_1 \cdots \to X_n = X_f.$$

Let $X_{i,ns}$ be the smooth locus of X_i.

Proposition 3.1. For $m \in \mathbb{N}$, there is an injection

$$H^0(X, (\Omega_X^1)^{[m]}) \hookrightarrow H^0(X_f, (\Omega_{X_f}^1)^{[m]}).$$

Let $f : X \to X_f$ be the composition of the sequence of the MMP, then $H^0(X_f, (\Omega_{X_f}^1)^{[m]}) \neq \{0\}$. By Proposition 2.1, X_f is a Mori fiber space and we have a fibration $p : X_f \to \mathbb{P}^1$. Let $\pi = p \circ f : X \to \mathbb{P}^1$.

3.1 Source of non-zero reflexive pluri-forms

In this subsection, we will find out the source of non-zero pluri-forms on X_f. By Proposition 1.1, we have

$$H^0(X_f, (\Omega_{X_f}^1)^{[m]}) \cong H^0(U, (\Omega_U^1)^{\otimes m}),$$

where $m \in \mathbb{N}$ and U is any open subset of $X_{f,ns}$, the smooth locus of X_f, such that $X_f \setminus U$ has codimension at least 2 in X_f.

On the other hand, we have a natural morphism of locally free sheaves on $X_{f,ns}$:

$$(p|_{X_{f,ns}})^* \Omega_{\mathbb{P}^1}^1 \longrightarrow \Omega_{X_{f,ns}}^1.$$

Furthermore, if R is the ramification divisor of $p|X_{f,ns}$ there exist a factorisation:

$$(p|_{X_{f,ns}})^*\Omega^1_{\mathbb{P}^1} \longrightarrow ((p|_{X_{f,ns}})^*\Omega^1_{\mathbb{P}^1}) \otimes \mathscr{O}_{X_f,ns}(R) \xrightarrow{\rho} \Omega^1_{X_{f,ns}}$$

Moreover, $\rho \otimes k_x$ is injective for x in an open subset $V \subseteq X_{f,ns}$ such that $X_{f,ns} \setminus V$ is a finite set of points, where k_x is the residue field of x.

Thus we have an exact sequence

$$0 \longrightarrow ((p|_V)^*\Omega^1_{\mathbb{P}^1}) \otimes \mathscr{O}_V((R|_V)) \longrightarrow \Omega^1_V \longrightarrow \mathscr{G} \longrightarrow 0,$$

where $\mathscr{G} = \Omega^1_{V/\mathbb{P}^1}/(\text{torsion of } \Omega^1_{V/\mathbb{P}^1})$ is an invertible sheaf on V, for $\mathscr{G} \otimes k_x$ is of rank 1 at every point x of V, where k_x is the residue field of x.

Proposition 3.1.1. With the notations above, we have a natural isomorphism $H^0(X_f, (\Omega^1_{X_f})^{[m]}) \cong H^0(\mathbb{P}^1, \mathscr{O}_{\mathbb{P}^1}(-2m) \otimes (p|_{X_{f,ns}})_*\mathscr{O}_{X_{f,ns}}(mR))$.

Remark 3.1.2. $(p|_{X_{f,ns}})_*\mathscr{O}_{X_{f,ns}}(mR)$ is a torsion-free sheaf of rank 1 over $\mathbb{P}^1$, thus it is an invertible sheaf and there is a $k \in \mathbb{Z}$ such that $\mathscr{O}_{\mathbb{P}^1}(k) \cong (p|_{X_{f,ns}})_*\mathscr{O}_{X_{f,ns}}(mR)$. If this k is not less than $2m$, $H^0(\mathbb{P}^1, \mathscr{O}_{\mathbb{P}^1}(-2m) \otimes \mathscr{O}_{\mathbb{P}^1}(k)) \neq \{0\}$ and there exist non-zero reflexive pluri-forms over X_f.

3.2 Back to the initial variety

We have studied X_f and now we have to reverse the MMP and pull back reflexive pluri-forms to the initial variety X. In this subsection, we will prove that $H^0(X, (\Omega^1_X)^{[m]}) \cong H^0(X_f, (\Omega^1_{X_f})^{[m]})$ which ends the proof of Theorem 1.16.

Let $f : X \to X_f$ be the composition of the sequence in the MMP and $\pi = p \circ f : X \to \mathbb{P}^1$. Assume that the non-reduced fibers of p are $p^*z_1, ..., p^*z_r$ and the ones of π are $\pi^*z_1, ..., \pi^*z_r$, $\pi^*z'_1, ..., \pi^*z'_t$.

