Domino tilings and the Gaussian free field

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Summary

In this report, our main purpose is to present a relatively recent theorem due to Richard Kenyon which states that the scaling limit of the height function defined by domino tilings of simply connected domains of $\mathbb{R}^2$ with smooth boundary is the Gaussian free field [3, 5].

In the introduction, we first present domino tilings of polyomino and we define the height function. Then, we give a concise definition of the Gaussian free field.

The proof of the theorem is divided in two parts. In the first part, more combinatorial, we construct the coupling function which allows to compute probabilities of domino tilings. We describe the combinatorial assumption we use on the boundary of polyominos to approximate simply connected domains in $\mathbb{R}^2$ and which allow us to see the coupling function as a boundary value problem. In the second part, more analytical, we compute the scaling limit of the coupling function as the lattice size tends to zero. We then show that the scaling limit of the multi-point expectation of the height function is conformally invariant and that the scaling limit of the height function is the Gaussian free field.

Finally, we discuss the impact this work has had on the field and we mention some open questions.

Disclaimer: This report mostly presents work by Kenyon published in [3, 5]. To make the text easier to read, we do not cite this two papers each time we present a result from them. When no citation is mentioned, reference to [3, 5] is assumed.

Contents

Chapter 1

Introduction

1.1 Domino tilings and height function

1.1.1 Basic definitions

Definition 1.1.1. A **polyomino** is a connected union of unit squares (faces) in $\mathbb{Z}^2$. Two squares are connected if they share one common edge. Polyominos are colored in black and white in a checkerboard fashion.

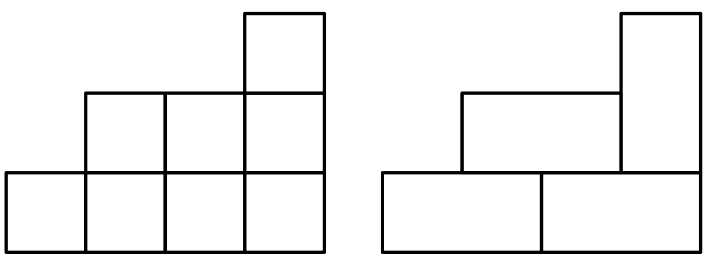

Figure 1.1: Example of polyomino and its tiling with dominos

Definition 1.1.2. A **domino tiling** of a polyomino P is a tiling of P with 2×1 and 1×2 rectangles (each face of the polyomino belongs to exactly one domino). We say that a polyomino is **tileable** if there exist a domino tiling of the polyomino.

For each finite polyomino P, there is a dual graph P^* : vertices of P^* are the unit squares (faces) of P and two vertices in P^* are connected by an edge if and only if the two corresponding unit squares in P share a common edge. When we say domino tiling of P^*, we refer to the corresponding domino tiling of P.

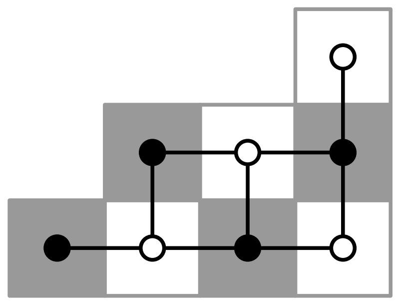

Figure 1.2: Example of polyomino (in grey) and its dual graph (in black) which black and white vertices define black and white faces of the polyomino

1.1.2 Tileability of polyominos by dominos

Not all polyominos are tileable. For example, a polyomino where the number of black faces differ from the number of white faces is clearly not tileable as each domino occupies a black and a white square. On the other hand, a polyomino with a number of white faces equal to the number of black faces is not always tileable. A tileable polyomino may have one or more possible tiling. In the example below (figure 1.3), the two polyomino have the same number of faces but the one on the left has a unique possible tiling while the one on the right has five. This illustrates the fact that the number of possible tilings of a polyomino depends on the shape of its boundary. In this report, we are interested in polyominos which are tileable and which have numerous possible tilings. In section 2.2, we will see that certain boundary conditions on the finite polyominos guarantee these properties. In this report, all polyominos are assumed to have an equal number of black and white faces as it is a necessary condition for tileability.

1.1.3 Height function

For a tileable polyomino, we can use a tiling to define a height function on the vertices of the polyomino. Furthermore, when the polyomino has multiple

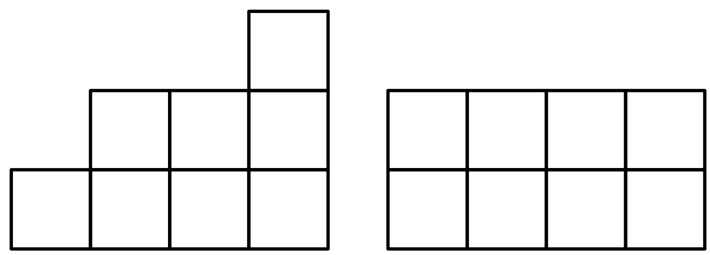

Figure 1.3: Two polyominos with different number of possible tilings by dominos (one on the left, five on the right)

possible tilings and if we assume that all tilings have uniform probability, this gives us a way to construct a discrete random surface.

Given a tiling of a simply connected finite polyomino, we define the height function h on the vertices to $\mathbb{Z}$ as following. We first choose a reference vertex V_0 to which we assign the value c. The height of a vertex V is obtained by choosing a path on the edges of the tiling (i.e. on the edges of the polyomino with no crossing of dominos) starting from the reference vertex and ending on V and computing the height along the path according to the following rule : let's $V_0, V_1, ..., V_n = V$ be the sequence of vertices encountered along the path, for $i \in 1, ..., n$, $h(V_i) = h(V_{i-1}) + 1$ if the edge going from V_{i-1} to V_i has a white face on the left and $h(V_i) = h(V_{i-1}) - 1$ if the edge going from V_{i-1} to V_i has a black face on the left. The height of V does not depend on the path chosen because the polyomino is simply connected. This can easily be seen using the following fact : given two vertices A and B on a domino (a domino has six vertices), the two paths going from A to B on the edges of the domino give the same height difference between A and B. Thus the height function is well defined.

Remarks:

1. At every boundary vertex, the value of h does not depend on the tiling because for any two vertices on the boundary, the height difference between them can be computed using the path following the boundary which does not cross any domino.
2. At every vertex, the value $\mod 4$ is fixed and does not depend on the tiling. Moreover, when going along an edge V_1V_2, the difference $h(V_2) - h(V_1)$ is $1 \mod 4$ if a white face is on the left and $-1 \mod 4$

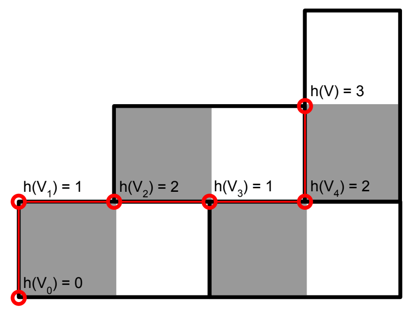

Figure 1.4: Height function : computation of the height in V with an example path

if a black face is on the left.

3. Assuming h defined on the vertices of a polyomino Ω with correct values $h \mod 4$ and such that at every pair of neighboring vertices A and B with a white face to the left of AB, either $h(b) = h(a) + 1$ or $h(b) = h(a) - 3$, there is a tiling which corresponds to h. This can easily be seen by noticing that with these conditions, each face belongs to exactly one domino. Thus we have a bijection between such function h and the set of tilings.
4. If the polyomino is not simply connected, the height function is not necessarily defined (see fig 1.5).

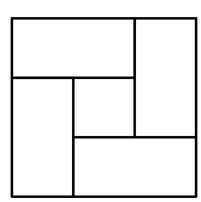

Figure 1.5: Example of a tiling of a not simply connected polyomino where the height function is not defined

If a polyomino is tileable and has multiple possible tilings, we have seen that each tiling define a height function. If we assign a probability to each tiling, we have constructed a discrete random surface.

On a simply connected domain polyomino, once we know the value of h

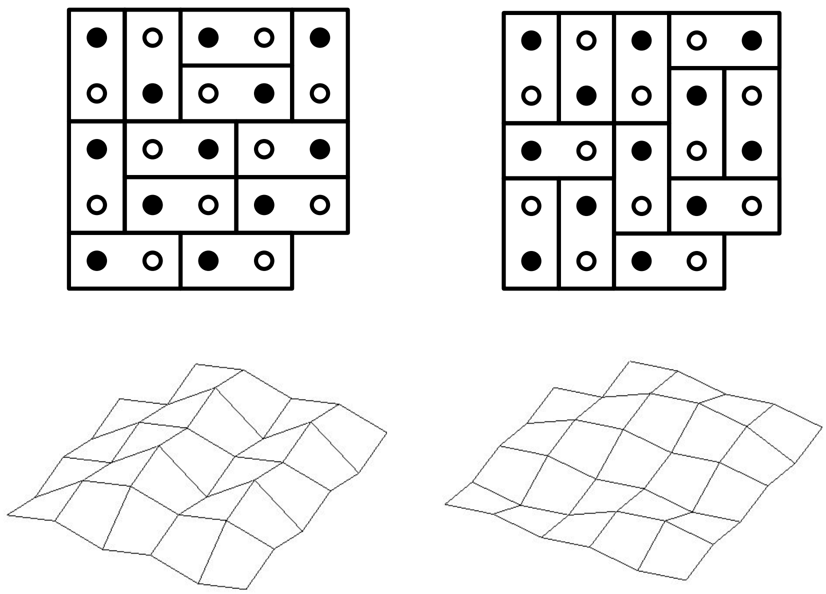

Figure 1.6: Two tilings of the same polyomino defining two height function.

at one vertex, we know the value of h on all vertices. Thus h is determined by its values on the boundary. We give a criteria for the tileability with respect to the boundary values here:

Theorem 1.1.3. *Let the domain Ω be a simply connected finite graph in $\mathbb{Z}^2$ and h be a function on the boundary vertices to $\mathbb{Z}$ such that for a checkerboard coloring of Ω, the number of black squares is equal to the number of white squares. Then, the following statement are equivalent:*

1. *There exists a domino tiling of Ω,*
2. *For all boundary vertices u and v, $h(v) \leq h(u)$+length of the shortest path from u to v inside Ω such that it has black squares to the right.*

Proof. If there exists a tiling and if h is its height function, for all boundary vertices u and v, $h(v) = h(u)$+length of the shortest path from u to v inside the tiling such that it has black squares to the right.

Reciprocally, if we assume (*2.*), we can construct a well-defined height function based on the boundary values of h.

Given values of h on the boundary of Ω, for an inner vertex v, let $H(x) := \min_{a\in\partial\Omega}\{h(a)+$length of the shortest path from u to v inside Ω such that it has black squares to the right$\}$. We just need to verify Remark (c). We define H by breadth-first-search so that it satisfies the condition. Here is how we proceed:

First, we mark the vertices on the boundary and define $H(x) := h(x)$ for all vertices x on the boundary. Then each time, we choose a vertex u with the smallest H from the marked vertices. Then we extend the path we have found from u by one such that the extended path has a black face to the right. If the path reaches a vertex that has not been marked, we can define H to be $H(u)+1$ at this vertex and mark it. If the path reaches a vertex that has been marked, from the way we define H and (*2.*), this value is either $H(u)+1$ or $H(u)-3$. Then H modulo 4 is defined correctly. This completes the proof. $\square$

1.2 Gaussian free field

In this section, we will present a concise construction of the Gaussian free field given by Scott Sheffield in [6].

Let v be a measurable random variable in $\mathbb{R}^d$. v is called the standard Gaussian variable in $\mathbb{R}^d$ if the distribution of v is the same with $\sum_i^d \alpha_j v_j$, where $v_1, \cdots, v_d$ is an orthonormal basis in $\mathbb{R}^d$, $\alpha_j, j = 1, \cdots, d$ are independent identical Gaussian variables with mean 0 and variance 1.

Consider $H_c(U)$ formed by the smooth functions with compact support in a domain $U \subset \mathbb{R}^2$ (so that their fist-order derivatives are in $L^2(U)$), which has the L^2 norm. Denote Hilbert space $H(U)$ the completion of $H_c(U)$. This is a Sobelev space (real vector space). In particular, the function in $H(U)$ can be a Schwartz distribution, which is a continuous linear functional on $C^\infty(U)$. To be more precise, the Gaussian free field is the Gaussian variables on $H(U)$, whose dimension is infinite. To define the Gaussian variable on such a space, we need to express it on an orthonormal basis.

If U is a Jordan region with smooth boundary, the eigenvectors of the Laplacian on U with Dirichlet boundary conditions ($f_i(x) \equiv 0,\ x \in \partial U$) form an orthonormal basis for $H(U)$. Let λ_i be the eigenvalue of f_i, then the Gaussian free field F has the form:

$$F = \sum_{i \geq 1} \frac{\alpha_i f_i}{(-\lambda_i)^{1/2}},$$

where α_j are i.i.d. Gaussian random variables of mean 0 and variance 1.

However, this expression is meaningless, because it diverges almost everywhere. In fact, In [2], R. Kenyon has proved that for x in P_ϵ, the height $h(x)$ converges to a Gaussian random variable with variance $\frac{c}{\log(\frac{1}{\epsilon})}$, where c is a constant. So here, this expression should be interpreted as a Schwartz distribution, which is a generalized function satisfying, for any function $\varphi \in C_c^1(U)$,

$$\int_U \varphi F = \sum_{i \geq 1} \frac{\alpha_i}{(\lambda_i)^{1/2}} \int_U \varphi f_i$$

which converges almost surely. Recall that by Weyl's formula, $i^{2/d}/(-\lambda_i)$ converges to a constant as i tends to infinity.

Chapter 2

Constructing the coupling function

2.1 Kasteleyn theorem

For a given finite polyomino P, Kastelelyn's theorem gives the number of possible tilings by dominos. Let μ be the uniform probability measure on domino tilings of P, Kasteleyn's theorem allows as to compute the probability of a particular tiling. In addition, from a corollary of the theorem, the μ-measures of cylinder sets are determined by the inverse of the Kasteleyn matrix of the polyomino.

Definition 2.1.1. A **Kasteleyn weighting** of P^* (dual graph of P) is a function α which assigns to each edge in P^* a value in $\{\pm 1, \pm i\}$ such that for every cycle in P^*, $b_1 w_1, w_1 b_2, \cdots, b_k w_k, w_k b_1$ (b_i, $\forall i \in 1, \cdots, k$ are black vertices and w_i, $\forall i \in 1, \cdots, k$ are white vertices), we have :

$$\frac{\alpha(b_1, w_1) \cdots \alpha(b_k, w_k)}{\alpha(b_2, w_1) \cdots \alpha(b_1, w_k)} = (-1)^{k-1} \tag{2.1}$$

The existence of a Kasteleyn weighting can be established using spanning trees: once allocating required values of α for a square, we can inductively allocate values of other three edges sharing the same square such that the condition is satisfied.

Definition 2.1.2. A **Kasteleyn matrix** K of a domino tiling on a bipartite planar graph G^* is a Kasteleyn weighted adjacency matrix. Let K' be defined as following : each row b representing a black vertex and each column w representing a white vertex, $K'(b, w) = 0$ if b is not adjacent to w and

$K'(b,w) = \alpha(b,w)$ if b is adjacent to w, α being the Kasteleyn weighting of the edges.

$$K = \begin{pmatrix} 0 & K'^T \\ K' & 0 \end{pmatrix} \tag{2.2}$$

Theorem 2.1.3 (Kasteleyn)**.** *Let G^* be a bipartite planar graph with equal number of black and white vertices and K its associated Kasteleyn matrix. Then,*

$$\textit{number of domino tilings of } G^* = \sqrt{|det(K)|} \tag{2.3}$$

Proof. Let K' be defined as in 2.1.2.

$$\begin{aligned} \det K' &= \sum_{\sigma:\text{permutations}} \operatorname{sgn}(\sigma) \prod_j K(b_j, \sigma(w_j)) \\ &= \sum_{\sigma:\text{domino tilings s}} \operatorname{sgn}(\sigma) \prod_j \alpha(b_j, \sigma(w_j)) \end{aligned}$$

The second equality is obtained by noticing that only permutations corresponding to a domino tiling give non-zero products.
To complete the proof, we have to show that the terms in the sum are all equal and of module 1. Given two domino tilings σ_1 andσ_2, we can draw them simultaneously on G^*: we obtain a set of cycles. On a cycle of $2k$ edges, the difference between σ_1 and σ_2 is $k-1$ transpositions ; on the other hand, (2.1) tell us that the ratio of the Kasteleyn weights on this cycle is $(-1)^{(k-1)}$. Multiplying all the cycles together, we get :

$$\frac{\operatorname{sgn}(\sigma_1) \prod_j \alpha(b_j, \sigma_1(b_j))}{\operatorname{sgn}(\sigma_2) \prod_j \alpha(b_j, \sigma_2(b_j))} = 1 \tag{2.4}$$

□

If P is a tileable polyomino and K its associated Kasteleyn matrix, let us assign a uniform probability measure μ on all possible tilings. Each tiling thus has probability $1/\sqrt{|\det K|}$. Furthermore, we have the following corollary :

Corollary 2.1.4. *For a uniformly chosen domino tiling of G and a collection of edges $w_1 b_1, \cdots, w_k b_k$, the probability of $w_1 b_1, \cdots, w_k b_k$ belonging to a domino tiling is*

$$|\det K^{-1}(w_i, b_j)_{1 \le i,j \le k}|$$

Proof. Let K' be defined as in 2.1.2.

$$\Pr(\text{all } w_1b_1, \cdots, w_kb_k \text{ belong to a domino tiling}) \tag{2.5}$$

$$= \frac{\#\{\text{domino tilings with } w_1b_1, \cdots, w_kb_k\}}{\#\{\text{domino tilings}\}} \tag{2.6}$$

$$= \frac{|\det K'(w_i, b_j)_{i,j>k}|}{|\det K'|} \tag{2.7}$$

$$= |\det K'^{-1}(w_i, b_j)_{1\leq i,j\leq k}| \tag{2.8}$$

$$= |\det K^{-1}(w_i, b_j)_{1\leq i,j\leq k}| \tag{2.9}$$

The last equality is given by Jacobi's equality. □

2.2 Temperley domains

Given a simply connected polyomino P, let us consider its dual graph P^* which can be seen as a subgraph of $\mathbb{Z}^2$. we are going to refine the black and white coloring of P^* with the following rule :

- if the coordinate of a vertex is (0, 0) mod 2, it has color W_0
- if the coordinate of a vertex is (1, 1) mod 2, it has color W_1
- if the coordinate of a vertex is (1, 0) mod 2, it has color B_0
- if the coordinate of a vertex is (0, 1) mod 2, it has color B_1

For clarity, we will use the following notation. For any subset D of $\mathbb{Z}^2$, $\mathbf{B_0}(D)$, $\mathbf{B_1}(D)$, $\mathbf{W_0}(D)$, $\mathbf{W_1}(D)$ are the $\mathbf{B_0}$, $\mathbf{B_1}$, $\mathbf{W_0}$ and $\mathbf{W_1}$ vertices in D respectively. $\mathbf{B}(D) = \mathbf{B_0}(D) \cup \mathbf{B_1}(D)$ and $\mathbf{W}(D) = \mathbf{W_0}(D) \cup \mathbf{W_1}(D)$.

Definition 2.2.1. A **Temperley domain** is a polyomino with specific boundary combinatorics. It is a polyomino where all convex corners are around a B_1 face and all concave corners are opposed to a B_1 face. One B_1 face at a corner is removed so that the number of white (W_0 and W_1) and blacked faces (B_0 and B_1) is equal (the bottom right face in the example). This removed corner is called the root.

Proposition 2.2.2 (Temperley bijection)**.** *Let polyomino P be a Temperley domain. A spanning tree on the black squares is a tree whose vertices are all the black squares and the edges connect black squares are separated horizontally or vertically by a single white square. There is a bijection between the spanning trees on black squares and the domino tilings of P.*

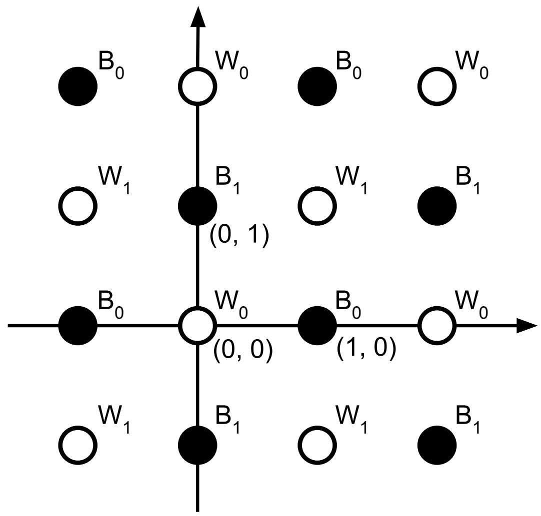

Figure 2.1: Coloring of $\mathbb{Z}^2$

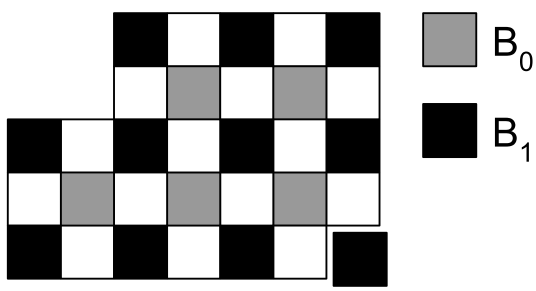

Figure 2.2: An example of Temperley domain. B_0 faces are grey, B_1 faces are black, W_0 and W_1 faces are white. The B_1 face on the bottom right is the root.

Proof. Consider the following mapping from the domino tilings of P to the spanning trees on black squares of P: two black squares separated horizon-

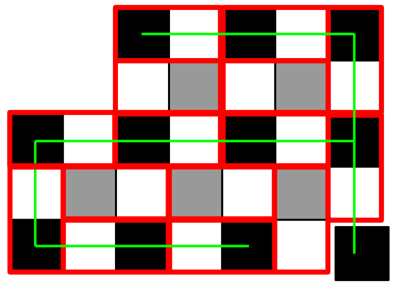

Figure 2.3: An example of spanning tree on black squares in green and the corresponding domino tiling in red

tally or vertically by a single white square are connected by an edge if and only if the white square belongs to a domino which direction is aligned with the edge.

Let us show that this mapping is injective and surjective. In other words, let us show that for each spanning tree, there is a unique corresponding domino tiling. Let us partition P in two parts : the first part is made of all the squares through which the spanning tree goes; the second part is made of the remaining squares.

We show that the first part has a unique tiling. Consider a terminal chain of a spanning tree, i.e. the chain starting at an extreme black square of the tree and ending at the first branching encountered (the branching square not included). This is a chain of squares with an equal number of white and colored squares and has thus a unique domino tiling. We can thus tile all the terminal chains of the tree. Then, removing the already tiled extreme chains, we get a smaller tree and we repeat the procedure. This gives us a unique domino tiling of the first part.

The second part consists of disconnected chains, each having an equal number of white and colored squares. These chains have thus a unique tiling. □

Corollary 2.2.3. *A Temperley domain is tileable.*

Proof. It suffices to show that for all Temperley domain, there is a spanning tree on the black squares. Let P be a Temperley domain. Let us define the graph G as follows : the vertices are the black squares and two black squares are connected by en edge if and only if they are separated horizontally or vertically by a single white square. G is a connected graph. If G contains a cycle, we remove one edge of the cycle. If the remaining graph contains a cycle, we repeat the procedure until the graph contains no cycle. The graph we get is a spanning tree on the black squares. □

Remarks:

1. The definition of Temperley domains implies that the boundary has no "staircase" or "zigzag". These motifs tend to impose a unique tiling near them which we would like to avoid as we are interested in constructing a random object.