Proposition 3.2.1. For $m \in \mathbb{N}$, we have

$$(p|_{X_{f,ns}})_*\mathscr{O}_{X_{f,ns}}(mR) \cong \mathscr{O}_{\mathbb{P}^1}([\frac{m}{2}](z_1 + ... + z_r)) \cong \mathscr{O}_{\mathbb{P}^1}([\frac{m}{2}]r)$$

where [ ] is the integer part. In particular,

$$H^0(X_f, (\Omega^1_{X_f})^{[m]}) \cong H^0(\mathbb{P}^1, \mathscr{O}_{\mathbb{P}^1}(-2m + [\frac{m}{2}]r)).$$

We note that the fibers of $\pi : X \to \mathbb{P}^1$ have reduced components over $z'_1, \ldots, z'_t \in \mathbb{P}^1$ since $p : X_f \to \mathbb{P}^1$ has reduced fibers over these points. Thus the ramification divisor over these points will not give contribution to non-zero reflexive pluri-forms. Our aim is now to prove that $H^0(X, (\Omega^1_X)^{[m]}) \cong H^0(X_f, (\Omega^1_{X_f})^{[m]})$. To achieve this, it's enough to prove that the fibers of $\pi : X \to \mathbb{P}^1$ over $z_1, \ldots, z_r$ are non-reduced along each of their components, *i.e.* the coefficient of any component in $\pi^*(z_1 + \cdots + z_r)$ is larger than 1.

Proposition 3.2.2. The birational morphism $f : X \to X_f$ is an isomorphism around every singular point of X_f.

From Proposition 3.2.2, every exceptional divisor of $f : X \to X_f$ is over a smooth point of X_f.

Proposition 3.2.3. The fibers of $\pi : X \to \mathbb{P}^1$ over $z_1, \ldots, z_r \in \mathbb{P}^1$ are non-reduced along each of their components.

From Proposition 3.2.3, we obtain Theorem 1.16

References

[Cam91] F.Campana: *On twistor spaces of class C*, J. Diff. Geom **33** (1991) 541-549.

[CP11] F.Campana and T.Peternell: *Geometric stability of the cotangent bundle and the universal cover of a projective manifold*, Bull. Soc. Math. France **139**(2011), no.1, 41Â¨C74. **MR 1440180 (98c:14001)**.

[GKK10] D.Greb, S.Kebekus and S.J.Kovács: *Extension theorems for differential forms, and Bogomolov-Sommese vanishing on log canonical varieties*, Compos. Math. **146** (2010), 193-219, DOI:10.1112/S0010437X09004321.

[GKKP11] D.Greb, S.Kebekus, S.J.Kovács, and T.Peternell: *Differential forms on log canonical spaces*, Inst. Hautes Étude Sci. Publ. Math. **114**(2011), no.1, 87-169, DOI:10.1007/s10240-011-0036-0.

[GKP12] D.Greb, S.Kebekus and T.Peternell: *Reflexive differential forms on singular spaces – Geometry and Cohomology*, **arXiv:1202.3243v1 [math.AG]**.

[Har77] R.Hartshorne: *Algebraic Geometry*, Springer-Verlag, New York, 1977, Graduate Texts in Mathematics, No.52. **MR 0463157 (57 #3116)**.

[Har80] R.Hartshorne: *Stable Reflexive Sheaves*, Math. Ann. **254** (1980), 121-176. **MR 597077 (82b:14011)**.

[HM07] C.D.Hacon and J.McKernan: *On Shokurov's rational connectedness conjecture*, Duke Math. J. **138** (2007), no.1, 119-136. **MR 2309156 (2008f:14030)**.

[KK94] J.Kollár and S.J.Kovács: *Birational Geometry of Log Surfaces*, available from **https://web.math.princeton.edu/~kollar/**.

[KM98] J.Kollár and S.Mori: *Birational Geometry of Algebraic Varieties*, Cambridge Tracts in Mathematics, vol 134, Cambridge University Press, Cambridge, 1998, With the collaboration of C.H.Clemens and A.Corti, Translated from the 1998 Japanese original **2000b:14018**.

[KMM92] J.Kollar, Y.Myyaoka and S.Mori: *Rationally connected varieties*, J. Alg. Geom. **1** (1992), 429-448.

[Kol96] J.Kollár: *Rational Curves on Algebraic Varieties*, Ergebnisse der mathematik und ihrer Grenzgebiete.3. Folge. A Series of Modern Survey in Mathematics, vol.32, Springer-Verlag, Berlin, 1996. **MR 1440180 (98c:14001)**.

[Kol97] J.Kollár: *Singularities of Pairs*, Proc. Symp. Pure Math. **62.1**, 221Â¨C287 (1997).

[Kol07] J.Kollár: *Lectures on Resolution of Singularities*, Princeton: Princeton University Press. **ISBN 0691129231**.

[MP97] Y.Miyaoka and T.Peternell: *Geometry of higher-dimensional algebraic varieties*, DMV Seminar, vol.26, Birkhäuser Verlag, Basel, 1997. **MR 1468476 (98g:14001)**.

[Reid] M.Reid: *The Du Val singularities* A_n, D_n, E_6, E_7, E_8, available from **http://homepages.warwick.ac.uk/∼masda**.

[Zhang06] Q.Zhang: *Rational connectedness of log Q-Fano varieties*, Crelle J.**590**, 131-142 (2006).