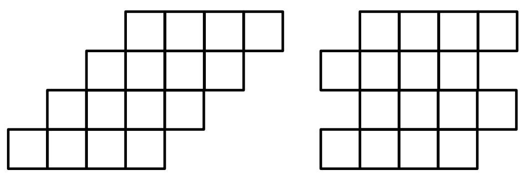

Figure 2.4: Examples of non Temperley domain with "staircase" and "zigzag" motifs on the boundary. Both have a unique tiling.

2. Temperley domains guarantee certain boundary combinatorics which are going to be used in the next section.

2.3 Discrete analytic functions

Before defining discrete analytic functions, let us first define some operators $\mathbb{C}^{\mathbb{Z}^2} \to \mathbb{C}^{\mathbb{Z}^2}$:

- $\partial_x f(v) = f(v+1) - f(v-1)$
- $\partial_y f(v) = f(v+i) - f(v-i)$
- $\partial_z = \partial_x - i\partial_y$
- $\partial_{\bar{z}} = \partial_x + i\partial_y$

Definition 2.3.1. Using the same coloring as for Temperley domains, a function F is **discrete analytic** on $\mathbb{Z}^2$ if it is real on $\mathbf{B_0}(\mathbb{Z}^2)$ vertices, purely imaginary on $\mathbf{B_1}(\mathbb{Z}^2)$ vertices, null on $\mathbf{W}(\mathbb{Z}^2)$, and if $\partial_{\bar{z}} F = 0$.

Remarks:

1. Let F be a discrete analytic function. $F = f + ig$, f and g being the real and purely imaginary part of F respectively. By definition, f is real on $\mathbf{B_0}(\mathbb{Z}^2)$ and null everywhere else whereas g is real $\mathbf{B_1}(\mathbb{Z}^2)$ and null everywhere else.
2. An equivalent definition for discrete analytic functions is obtained by replacing the condition $\partial_{\bar{z}} F = 0$ by the discrete Cauchy-Riemann equations :
$$\partial_x f(v) = \partial_y g(v) \quad \text{for} \quad v \in W_0 \tag{2.10}$$
$$\partial_y f(v) = -\partial_x g(v) \quad \text{for} \quad v \in W_1 \tag{2.11}$$
3. Let D be a subset of $\mathbb{Z}^2$. The operators ∂_x, ∂_y, ∂_z, $\partial_{\bar{z}}$ we defined on $\mathbb{C}^{\mathbb{Z}^2}$ can be defined on $\mathbb{C}^D$ using the convention that all elements outside of D have value 0 : we compute the value of the operator on $\mathbb{C}^{\mathbb{Z}^2}$ and then take the restriction on $\mathbb{C}^D$.
4. If P is a finite subset of D such that F is discrete harmonic on $D \setminus P$, we say that F is discrete harmonic with poles in P.

Let A be a function $\mathbb{Z}^2 \to \mathbb{C}$, we define the discrete Laplacian operator :

$$\Delta A = -\partial_z \partial_{\bar{z}} A = 4A(v) - A(v+2) - A(v+2i) - A(v-2) - A(v-2i)$$

Proposition 2.3.2. *Let F be a discrete analytic function. $F = f + ig$, f and g being the real and purely imaginary part of F respectively. Then,*

$$\forall v \in \mathbf{B_0}(\mathbb{Z}^2), \quad \Delta f(v) = 0$$

$$\forall v \in \mathbf{B_1}(\mathbb{Z}^2), \quad \Delta g(v) = 0$$

Consequently,

$$\forall v \in \mathbb{Z}^2, \quad \Delta F(v) = 0$$

Proof. This results from simple computations. □

Remarks:

1. Let F be a discrete analytic function on a bounded subset D of $\mathbb{Z}^2$ and f and g its real and purely imaginary parts respectively. As a direct consequence of the proposition, we have $\{w \in \mathbf{B_0}(D) \mid w = \max_{v \in \mathbf{B_0}(D)} f(v)\} \subset \partial \mathbf{B_0}(D)$ and $\{w \in \mathbf{B_1}(D) \mid w = \max_{v \in \mathbf{B_1}(D)} g(v)\} \subset \partial \mathbf{B_1}(D)$. A similar result holds for the minimal value.

2. Let D be a bounded subset of $\mathbb{Z}^2$. Let a and b be two functions $D \to \mathbb{R}$ such that $\forall v \in D$, $\Delta a(v) = 0$ and $\Delta b(v) = 0$ and $\forall v \in \partial D$, $a(v) = b(v)$. Then $a = b$. This results from the previous remark considering the function $a - b$ which is null on ∂D and which has Laplacian zero on D.

2.4 Properties of the coupling function

Let D be a dual graph of a tileable subset of $\mathbb{Z}^2$. The coupling function C is a complex-valued function on $D \times D$ which allows to compute the probability of a finite set of dominos being present in a tiling if the tiling is uniformly chosen among all possible tilings, for any set of dominos in D. We have seen in section 2 that if D is bounded, the coupling function is related to the inverse Kasteleyn matrix. Precisely, if we order the vertices of D placing all the white vertices before the black vertices and if K is the Kasteleyn matrix associated with D, Then $C = K^{-1}$.

We are going to show that on the dual graph of Temperley domains, the coupling function has certain properties which allows it to be seen as a boundary value problem.

First, we are going to choose Kasteleyn weights as in figure 2.5. Let D

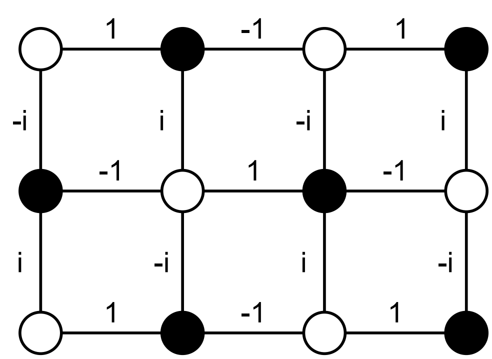

Figure 2.5: Kasteleyn weights

be the dual graph of a Temperley domain and K its associated Kasteleyn

matrix. K can be seen as on operator on $\mathbb{C}^D$ which is equivalent to $-\partial_{\bar{z}}$ on $\mathbf{B}(D)$ and equivalent to $\partial_{\bar{z}}$ on $\mathbf{W}(D)$. In addition, on D, K^*K is equivalent to the Laplacian (K^* being the Hermitian conjugate of K).
For a continuous analytic functions, we know that the imaginary part of the function is determined, up to a constant, by its real part. We have a similar result for discrete analytic function on simply connected Temperley domains :

Lemma 2.4.1. *Let D be the dual graph of a simply connected Temperley domain and F a discrete analytic function on D. The imaginary part of F is determined, up to an imaginary constant, by the real part of F.*

Proof. Let f be the real part of a discrete analytic function F on D. We know that f is real-valued on $\mathbf{B_0}(D)$, null everywhere else and $\forall v \in \mathbf{B_0}(D)$, $\Delta f(v) = 0$. We are going to compute g, the purely imaginary part of F. Let us take a reference vertex $v_0 \in \mathbf{B_1}(D)$ and assign a real value c to it. $\forall v \in \mathbf{B_1}(D)$, we choose a path in $\mathbf{B_1}(D)$ going from v_0 to v. Along the path, at each crossing of an edge of $\mathbf{B_0}(D)$, we add what is given by the discrete Cauchy-Riemann equations (2.10) and (2.11). This gives us the value of g at v as this process does not depend on the path chosen because $\Delta f = 0$ on $\mathbf{B_0}(D)$. □

As mentioned earlier, on tileable polyomino, the coupling function C is equal to the inverse Kasteleyn matrix K^{-1}. This gives us immediately a crucial property :

Lemma 2.4.2. *Let D be the dual graph of a finite tileable polyomino and K its associated Kasteleyn matrix. Let us define the delta function :*

$$\forall (v_1, v_2) \in D^2, \quad \delta_{v_1}(v_2) = \begin{cases} 1 & \text{if } v_1 = v_2 \\ 0 & \text{otherwise} \end{cases}$$

Then,

$$\forall (v_1, v_2) \in D^2, KC(v_1, v_2) = \delta_{v_1}(v_2)$$

Proof. $KC = I$ □

Lemma 2.4.3. *Let D be the dual graph of a finite tileable polyomino and C its associated coupling function. We have the following properties :*

1. *$\forall (v_1, v_2) \in D^2$, $C(v_1, v_2) = C(v_2, v_1)$*
2. *If $(v_1, v_2) \in \mathbf{B}(D)$ or $(v_1, v_2) \in \mathbf{W}(D)$, $C(v_1, v_2) = 0$.*
3. *If $v_1 \in \mathbf{W_0}(D)$, $C(v_1, v_2)$ is a discrete analytic function of v_2 with a pole at v_1.*

Proof. Let K be the Kasteleyn matrix associated with D. By definition, K can be seen as a symmetrical matrix of size $|D|$. If the indices of the matrix are ordered such that the $\mathbf{W_0}(D)$ are first, followed by the $\mathbf{W_1}(D)$, followed by the $\mathbf{B_0}(D)$ and finally the $\mathbf{B_1}(D)$, then K can be written as :

$$K = \begin{pmatrix} 0 & 0 & K_1 & iK_2 \\ 0 & 0 & iK_3 & K_4 \\ K_1^T & iK_3^T & 0 & 0 \\ iK_2^T & K_4^T & 0 & 0 \end{pmatrix}$$

Where K_1, K_2, K_3 and K_4 are real-valued square matrices. This follows directly from our choice of Kasteleyn weights (figure 2.5). From block matrix multiplication, we deduce that

$$C = K^{-1} = \begin{pmatrix} 0 & \begin{pmatrix} K_1^T & iK_3^T \\ iK_2^T & K_4^T \end{pmatrix}^{-1} \\ \begin{pmatrix} K_1 & iK_2 \\ iK_3 & K_4 \end{pmatrix}^{-1} & 0 \end{pmatrix}$$

The formula for block matrix inversion gives us :

$$\begin{pmatrix} K_1 & iK_2 \\ iK_3 & K_4 \end{pmatrix}^{-1} = \begin{pmatrix} (K_1 + K_2K_4^{-1}K_3)^{-1} & -i(K_1 + K_2K_4^{-1}K_3)^{-1}K_2K_4^{-1} \\ -iK_4^{-1}K_3(K_1 + K_2K_4^{-1}K_3)^{-1} & K_4^{-1} - K_4^{-1}K_3(K_1 + K_2K_4^{-1}K_3)^{-1}K_2K_4^{-1} \end{pmatrix}$$

(1.) and (2.) are thus proven.

For (3.), we first observe that for $v_1 \in \mathbf{W_0}(D)$ fixed, $C(v_1, v_2)$ is real when $v_2 \in \mathbf{B_0}(D)$ and purely imaginary when $v_2 \in \mathbf{B_1}(D)$. In addition, $KC(v_1, v_2) = \delta_{v_1}(v_2)$. This completes the proof. □

Let P be the dual graph of a Temperley domain and C its associated coupling function. Let us call Y the B_0 vertices outside of P which are at a distance of 2 of $B_0(P)$ (distance on a graph). We can thus extend $B_0(P)$ to $B_0'(P) = B_0(P) \cup Y$. The boundary vertices of $B_0'(P)$, noted $\partial B_0'(P)$, are the vertices Y. As mentioned earlier, the operator $\partial_{\hat{z}}$ is well defined on P by computing $\partial_{\hat{z}}$ in $\mathbb{Z}^2$ setting all the values in $\mathbb{Z}^2 \setminus P$ to zero and then restricting the operator to P. Lemma 2.4.3 tells us that if $v_1 \in W_0(P)$, $C(v_1, v_2)$ is discrete analytic in v_2 and thus, $Re(C(v_1, .))$ seen as a real valued function on $B_0(P)$ can be extended to $B_0'(P)$ and $\forall v_2 \in \partial B_0'(P)$, $ReC(v_1, v_2) = C(v_1, v_2) = 0$. With some simple calculations we get the following lemma :

Lemma 2.4.4. *For $v_1 \in W_0(P)$ fixed. Consider $C(v_1, v_2)$ as a function of v_2 on $B_0(P)$, extended to be zero on Y.*

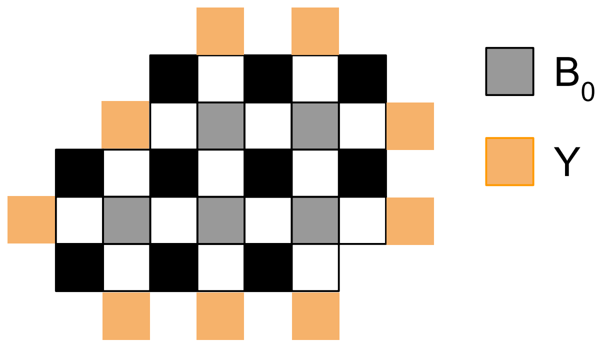

Figure 2.6: Temperley domain extension on the neighboring B_0 vertices Y

1. $C(v_1,.)$ *is discrete harmonic on* $B_0(P) \setminus \{v_1+1, v_1-1\}$

2. $\Delta ReC(v_1, v_1 \pm 1) = \pm 1$

For $v_1 \in W_0(P)$ *fixed. Consider* $C(v_1, v_2)$ *as a function of* v_2 *on* $B_0(P)$*, extended to be zero on* Y*.*

a. $C(v_1,.)$ *is discrete harmonic on* $B_0(T) \setminus \{v_1+i, v_1-i\}$

b. $\Delta ImC(v_1, v_1 \pm i) = \mp 1$

From Lemma 2.4.4, we deduce easily that the coupling function is the unique solution to the following boundary value problem : f is a solution if it satisfies (*1.*) and (*2.*) of the lemma and Ref is null on $\partial B_0'(P)$. To show that the solution is unique, Let F be a solution to the b.v.p. If $G = F - ReC(v_1,.)$, G is harmonic on $B_0(P)$ and ReG is null on $\partial B_0'(P)$. According to the remarks on page 16, $F = ReC$.

2.5 Approximating bounded simply connected domains with Temperley domains

Let U be a bounded simply connected domain in $\mathbb{R}^2$ with smooth boundary. In order to consider the scaling limit of the height function defined on polyominos approximating U as the lattice size tends to 0, we first need to define what is an appropriate approximation of U. We are going to use Temperley domains to approximate U because Temperley domains are always tileable (Corollary 2.2.3) and because, on Temperley domains, the coupling function can be seen as the unique solution to a b.v.p. More precisely, let d be a point on ∂U, called the marked point, an appropriate approximation P_ϵ (polyomino of lattice size ϵ) of U has the satisfy the following conditions :

- P_ϵ is a Temperley domain and its root b_ϵ (see Definition 2.2.1) is at a distance $O(\epsilon)$ of d.
- The boundary of P_ϵ is within $O(\epsilon)$ of ∂U and, locally, the counterclockwise path on the boundary of P_ϵ points in the same half-plane as the counterclockwise tangent of ∂U.
- The boundary of P_ϵ contains a straight segment, either vertical or horizontal, of length δ such that, when $\epsilon \to 0$, $\delta/\epsilon \to \infty$.

Chapter 3

Scaling limit of the coupling function

In this section we will show that, as ϵ tends to zero, $\frac{1}{\epsilon}C(v_1, \cdot)$ converges to a couple of complex analytic functions, F_0 and F_1 (when $v_1 \in W_0$, it converges to F_0; when $v_1 \in W_1$, it converges to F_1), which transform analytically under conformal mappings of the domain U. In the meantime, we will show that these functions are solutions to a Dirichlet boundary value problem.

First, we will introduce a few properties of the discrete Green's function on the whole plane to show that the asymptotic values of the function $C_0(v_1, .)$ on the whole plane have the same properties as in Lemma 2.4.4 (notice that we haven't defined the coupling function on an unbounded domain). We will see in the following part that for any coupling function $C(v_1, .)$ on a region U, the singular part is $C_0(v_1, .)$.

3.1 On the whole plane

Consider the Fourier transform of the lattice function $H(n, m) := C_0(0, n-mi)$ in $\mathbb{Z}^2$: $\hat{H}(x, y) = \sum_{n,m\in\mathbb{Z}} H(n, m)e^{-inx-imy}$. It has the following property:

$$
\begin{aligned}
&\hat{H}(x,y)(e^{ix}-e^{-ix}+\frac{1}{i}(e^{iy}-e^{-iy}))\\
=&\sum_{n,m\in\mathbb{Z}}(H(n+1,m)-H(n-1,m)+\frac{1}{i}(H(n,m+i)-H(n,m-i)))e^{-inx-imy}\\
=&\,1
\end{aligned}
\tag{3.1}
$$

Where the first equality is obtained by the expression of $\hat{H}$, and the second equality follows from the fact that H is analytic with a pole at the origin. By the reciprocal Fourier transform,

$$H(m,n)=\frac{1}{4\pi^2}\int_0^{2\pi}\int_0^{2\pi}\frac{e^{i(n\theta+m\phi)}}{2i\sin(\theta)+2\sin(\phi)}d\theta d\phi.$$

Nevertheless, this integral diverges, but if we substract 1 in the numerator of the integrand, the integral $H(m,n)=\frac{1}{4\pi^2}\int_0^{2\pi}\int_0^{2\pi}\frac{e^{i(n\theta+m\phi)}-1}{2i\sin(\theta)+2\sin(\phi)}d\theta d\phi$ converges and still satisfies the property (3.1). From the uniqueness of the Green's function given boundary values we have

$$C_0(0,n+im)=\frac{1}{4\pi^2}\int_0^{2\pi}\int_0^{2\pi}\frac{e^{i(n\theta-m\phi)}-1}{2i\sin(\theta)+2\sin(\phi)}d\theta d\phi$$

.

In [2], there is an alternative method to define the coupling function on the whole plane: it is the limit of the coupling function on $2n\times 2n$ squares as n tends to infinity. Both definitions give the same expression.

Theorem 3.1.1. *As $|z|\to\infty$, C_0 tends to $\frac{1}{\pi z}$ asymptotically, that is to say :*

$$C_0(0,z)=\begin{cases}Re\frac{1}{\pi z}+O(\frac{1}{|z|^2}) & z\in B_0,\\ iIm\frac{1}{\pi z}+O(\frac{1}{|z|^2}) & z\in B_1\end{cases}$$

Proof. Lemma 2.4.4 gives us the relationship between the the real part and the purely imaginary part of C_0 (real on B_0 vertices and purely imaginary on B_1 vertices) and Green's function on the whole plane: the real part of C_0 is the unique function in $B_0(\mathbb{Z}^2)$ such that $\Delta ReC_0=\delta_1-\delta_{-1}$ and tends to zero as $|z|\to\infty$; the imaginary part of C_0 is the unique function in $B_1(\mathbb{Z}^2)$ such that $\Delta ImC_0=\delta_i-\delta_{-i}$ and tends to zero as $|z|\to\infty$.

Green's function in $\mathbb{Z}^2$, $G_0(v,w)$, is a symmetric function having the properties that $\Delta G_0(0,w) = \delta_0(w)$ and for a fixed v, $G_0(0,w) - G_0(v,w) \to 0$ as $w \to \infty$. Therefore we have

$$ReC_0(0,w) = G_0(0, \frac{w-1}{2}) - G_0(0, \frac{w+1}{2})$$

In order to obtain the asymptotic values of C_0, we need a classical result about G_0:

Lemma 3.1.2. *[9] There exists a constant c_0 such that*

$$G_0(0,v) = -\frac{1}{2\pi}\log|v| + c_0 + O(\frac{1}{|v|^2}).$$

Using Lemma 3.1.2 we have

$$\begin{aligned}
ReC_0(0,w) &= G_0(0, \frac{w-1}{2}) - G_0(0, \frac{w+1}{2}) \\
&= \frac{1}{2\pi}\log|\frac{w+1}{2}| - \frac{1}{2\pi}\log|\frac{w-1}{2}| + O(\frac{1}{|v|^2}) \\
&= \frac{1}{2\pi} Re \log(\frac{w+1}{w-1}) + O(\frac{1}{|v|^2}) \\
&= \frac{1}{2\pi} Re \frac{2}{w-1} + O(\frac{1}{|v|^2}) \\
&= Re \frac{1}{\pi w} + O(\frac{1}{|v|^2}),
\end{aligned}$$

where we used $\log(1+w) = z + O(|z|^2)$. Similarly, when $w \in B_1(\mathbb{Z}^2)$,

$$\begin{aligned}
ImC_0(0,w) &= G_0(0, \frac{w-i}{2}) - G_0(0, \frac{w+i}{2}) \\
&= \frac{1}{2\pi}\log|\frac{w+i}{2}| - \frac{1}{2\pi}\log|\frac{w-i}{2}| + O(\frac{1}{|v|^2}) \\
&= \frac{1}{2\pi} Re \log(\frac{w+i}{w-i}) O(\frac{1}{|v|^2}) \\
&= \frac{1}{2\pi} Re \frac{2i}{w-1} + O(\frac{1}{|v|^2}) \\
&= Im \frac{1}{\pi w} + O(\frac{1}{|v|^2})
\end{aligned}$$

□

Corollary 3.1.3. *As $\epsilon \to 0$, the scaled function $\frac{1}{\epsilon}C_0^\epsilon(0,z)$ on $\epsilon\mathbb{Z}^2$ tends to $\frac{1}{\pi z}$, that is to say :*

$$C_0(0,z) = \begin{cases} Re\frac{1}{\pi z} + O(\epsilon) & z \in B_0(\epsilon\mathbb{Z}^2), \\ iIm\frac{1}{\pi z} + O(\epsilon) & z \in B_1(\epsilon\mathbb{Z}^2) \end{cases}$$

Proof. On the scaled lattice $\epsilon\mathbb{Z}^2$, let us write $C_0^\epsilon(0,z) = C_0(0,\frac{z}{\epsilon})$. Using Theorem 3.1.1, when $z \in B_0(\epsilon\mathbb{Z}^2)$,

$$\begin{aligned} \frac{1}{\epsilon}C_0^\epsilon(0,z) &= \frac{1}{\epsilon}Re(\frac{\epsilon}{\pi z}) + \frac{1}{\epsilon}O(\frac{\epsilon^2}{|z^2|}) \\ &= Re\frac{1}{\pi z} + O(\epsilon) \end{aligned}$$

When $z \in B_1(\epsilon\mathbb{Z}^2)$, similarly, we have:

$$\begin{aligned} \frac{1}{\epsilon}C_0^\epsilon(0,z) &= \frac{1}{\epsilon}Im(\frac{\epsilon}{\pi z}) + \frac{1}{\epsilon}O(\frac{\epsilon^2}{|z^2|}) \\ &= Re\frac{1}{\pi z} + O(\epsilon) \end{aligned}$$

□

3.2 On bounded regions

One of the main results in this part is to show that the coupling function on a finite region converges, as the lattice size ϵ tends to zero, to a pair of analytic functions which transform analytically under conformal maps of the region. For a fixed bounded region U, we are going to prove this convergence when U is approximated by the Temperley domains P_ϵ as described in 2.5. At the moment, we are not able to prove this convergence when U is approximated by general polyominos because of our lack of understanding of the asymptotics of the discrete Green's functions near the boundary of the polyominos.

Let us define two functions $F_0(z_1,z_2)$ and $F_1(z_1,z_2)$: for a fixed z_1, $F_0(z_1,z_2)$ is an analytic function of z_2 with a simple pole of residue $\frac{1}{\pi}$ at $z_2 = z_1$ and has real part zero on the boundary of U; for a fixed z_1, $F_1(z_1,z_2)$ is an analytic function of z_2 with a simple pole of residue $\frac{1}{\pi}$ at $z_2 = z_1$ and has imaginary part zero on the boundary of U. The existence and uniqueness of the two functions as solutions of boundary value problems will be shown in the following proof.

We have a significant result concerning the convergence of the coupling function.

Theorem 3.2.1. *Let P_ϵ^* be the dual graph of P_ϵ. For any real $\xi > 0$, the coupling function $C(v_1, v_2)$ on P_ϵ^* satisfies: for $v_1 \in W_0$ and v_1, v_2 not within ξ of the boundary of M_ϵ and $\epsilon \cdot |z_1 - z_2|^{-1} = o(1)$*

$$\frac{1}{\epsilon} C(v_1, v_2) = F_0(v_1, v_2) + o(1).$$

If $v_1 \in W_1$, then

$$\frac{1}{\epsilon} C(v_1, v_2) = F_1(v_1, v_2) + o(1).$$

The equality should be interpreted as following: when $v_1 \in W_0$, $v_2 \in B_0$, $C(v_1, v_2)$ equals the real part of the right-hand side; when $v_1 \in W_0$, $v_2 \in B_1$, $C(v_1, v_2)$ equals the imaginary part of the right-hand side; when $v_1 \in W_1$, $v_2 \in B_1$, $C(v_1, v_2)$ equals the real part of the right-hand side; when $v_1 \in W_1$, $v_2 \in B_0$, $C(v_1, v_2)$ equals the imaginary part of the right-hand side.

When $v_1 \in W_0$, suppose that $G(w_1, w_2)$ is a discrete Green's function on $B_0'(P_\epsilon)$. ($\Delta G(w_1, w_2) = \delta_{w_1}(w_2)$ and taking value 0 on the boundary Y). By Lemma 2.4.4,

$$\Delta ReC(v_1, \cdot) = \delta_{v_1+\epsilon} - \delta_{v_1-\epsilon}.$$

So

$$ReC(v_1, v_2) = G(v_1 + \epsilon, v_2) - G(v_1 - \epsilon, v_2).$$

Therefore, we need the convergence for Green's functions. In fact, $\frac{1}{\epsilon} ReC(v_1, v_2)$ is twice the $x-$derivative of the discrete Green's function. This holds even when we take the limit. In order to prove the theorem, we need the following lemma :

Lemma 3.2.2. *Let $z_1 = x_1 + iy_1$ be a point in the interior of U and let $z_2 \in U$ be different from z_1. Let v_1 be a vertex in $B_0'(P_\epsilon)$ within $O(\epsilon)$ of z_1, v_2 be a vertex in $B_0'(P_\epsilon)$ within $O(\epsilon)$ of z_2 and $\epsilon \cdot |z_1 - z_2|^{-1} = o(1)$. Then, $\frac{1}{\epsilon} G(v_1 + \epsilon, v_2) - \frac{1}{\epsilon} G(v_1 - \epsilon, v_2)$ converges to $2\partial_{x_1} g_U(z_1, z_2)$; $\frac{1}{\epsilon} G(v_1 + i\epsilon, v_2) - \frac{1}{\epsilon} G(v_1 - i\epsilon, v_2)$ converges to $2\partial_{y_1} g_U(z_1, z_2)$.*

Proof. Let $H(v_1, v_2) = \frac{1}{\epsilon}(G(v_1 + \epsilon, v_2) - G(v_1 - \epsilon, v_2))$.

For Green's function on the whole plane, Corollary 3.1.3 tells us that

$$H_0(v_1, v_2) := \frac{1}{\epsilon}(G_0(v_1+\epsilon, v_2) - G_0(v_1-\epsilon, v_2)) = Re\frac{1}{\pi(v_2 - v_1)} + O(\frac{\epsilon}{|v_2 - v_1|^2}) \tag{3.2}$$

Notice that $H(v_1, v_2) - H_0(v_1, v_2)$ (considered as a function of v_2) is harmonic on $B_0'(P_\epsilon)$ because we eliminate the singularity at $H(v_1, v_2)$. Meanwhile, the boundary values are bounded: Corollary 3.1.3 tells us that $H_0(v_1, v_2)$ is bounded on a bounded domain and $H(v_1, v_2)$ has boundary value 0.

Let g be a continuous harmonic function with boundary values equal to the boundary values of the limit

$$\lim_{\epsilon \to 0} H(v_1, v_2) - H_0(v_1, v_2).$$

According to (3.2), these boundary values converge uniformly (the boundary values of $H(v_1, v_2) - H_0(v_1, v_2)$ are within $O(\epsilon)$ of the limit values), and the limit of boundary values are continuous. Restrict g on the vertices of $B_0'(P_\epsilon)$ (notice that this is a lattice function), the discrete Laplacian of g on $v \in B_0'(P_\epsilon)$ is:

$$\Delta_\epsilon g(v_1, v) = 4g(v) - g(v + \epsilon) - g(v - \epsilon) - g(v + i\epsilon) - g(v - i\epsilon).$$

When ϵ is sufficiently small, according to the Taylor expansion of g, the formula above is approximated by

$$\Delta_\epsilon = -\frac{\epsilon^4}{12}(\frac{\partial^4 g(v)}{\partial x^4} + \frac{\partial^4 g(v)}{\partial y^4}) + O(\epsilon^5)$$

Thus, the discrete Laplacian of $H(v_1, v_2) - H_0(v_1, v_2) - g(v_1, v_2)$ is $O(\epsilon^4)$ on $B_0'(P_\epsilon)$, $O(\epsilon)$ on the boundary. Next, we will prove that $H_0 - H$ converges to g. Construct two sequences of functions which are superharmonic and subharmonic respectively that converge to 0 on the boundary, where by the discrete version of Harnack's principle, pointwise convergence is enough. Since the limit of sub(super)harmonic functions is still sub(super)harmonic, the limit is 0. Since the discrete Laplacian of $x + iy \mapsto x^2$ is a constant, we can find B_1 and B_2, two constants sufficiently big such that

$$\Delta_\epsilon(B_2\epsilon^4 Re(v_2)^2 + H(v_1, v_2) - H_0(v_1, v_2) - g(v_1, v_2)) \geq 0,$$

and

$$\Delta_\epsilon(B_3\epsilon^4 Re(v_2)^2 + H(v_1, v_2) - H_0(v_1, v_2) - g(v_1, v_2)) \geq 0.$$

By the maximum principle of superharmonic functions, these functions must take their maximum value on the boundary of the domain $B_0'(P_\epsilon)$. Since $H(v_1, v_2) - H_0(v_1, v_2) - g(v_1, v_2) = O(\epsilon)$ on the boundary of $B_0'(P_\epsilon)$, we have

$$H(v_1, v_2) - H_0(v_1, v_2) - g(v_1, v_2) = O(\epsilon).$$

Therefore, $H(v_1, v_2)$ converges to the funtion $Re\frac{1}{\pi(z_2-z_1)} + g(z_1, z_2)$ (the discretization of H_0 and g converge to $Re\frac{1}{\pi(z_2-z_1)}$ and g respectively). Since $H(v_1, v_2)$ is 0 on the boundary of $B_0'(P_\epsilon)$, this is exactly $2\partial_{x_1} g_U(z_1, z_2)$.

By the same argument, we have that $\frac{1}{\epsilon}G(v_1 + i\epsilon, v_2) - \frac{1}{\epsilon}G(v_1 - i\epsilon, v_2)$ converges to $2\partial_{y_1} g_U(z_1, z_2)$. □

Now, we can prove Theorem 3.2.1:

$\frac{1}{\epsilon}(G(v_1 + \epsilon, v_2) - G(v_1 - \epsilon, v_2))$ coverge to $2\partial_{x_1} g_U(z_1, z_2)$ on U.

The C^0-convergence of ReC implies the convergence of its derivative for harmonic functions. Therefore, if we take a local integration of it on $U\backslash\{v_1\}$ with value 0 at d, the convergence of ImC holds. Since ImC is the harmonic conjugate of ReC, $ImC = Im\frac{1}{\pi(z_2-z_1)} + g'(z_1, z_2)$, where $g'(z_1, z_2)$ is the harmonic conjugate of g on the simply connected domain U. Thus, when $v_1 \in W_0$, $\frac{1}{\epsilon}C(v_1, v_2)$ converges to a complex-valued function of z_2, having a simple pole of residue $\frac{1}{\pi}$ at z_1 and has real part 0 on the boundary, which has the same property as the function F_0. The existence of F_0 follows.

The similar result holds for $v_1 \in W_1$ and F_1.

Let $F_+ = F_0 + F_1$ and $F_- = F_0 - F_1$. F_+ and F_- depend only on the conformal type of the domain U in the following sense (conformal covariance):

Proposition 3.2.3. *The function $F_+(z_1, z_2)$ is analytic as a function of z_1 and z_2. The function $F_-(z_1, z_2)$ is analytic as a function of z_2 and antianalytic as a function of z_1. If V is another domain with smooth boundary and if $f : U \to V$ is a bijective complex analytic map sending the marked point on U to the marked point of V, and if F_+^V and F_-^V are the functions defined as above for the region V, then*

$$F_+^U(z_1, z_2) = f'(z_1)F_+^V(f(z_1), f(z_2))$$
$$F_-^U(z_1, z_2) = \overline{f'(z_1)}F_-^V(f(z_1), f(z_2))$$

Proof. From the definition of $F_\pm$ and the properties of F_0 and F_1, it is analytic in the second variable.

To show the property in the first variable, we need to go back to the coupling function. For a fixed black vertex v_2 not adjacent to v_1, we have

$$-C(v_1 + \epsilon, v_2) + C(v_1 - \epsilon, v_2) - iC(v_1 + i\epsilon, v_2) + iC(v_1 - i\epsilon, v_2) = 0.$$

Write $v_1 = x_1 + iy_1$. If $v_2 \in B_0$, $v_1 \pm \epsilon \in W_0$, $v_1 \pm i\epsilon \in W_1$ and v_1 and v_2 are within ϵ of z_1 and z_2 respectively, by Theorem 3.2.1, multiplying the equation above with $\frac{1}{\epsilon}$ gives in the limit

$$-\partial_{x_1} ReF_0(z_1, z_2) + \partial_{y_1} ImF_1(z_1, z_2) = 0$$

and if $v_2 \in B_1$, $v_1 \pm \epsilon \in W_0$, $v_1 \pm i\epsilon \in W_1$ and v_1 and v_2 are within ϵ of z_1 and z_2 respectively, this gives

$$-\partial_{x_1} ImF_0(z_1, z_2) - \partial_{y_1} ReF_1(z_1, z_2) = 0$$

Combine the two equations above into a single complex-valued equation

$$-\partial_{x_1} F_0(z_1, z_2) - i\partial_{y_1} F_1(z_1, z_2) = 0.$$

Similarly, if $v_1 \pm \epsilon \in B_1$ and $v_1 \pm i\epsilon \in B_0$ we have

$$-\partial_{x_1} F_1(z_1, z_2) - i\partial_{y_1} F_0(z_1, z_2) = 0.$$

The sum of the two equations gives $\partial_{\overline{z_1}}(F_0 + F_1) = 0$ and their difference gives $\partial_{z_1}(F_0 - F_1) = 0$. This proves the first two statements.

As a function of z_2, $F_0^v(f(z_1), f(z_1))$ satisfies all the properties of F_0^U except that the residue at $z_2 = z_1$ is $\frac{1}{\pi f'(z_1)}$. Similarly, the function $F_1^V(f(z_1), f(z_2))$ has all the properties of F_1^U except that the residue at $z_2 = z_1$ is $\frac{1}{\pi f'(z_1)}$. But, $Re(f'(z_1))F_0^V(f(z_1), f(z_2)) + iIm(f'(z_1))F_1^V(f(z_1), f(z_2))$ has the same properties as F_0^U. We deduce from the uniqueness from F_0^U that

$$F_0^U = Re(f'(z_1))F_0^V(f(z_1), f(z_2)) + iIm(f'(z_1))F_1^V(f(z_1), f(z_2))$$

.

Similarly,

$$F_1^U = iIm(f'(z_1))F_0^V(f(z_1), f(z_2)) + Re(f'(z_1))F_1^V(f(z_1), f(z_2)).$$

Again, the sum and the difference of F_0^U and F_1^U completes the proof of the theorem. $\square$

Chapter 4

Scaling limit of the height function

4.1 Multi-point expectation

Let U be a bounded simply connected domain in $\mathbb{R}^2$ with smooth boundary. For any $\epsilon > 0$, let P_ϵ be a Temperley domain approximating U as defined in 2.5. Attributing uniform probability to all possible domino tilings of P_ϵ, the height function h can be seen as a random function. Let $\overline{h}$ be the mean value of h, we will show that the fluctuations of $h - \overline{h}$ converge to the Gaussian free field as ϵ tends to zero. It has a well-known one-dimensional analog: let X be the set of all maps from $0, \frac{1}{n}, \frac{2}{n}, \cdots, 1$ to $\mathbb{Z}$, with $h(0) = h(1) = 1, |h(\frac{i+1}{n} - h(\frac{i}{n}))| = 1$. If we take the uniform distribution on X, $\frac{X}{n}$ converges to the Brownian bridge (the Brownian bridge on $[0,1]$ is the Brownian motion $(B_t)_{t\in[0,1]}$ conditioned on the event $\{B_0 = B_1 = 0\}$). In the eigenbasis of the one-dimensional Laplacian $\frac{\partial^2}{\partial x^2}$ the coefficients of the Brownian bridge are independent Gaussians. One difference from the one-dimensional case is that it is not required that height function h is normalized. It is therefore more surprising that the integer-valued function h converges to a continuous-valued object.

Take any test function $\varphi \in C_0^\infty(U)$ on the domain, in order to prove that $\int_U \phi(h - \overline{h})$ is a Gaussian, we only need to show that its moments is the same as the moments of a Gaussian distribution.[3]

$$\begin{aligned}\mathbb{E}[(\int_U \varphi(h^\epsilon - \overline{h^\epsilon}))^n] &= \mathbb{E}[\int_{U\times\cdots\times U}\int \varphi(a_1)\cdots\varphi(a_n)(h^\epsilon - \overline{h^\epsilon})(a_1)\cdots(h^\epsilon - \overline{h^\epsilon})(a_n)] \\ &= \int_{U\times\cdots\times U}\int \varphi(a_1)\cdots\varphi(a_n)\mathbb{E}[(h^\epsilon - \overline{h^\epsilon})(a_1)\cdots(h^\epsilon - \overline{h^\epsilon})(a_n)]\end{aligned}$$

In fact, [2] has proved that for x inside P_ϵ, the height $h(x)$ converges to a Gaussian with variance $\frac{c}{\log(\frac{1}{\epsilon})}$, where c is a constant. When $a_1 = a_2 = \cdots = a_K$, $\mathbb{E}[(h^\epsilon - \overline{h^\epsilon})(a_1)\cdots(h^\epsilon - \overline{h^\epsilon})(a_n)]$ does not converge as $\epsilon \to 0$, but the function is still uniformly integrable with respect to ϵ. Therefore, we want to get that

$$\mathbb{E}[(h^\epsilon - \overline{h^\epsilon})(a_1)\cdots(h^\epsilon - \overline{h^\epsilon})(a_n)] \tag{4.1}$$

converges when $\{a_i\}$ are distinct. In order to know the height at a certain point a, we can find a path from the boundary of the domain to a, then counting the changes of height along the edges of the path which is determined by the domino tiling.

Theorem 4.1.1. *Let U be a bounded simply connected domain with smooth boundary in the plane, $\{a_1, a_2, \cdots, a_k\}$ be distinct points of U, and $\{\gamma_1, \gamma_2, \cdots, \gamma_k\}$ disjoint paths running from the boundary of U to $a_1, a_2, \cdots, a_k$. Let $h(z_i)$ denote the height of a point of P_ϵ lying within $O(\epsilon)$ of a_i. As $\epsilon \to 0$,*

$\mathbb{E}[(h^\epsilon - \overline{h^\epsilon})(a_1)(h^\epsilon - \overline{h^\epsilon})(a_2)\cdots(h^\epsilon - \overline{h^\epsilon})(a_k)]$ converges to

$$|\sum_{\epsilon_1,\cdots,\epsilon_k\in\{\pm1\}} \epsilon_1\epsilon_2\cdots\epsilon_k \int_{\gamma_1}\cdots\int_{\gamma_k} \det_{i,j\in\{1,k\}}(F_{\epsilon_i,\epsilon_j}(z_i,z_j))dz_1^{(\epsilon_1)}\cdots dz_k^{(\epsilon_k)}|. \tag{4.2}$$

where $dz_j^{(1)} = dz_j, dz_j^{(-1)} = \overline{z_j}$, and

$$F_{\epsilon_i,\epsilon_j} = \begin{cases} 0 & i = j \\ F_+(z_i,z_j) & (\epsilon_i,\epsilon_j) = (1,1) \\ F_-(z_i,z_j) & (\epsilon_i,\epsilon_j) = (-1,1) \\ \overline{F_-(z_i,z_j)} & (\epsilon_i,\epsilon_j) = (1,-1) \\ \overline{F_+(z_i,z_j)} & (\epsilon_i,\epsilon_j) = (-1,-1) \end{cases}.$$

Notice that in each of the 2^K multiple integrals in (4.2), by Proposition 3.2.3, each integrand is analytic or antianalytic in z_i according to $\epsilon_i = \pm1$. The integrand I is conformally invariant, in the sense that

$$\int_\gamma I(z)dz = \int_{f(\gamma)} I(f(z))dz$$

Therefore (4.2) is conformally invariant.

Proof. For each ϵ sufficiently small, let $\gamma_1^\epsilon, \cdots, \gamma_k^\epsilon$ be pairwise disjoint lattice paths in P_ϵ which start on the flat boundary near d (marked point). We require that each straight edge of γ_i^ϵ has even length (an even multiple of ϵ).

In a given tiling, the height change on γ_i^ϵ equals $4(A_i - B_i)$, where A_i is the number of dominos crossing γ_i with the black square on the right and B_i is the number of dominos crossing γ_i with the white square on the right. This is because, if γ_i does not cross any dominos, the height change is 0: the straight edges of γ_i have even length along which the number of black squares and the number of white squares on the right are the same. For each domino crossed by γ_i, the height difference changes along the edge from -1 to $+3$ if the domino has black square on the right, and from $+1$ to -3 if the black square is on the left.

Since $h_i = 4(A_i - B_i)$,

$$\mathbb{E}[(h^\epsilon - \overline{h^\epsilon})(a_1) \cdots (h^\epsilon - \overline{h^\epsilon})(a_k)] = 4^k \mathbb{E}[(A_1 - B_1 - \overline{A_1} + \overline{B_1}) \cdots (A_k - B_k - \overline{A_k} + \overline{B_k})]. \tag{4.3}$$

The rest of the proof involves expanding this out, cancelling various terms and recombining them the right way.

Let α_{it} denote the indicator functions of the presence of the t-th possible domino crossing γ_i whose black square is right of γ_i and let β_{it} denote the indicator functions of the presence of the t-th possible domino crossing γ_i whose black square is left of γ_i. Then,

$$A_i - B_i = \sum_t \alpha_{it} - \sum_{t'} \alpha_{it'}.$$

Let (w_it, b_it) be the white and black squares respectively of the domino α_{it} and $(w_i't, b_i't)$ be the white and black squares of the domino β_{it}.

Since the straight edges in the path γ_i have even length, we can pair the α_{it} dominos with adjacent $\beta_{it'}$ dominos which are parallel to α_{it}. It is then convenient to write

$$A_i - B_i - \overline{A_i} + \overline{B_i} = \sum_t (\alpha_{it} - \overline{\alpha_{it}} - \beta_{it} + \overline{\beta_{it}})$$

where α_{it} and β_{it} are paired. Thus we can write (4.3) as

$$\begin{aligned}
&\mathbb{E}[(A_1 - B_1 - \overline{A_1} + \overline{B_1}) \cdots (A_k - B_k - \overline{A_k} + \overline{B_k})] \\
&= \sum_{t_1,\cdots,t_k} \mathbb{E}[(\alpha_{1t_1} - \overline{\alpha_{1t_1}} - \beta_{1t_1} + \overline{\beta_{1t_1}}) \cdots (\alpha_{kt_k} - \overline{\alpha_{kt_k}} - \beta_{kt_k} + \overline{\beta_{kt_k}})] \\
&= \sum_{t_1,\cdots,t_k} (\mathbb{E}[(\alpha_{1t_1} - \overline{\alpha}_{1t_1}) \cdots (\alpha_{kt_k} - \overline{\alpha}_{kt_k})] + \cdots + (-1)^k \mathbb{E}[(\beta_{1t_1} - \overline{\beta}_{1t_1}) \cdots (\beta_{kt_k} - \overline{\beta}_{kt_k})]).
\end{aligned} \tag{4.4}$$

where the sums are over all the pairs $\{\alpha_{1t_1}, \beta_{1t_1}\}, \cdots, \{\alpha_{k,t_k}, \beta_{kt_k}\}$. To expand this, we need the following lemma.

Lemma 4.1.2. *Let $e_i = (w_i, b_i), i = 1, \cdots, n$ be a set of n disjoint edges; then*

$$\mathbb{E}[(e_1 - \overline{e_i}) \cdots (e_n - \overline{e_n})] = a_E \det \begin{pmatrix} 0 & C(w_1, b_2) & \cdots C(w_1, b_n) & \\ C(w_2, b_1) & 0 & & \vdots \\ \vdots & & & C(w_{n-1}, b_n) \\ C(w_n, b_1) & \cdots & C(w_n, b_{n-1}) & 0 \end{pmatrix}.$$

where a_E is the product of the edge weights of the e_i.

Proof. We will proceed by induction on n. When $n = 1$, both sides of the equation are zero.

Suppose that the equation holds for integers k less than $n+1$. Note that

$$\begin{vmatrix} 0 & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & & \\ \vdots & & \ddots & \\ a_{n1} & & & a_{nn} \end{vmatrix} = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & & \\ \vdots & & \ddots & \\ a_{n1} & & & a_{nn} \end{vmatrix} - \begin{vmatrix} a_{11} & 0 & \cdots & 0 \\ 0 & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ 0 & a_{n2} & \cdots & a_{nn} \end{vmatrix},$$

we have

$$
\begin{aligned}
&\mathbb{E}[(e_1 - \overline{e_1}) \cdots (e_n - \overline{e_n})] \\
&= \mathbb{E}[e_1 \cdots e_n] - \sum_{S \subset \{1,\cdots,n\}, S \neq \emptyset} \prod_{i \in S} a_{e_i} \overline{e_i} \mathbb{E}[\prod_{i \neq S} (e_i - \overline{e_i})] \\
&= a_E \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & & \\ \vdots & & \ddots & \\ a_{n1} & & & a_{nn} \end{vmatrix} - a_E \prod_{i \in S} \overline{e_i} \det_{\{1,\cdots,n\} \setminus S} \begin{pmatrix} 0 & & \star \\ & \ddots & \\ \star & & 0 \end{pmatrix} \\
&= a_E \det \begin{pmatrix} 0 & a_{12} & \cdots a_{1n} & \\ a_{21} & 0 & & \vdots \\ \vdots & & & a_{n-1,n} \\ a_{n,1} & \cdots & a_{n,n-1} & 0 \end{pmatrix}.
\end{aligned}
$$

where $a_{i,j} = C(w_i, b_j)$, $i, j = 1, 2, \cdots, n$. □

Each terms in the matrix has the form

$$a_E sgn(\sigma) C(w_1, b_{\sigma(1)} C(w_2, b_{\sigma(2)}) \cdots C(w_k, b_{\sigma(k)})). \tag{4.5}$$

where σ is a permutation without fixed point.

Reorder the vertices such that (4.5) is of the form

$$a_E sgn(\sigma) C(w_1, b_2) C(w_2, b_3) \cdots C(w_k, b_1) \tag{4.6}$$

To expand the products of the coupling functions in (4.6), let us define $r_i = \pm 1$ according to whether $w_{it_i} \in W_0$ or $w_{it_i} \in W_1$, and $s_i = \pm 1$ according to whether $b_{it_i} \in B_0$ or $b_{it_i} \in B_0$. If we assume that neither w_1 or b_2 is close to the boundary, by Theorem 3.2.1,

$$
\begin{aligned}
C(w_1, b_2) &= \epsilon(\frac{1 - r_1 s_2}{2} iIm + \frac{1 + r_1 s_2}{2} Re)(\frac{1 + r_1}{2} F_0(w_1, b_2) + \frac{1 - r_1}{2} F_1(w_1, b_2)) + o(\epsilon) \\
&= \frac{\epsilon}{4}(F_+(w_1, b_2) + r_1 F_-(w_1, b_2) + s_2 \overline{F_-}(w_1, b_2) + r_1 s_2 \overline{F_+}(w_1, b_2)) + o(\epsilon).
\end{aligned}
$$

For each fixed $\xi > 0$, when neither of w_1 and b_2 are within ξ of the boundary, this approximation holds for sufficiently small ϵ. Next, we first fix ξ, let $\epsilon \to 0$, then let $\xi \to 0$. When one of w_1, b_2 is within ξ of the boundary, we need only to show, $\frac{1}{\epsilon} C(w_1, b_2)$ is controlled by a constant irrelevant to ϵ and ξ. Then in the integral, we can ignore them as it is at most $O(\epsilon)$. By Theorem 3.2.1, we have the boundedness of $\frac{1}{\epsilon} C(w_1, b_2)$.

Now we can write (4.6) as:

$$4^{-k}\epsilon^k a_E sgn(\sigma)((F_+(w_1,b_2)+r_1F_-(w_1,b_2)+s_2\overline{F_-}(w_1,b_2)+$$

$$r_1s_2\overline{F_+}(w_1,b_2))\cdots(F_+(w_k,b_1)+r_kF_-(w_k,b_1)+s_1\overline{F_-}(w_k,b_1)+r_ks_1\overline{F_+}(w_k,b_1)))+o(\epsilon^k). \tag{4.7}$$

Replace (w_1,b_1) by (w_1',b_1'), then a similar expression follows except that the signs of r_1 and s_1 are reversed. If we sum over all the 2^k choices of α_j,β_j, we get 2^k times the sum of those terms in (4.7) which have r_i to the same power as s_i (0 or 1) for each i. Therefore, this sum can be written as an error $o(\epsilon^k)$ plus

$$2^{-k}\epsilon^k sgn(\sigma)a_E\sum_{\epsilon_1,\cdots,\epsilon_k\in\{-1,1\}}(r_1s_1)^{\frac{1-\epsilon_1}{2}}\cdots(r_ks_k)^{\frac{1-\epsilon_k}{2}}F_{\epsilon_1,\epsilon_2}(z_1,z_2)F_{\epsilon_2,\epsilon_3}(z_2,z_3)\cdots F_{\epsilon_k,\epsilon_1}(z_k,z_1). \tag{4.8}$$

When ϵ is sufficiently small, we get an expression in the form of an integral by replacing 2ϵ by a certain phase times dz_i or $d\overline{z_i}$ as following:

$$2\epsilon=\begin{cases} dz_j=d\overline{z}_j & \gamma_j \text{ is going east, the edge weight is } -i,\ r_js_j=-1\\ -dz_j=-d\overline{z}_j & \gamma_j \text{ is going west, the edge weight is } i,\ r_js_j=-1\\ -idz_j=id\overline{z}_j & \gamma_j \text{ is going north, the edge weight is } 1,\ r_js_j=1\\ idz_j=-id\overline{z}_j & \gamma_j \text{ is going south, the edge weight is } -1,\ r_js_j=1\end{cases}.$$

Notice that 2ϵ times the edge weight times $(r_js_j)^{\frac{(1-\epsilon_j)}{2}}$ is $-\epsilon_j i dz_j^{\epsilon_j}$, and a_E is the product of the edge weights. For any choice of the ϵ_j we have

$$a_E(2\epsilon)^k(r_1s_1)^{\frac{(1-\epsilon_1)}{2}}\cdots(r_ks_k)^{\frac{(1-\epsilon_k)}{2}}=(-i)^k\epsilon_1\cdots\epsilon_k dz_1^{\epsilon_1}\cdots dz_k^{\epsilon_k}.$$

The sum (4.8) is therefore

$$4^{-k}(-i)^k sgn(\sigma)\sum_{\epsilon_1,\cdots,\epsilon_k\in\{-1,1\}}\epsilon_1\cdots\epsilon_k F_{\epsilon_1,\epsilon_2}(z_1,z_2)F_{\epsilon_2,\epsilon_3}(z_2,z_3)\cdots F_{\epsilon_k,\epsilon_1}(z_k,z_1)dz_1^{\epsilon_1}\cdots dz_k^{\epsilon_k}.$$

When σ is a product of disjoint cycles we can treat each cycle separately and the result is the product of terms involving disjoint sets of indices. Thus when we sum over all (fixed-point free permutations) σ, we obtain the formula (4.2) without the integral. Summing over $t_1,\cdots,t_k$ (notice that 4^k cancels) we get the formula. ☐

Corollary 4.1.3. *When $k=2$, the expression above is the Green's function on U.*

Proof. To prove that this expression is the Green's function on U, we first prove that this is the case on the half plane, then we use the conformal invariance.

$$\begin{aligned}
&\mathbb{E}[(h-\overline{h})(a)(h-\overline{h})(b)] \\
=&\int_{\gamma_1,\gamma_2}\begin{vmatrix} 0 & F_+(z_1,z_2) \\ F_+(z_2,z_1) & 0\end{vmatrix} dz_1dz_2 - \int_{\gamma_1,\gamma_2}\begin{vmatrix} 0 & F_-(z_1,z_2) \\ \overline{F_+(z_2,z_1)} & 0\end{vmatrix} d\overline{z_1}dz_2 \\
-&\int_{\gamma_1,\gamma_2}\begin{vmatrix} 0 & \overline{F_-(z_1,z_2)} \\ F_+(z_2,z_1) & 0\end{vmatrix} dz_1d\overline{z_2} + \int_{\gamma_1,\gamma_2}\begin{vmatrix} 0 & \overline{F_+(z_1,z_2)} \\ \overline{F_+(z_2,z_1)} & 0\end{vmatrix} d\overline{z_1}d\overline{z_2}
\end{aligned}$$

On the half plane, we have $F_0(z_1,z_2)=\frac{1}{\pi(z_2-z_1)}-\frac{1}{\pi(z_2-\overline{z_1})}$, $F_1(z_1,z_2)=\frac{1}{\pi(z_2-z_1)}+\frac{1}{\pi(z_2-\overline{z_1})}$. So, $F_+(z_1,z_2)=\frac{2}{\pi(z_2-z_1)}$, $F_-(z_1,z_2)=-\frac{2}{\pi(z_2-\overline{z_1})}$.

$$\begin{aligned}
&\mathbb{E}[(h-\overline{h})(a)(h-\overline{h})(b)] \\
=&-\frac{4}{\pi^2}\int_{\gamma_1}\int_{\gamma_2}\frac{1}{(z_2-z_1)^2}dz_1dz_2+\frac{4}{\pi^2}\int_{\gamma_1}\int_{\gamma_2}\frac{1}{(z_2-\overline{z_1})^2}d\overline{z_1}dz_2 \\
+&\frac{4}{\pi^2}\int_{\gamma_1}\int_{\gamma_2}\frac{1}{(\overline{z_2}-z_1)^2}dz_1d\overline{z_2}-\frac{4}{\pi^2}\int_{\gamma_1}\int_{\gamma_2}\frac{1}{(\overline{z_2}-\overline{z_1})^2}d\overline{z_1}d\overline{z_2}.
\end{aligned}$$

The first integral is equal to

$$-\frac{4}{\pi^2}\log\frac{(a-b)(Re(a)-Re(s))}{(a-Re(b))(Re(a)-b)}.$$

Therefore,

$$\begin{aligned}
\mathbb{E}[(h-\overline{h})(a)(h-\overline{h})(b)] &= \frac{4}{\pi^2}\Big(-2Re\log\frac{(a-b)(Re(a)-Re(s))}{(a-Re(b))(Re(a)-b)} \\
&\quad +2Re\log\frac{(\overline{a}-b)(Re(a)-Re(s))}{(\overline{a}-Re(b))(Re(a)-b)}\Big) \\
&= \frac{8}{\pi^2}Re\log\Big(\frac{\overline{p}-q}{p-q}\Big).
\end{aligned}$$

This is exactly the Green's function on the half plane. □

4.2 Convergence to the Gaussian free field

In this part, we will show that the scaling limit when ϵ tends to 0 of height functions on P_ϵ which approximate a bounded simply connected domain in $\mathbb{R}^2$ with smooth boundary is the Gaussian free field.

Theorem 4.2.1. *Let U be a bounded simply connected domain in $\mathbb{R}^2$ with smooth boundary, with a marked point $b \in \partial U$. For each $\epsilon > 0$ sufficiently small let P_ϵ be a Temperley domain approximating U as defined in 2.5. Then, as $\epsilon \to 0$, $h_\epsilon - \overline{h}_\epsilon$ converges weakly to $\frac{4}{\sqrt{\pi}}$ times the Gaussian free field F on U. That is to say, for any $\varphi \in C_0^\infty(U)$, the random variable $\epsilon^2 \sum_{x \in V_\epsilon} \varphi(x)(h_\epsilon(x) - \overline{h}_\epsilon(x))$ converges in distribution to $\frac{4}{\sqrt{\pi}} \int_U \varphi F dxdy$.*

To prove this, we first show that the limit of $\mathbb{E}[(h^\epsilon - \overline{h^\epsilon})(a_1) \cdots (h^\epsilon - \overline{h^\epsilon})(a_n)]$ has a simple expression in terms of Green's function.

When U is the upper half plane, $F_+(z_1, z_2) = \frac{2}{\pi(z_2 - z_1)}$ and $F_-(z_1, z_2) = -\frac{2}{\pi(z_2 - \overline{z_1})}$. The matrix in the integral of equation (4.2) is $(\frac{2\epsilon_i \epsilon_j}{\pi(z_j^{\epsilon_j} - z_i^{\epsilon_i})})_{i,j}$. Factoring a ϵ_i out of the ith column for each i, the matrix has the same determinant as the matrix $(\frac{2}{\pi(z_j^{\epsilon_j} - z_i^{\epsilon_i})})$.

Such a matrix has a simple determinant.

Lemma 4.2.2. *If $M = (m_{ij})$ is a $k \times k$ matrix, where $m_{ii} = 0$, $m_{ij} = \frac{1}{x_j - x_i}$, $i \neq j$. Then when k is odd, $\det M = 0$ and when k is even, we have*

$$\det(M) = \sum \frac{1}{(x_{\sigma(1)} - x_{\sigma(2)})^2 (x_{\sigma(3)} - x_{\sigma_4})^2 \cdots (x_{\sigma(k-1)} - x_{\sigma(k)})^2} \tag{4.9}$$

where the sum is over all $(k-1)!!$ possible pairings $\{\{\sigma(1), \sigma(2)\}, \cdots, \{\sigma(k-1), \sigma(k)\}\}$ of $\{1, \cdots, k\}$.

Proof. When k is odd $\det M = 0$ because M is antisymmetric. When k is even, we proceed by induction on k:

Clearly the formula holds when $k = 2$. For $k > 2$ and even, the determinant is a rational function of x_1 with a double pole at $x_1 = x_2$. Thus

$$\det(M) = \frac{c_{-2}}{(x_1 - x_2)^2} + \frac{c_{-1}}{(x_1 - x_2)} + c_0 + O(x_1 - x_2).$$

Because the determinant is symmetric in x_1 and x_2, $c_{-1} = 0$. c_{-2} is the determinant of M_{12}, the matrix obtained from M by deleting the first two rows and columns. Therefore, the right- and left-hand sides of (4.9) both represent functions (in each variable) with the same poles and residues; hence they differ by a constant. which is zero by homogeneity. If we replace all x_i with λx_i for each i, the determinant is multiplied by λ^{-k}. By homogeneity, the constant is zero. □

In the following, we will use Wick's rule to show that height functions converge to the Gaussian free field.

Theorem 4.2.3. *(Wick)[1] For a set of random variables $\{X_i\}$, if for any $X_1, \cdots, X_{2n}$ (not necessarily distinct), $\mathbb{E}[X_1 \cdots X_{2n-1}] = 0$*

$$\mathbb{E}[X_1 X_2 \cdots X_{2n}] = \sum_{pairings \sigma} \mathbb{E}[X_{\sigma(1)} X_{\sigma(2)}] \cdots \mathbb{E}[X_{\sigma(2n-1)} X_{\sigma(2n)}],$$

then $(X_1, \cdots, X_{2n})$ is Gaussian with mean zero.

To verify the conditions of Wick's rule, we need the following proposition:

Proposition 4.2.4. *Let U be a bounded simply connected domain in $\mathbb{R}^2$ with smooth boundary and $p_1, \cdots, p_k$ be distinct points in U. If k is odd, we have*

$$\lim_{\epsilon \to 0} \mathbb{E}[h_0(p_1) \cdots h_0(p_k)] = 0.$$

If k is even, we have

$$\lim_{\epsilon \to 0} \mathbb{E}[h_0(p_1) \cdots h_0(p_k)] = (-\frac{16}{\pi})^{k/2} \sum_{pairings\ \sigma} g(p_{\sigma(1)}, p_{\sigma(2)}) \cdots g(p_{\sigma(k-1)}, p_{\sigma(k)}).$$

Proof. When U is the upper half plane, by combining 4.2 with Corollary 4.1.3, we have

$$\begin{aligned}\lim_{\epsilon \to 0} \mathbb{E}[h_0(p_1) \cdots h_0(p_k)] = & \sum_{\epsilon_1, \cdots, \epsilon_k = \pm 1} \epsilon_1 \cdots \epsilon_k \int_{\gamma_1} \cdots \int_{\gamma_k} \sum_{\text{pairings } \sigma} \det_{i,j \in \{\sigma(1), \sigma(2)\}} (F_{\epsilon_i, \epsilon_j}) \cdots \\ & \det_{i,j \in \{\sigma(k-1), \sigma(k)\}} (F_{\epsilon_i, \epsilon_j}) dz_1^{(\epsilon_1)} \cdots dz_k^{(\epsilon_k)} \\ & = \sum_{\sigma} (\sum_{\epsilon_{\sigma(1)}, \epsilon_{\sigma(2)} = \pm 1} \det_{i,j \in \{\sigma(1), \sigma(2)\}} (F_{\epsilon_i, \epsilon_j}) dz_{\sigma(1)}^{\epsilon_{\sigma(1)}} dz_{\sigma(2)}^{\epsilon_{\sigma(2)}}) \cdots \\ & (\sum_{\epsilon_{\sigma(k-1)}, \epsilon_{\sigma(k)} = \pm 1} \det_{i,j \in \{\sigma(k-1), \sigma(k)\}} (F_{\epsilon_i, \epsilon_j}) dz_{\sigma(k-1)}^{\epsilon_{\sigma(k-1)}} dz_{\sigma(k)}^{\epsilon_{\sigma(k)}}) \\ & = (-\frac{16}{\pi})^{k/2} \sum_{\sigma} g(p_{\sigma(1)}, p_{\sigma(2)}) \cdots g(p_{\sigma(k-1)}, p_{\sigma(k)}).\end{aligned}$$

For an arbitrary U, we use conformal invariance and Corollary 4.1.3 to show that $\mathbb{E}[h_0(p_1)h_0(p_2)] = -\frac{16}{\pi}g^U(p_1,p_2)$, where g^U is the Green's function on U. The proof is completed by summing over all the pairs. $\square$

Next, we will complete the proof of Theorem 4.2.1.

Let $f_{n_1}, \cdots, f_{n_k}$ be (not necessarily distinct) eigenvectors of Δ with Dirichlet boundary conditions. Let $C^\epsilon_{n_j}$ be the real-valued random variable $C^\epsilon_{n_j} = \epsilon^2 \sum_{x\in V_\epsilon} h_0(x)f_{n_j}(x)$, where the sum is over the vertices V_ϵ of P_ϵ, and $f_{n_j}(x)$ is f_{n_j} evaluated at the vertex x. We have

$$
\begin{aligned}
\lim_{\epsilon\to 0}\mathbb{E}[C^\epsilon_{n_1}\cdots C^\epsilon_{n_k}] &= \lim_{\epsilon\to 0}\mathbb{E}[\sum_{x_1\in V_\epsilon}\epsilon^2 h_0(x_1)f_{n_1}(x_1)\cdots\sum_{x_k\in V_\epsilon}\epsilon^2 h_0(x_k)f_{n_k}(x_k)] \\
&= \lim_{\epsilon\to 0}\sum_{x_1\in V_\epsilon}\cdots\sum_{x_k\in V_\epsilon}\epsilon^2 f_{x_1}(x_1)\cdots\epsilon^2 f_{n_k}(x_k)\mathbb{E}[h_0(x_1)\cdots h_0(x_k)] \\
&= (-\frac{16}{\pi})^{k/2}\int_U\cdots\int_U f_{n_1}(x_1)\cdots f_{n_k}(x_k)\sum_\sigma g(x_{\sigma(1)},x_{\sigma(2)})\cdots g(x_{\sigma(k-1)},x_{\sigma(k)}) \\
&= (-\frac{16}{\pi})^{k/2}\sum_\sigma\int_U\cdots\int_U f_{n_1}(x_1)\cdots f_{n_k}(x_k) \\
&\qquad \sum_{m_1,\cdots,m_{k/2}}\frac{f_{m_1}(x_{\sigma(1)})f_{m_1}(x_{\sigma(2)})}{\lambda_{m_1}}\cdots\frac{f_{m_{k/2}}(x_{\sigma(k-1)})f_{m_{k/2}}(x_{\sigma(k)})}{\lambda_{m_{k/2}}} \\
&= (-\frac{16}{\pi})^{k/2}\sum_\sigma\frac{\delta_{n_{\sigma(1)},n_{\sigma(2)}}}{-\lambda_{n_{\sigma(1)}}}\cdots\frac{\delta_{n_{\sigma(k-1)},n_{\sigma(k)}}}{-\lambda_{n_{\sigma(k-1)}}}
\end{aligned}
$$

where we use an expression of the Green's function in terms of the eigenbasis $\{f_n\}$.

Lemma 4.2.5. $g_U(z_1,z_2) = \sum\limits_{i\geq 1}\frac{1}{\lambda_i}f_i(z_1)f_i(z_2)$

Proof. Let f_n be the normalized eigenvectors of Δ with Dirichlet boundary conditions. $\{f_n\}$ is an orthonormal basis of $L^2(U)$. Because

$$\begin{aligned}\langle f_i(z_2), g_U(z_1,z_2)\rangle &= \frac{1}{\lambda_i}\langle \lambda_i f_i(z_2), g_U(z_1,z_2)\rangle\\ &= \frac{1}{\lambda_i}\langle \Delta f_i(z_2), g_U(z_1,z_2)\rangle\\ &= \frac{1}{\lambda_i}\langle f_i(z_2), \Delta g_U(z_1,z_2)\rangle\\ &= \frac{1}{\lambda_i}\langle f_i(z_2), \delta_{z_1}(z_2)\rangle\\ &= \frac{1}{\lambda_i} f_i(z_1),\end{aligned}$$

$g_U(z_1,z_2) = \sum_{i\geq 1} \frac{f_i(z_1)f_i(z_2)}{\lambda_i}$. □

These are exactly the moments for a set of independent Gaussians of mean 0 and variances $-\frac{16}{\pi\lambda_i}$. To conclude we invoke the following standard probability lemma.

Lemma 4.2.6. *[1] A sequence of multidimensional random variables whose moments converge to the moments of a Gaussian, converges itself to a Gaussian.*

Now, we have to show that the Gaussian free field F is conformally invariant in the following sense (if we consider F to be a continuous linear functional on the space of smooth $2-$forms on U):

Proposition 4.2.7. *Let ω be a smooth $2-$form on U and $f : V \to U$ be a conformal bijection. Let $X = \int_U F_U(z)\omega(z)$, $Y = \int_V F_V(z) f^*\omega(z)$, where $f^*\omega$ is the pullback of ω to V. Then the random variables X and Y are equal in distribution.*

Proof. Since X and Y are Gaussians with mean 0, it suffices to compute their variances. In fact,

$$\begin{aligned}\mathbb{E}[X^2] &= \int_U\int_U \omega(z_1)\omega(z_2)\mathbb{E}[F(z_1)F(z_2)]\\ &= \int_U\int_U \omega(z_1)\omega(z_2) g^U(z_1,z_2)\\ &= \int_V\int_V f^*\omega(y_1)f^*\omega(y_2) g^U(f(y_1),f(y_2))\\ &= \int_V\int_V f^*\omega(y_1)f^*\omega(y_2) g^V(y_1,y_2)\\ &= \mathbb{E}[Y^2]\end{aligned}$$

where we used the conformal invariance of the Green's function: $g^U(f(y_1), f(y_2)) = g^V(y_1, y_2)$. $\square$

Chapter 5

Conclusion

At the time Kenyon published [3, 5], conformal invariance in the scaling limit of various 2D lattice model was conjectured by physicists and supported by numerical simulations but formal mathematical proofs remained elusive. These papers presented a machinery capable of proving some conformal invariance in the scaling limit of a particular 2D lattice model, namely domino tilings. Since then, many other 2D lattice models have been proven to have some conformal invarance in their scaling limit (for example the critical percolation on triangular lattice [7] and the 2D Ising model [8])

The work presented also paved the way for the proof of the conformal invariance of the scaling limit of other dimer models (models describing random perfect matching of a graph, a perfect matching being a set of edge such that all vertices belong to exactly one edge). Relatively recently, Kenyon proved the conformal invariance of loops in the double-dimer model (union of two perfect matching of the same graph) on $\mathbb{Z}^2$ [4].

Since the publication of Kenyon seminal papers [3, 5] in 1999 and 2001, impressive advances have been achieved. Nevertheless, several open problems remain to be solved. To mention only one example, at the moment, we are not able to adapt the proofs presented in this report if the polyominos are not Temperley domains. In other words, the proofs rely heavily on specific conditions on the boundary combinatorics of the polyominos. For an example, if the polyomino is even (i.e. the union of 2x2 blocs), it is believed that the height function converges to the Gaussian free field but no proof has been proposed yet.

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