Hodge theory

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October 2, 2020

Contents

1 History

First a brief history of Hodge theory, extracted from [16]. Hodge theory has origin in works by Abel, Jacobi, Gauss, Legendre and Weierstrass among many others on the periods of integrals of rational one-forms. The relative theory appeared in the late 1960's with the work of Griffiths [10, 11]. He found that higher weights generalization of the ordinary Jacobian, the intermediate Jacobian, need not be polarized. He generalized Abel-Jacobi maps in this set-up and used these to explain the difference of cycles and divisors. The important insight that any algebraic variety has a generalized notion of Hodge structure was worked out in Hodge II,III [5, 6]. In the relative setting, if the family acquires singularities, the Hodge structure on the cohomology of fiber may degenerate when the base point goes to the singular locus, leading to the so-called limit mixed Hodge structure. Morihiko Saito introduced the theory of Mixed Hodge Modules around 1985, which unifies many different theories: algebraic D-modules and perverse sheaves.

2 Hodge structure

Classically, for a compact orientable Riemann manifold, Hodge theory interprets $H^*_{dR}(X,\mathbb{R})$ in terms of the kernel of the Laplacian Δ_d, i.e., the space of harmonic forms. On a complex manifold, a Hermitian metric is a complex analogue of Riemannian metric. To be precise, a Hermitian metric h is a smoothly varying Hermitian inner product on each holomorphic tangent space. Note that its real part Reh is a usual Riemannian metric. With regard to h, we may define $\overline{\partial}$-Laplacian (resp. ∂-Laplacian) $\Delta_{\overline{\partial}}$ (Δ_∂). One may expect a refinement of the classical Hodge theory in this case. A priori, $\Delta_{\overline{\partial}}$ is not related to Δ_d. The remedy is the introduction of the following definition.

Definition 1 (Kähler manifold)**.** *A Kähler manifold is a complex manifold with a Hermitian metric h such that the 2-form ω defined by $\omega(u,v) = \mathrm{Imh(u,v)}$ is closed. Such a special metric h is called a Kähler metric and ω a Kähler form.*

For a Kähler manifold (X,ω), ω is a symplectic form. Hence we find three mutually compatible structures: a complex structure, a Riemannian structure, and a symplectic structure. Every complex submanifold of a Kähler manifold is again Kähler with the induced metric. The Fubini-Study metric [12, Example 3.1.9] on a projective space $\mathbb{C}P^n$ is Kähler. As a result, a complex projective manifold is Kähler. Any Hermitian metric on a Riemann surface is Kähler, since $d\omega \in A^3(X) = 0$. Thus we find various examples of Kähler manifold.

In the rest of the present section, let (X,w) be a compact Kähler manifold with $\dim_{\mathbb{C}} X = n$ unless otherwise stated. A landmark in Hodge theory is the following theorem.

Theorem 1 (Hodge's decomposition theorem)**.** *[12, Corollary 3.2.12]Let X be a compact Kähler manifold. Define $H^{p,q}(X) = \{[\alpha] \in H^{p+q}_{dR}(X,\mathbb{C}) : \alpha \in A^{p,q}(X), d\alpha = 0\}$ to be the subspace represented by (p,q)-forms. Then*

$$H^m(X,\mathbb{C}) = \oplus_{p+q=m} H^{p,q}(X).$$

Furthermore, we have Hodge symmetry: $\overline{H^{p,q}(X)} = H^{q,p}(X)$.

A key argument in the proof is $\Delta_d = 2\Delta_{\overline{\partial}}$. The Hodge decomposition is actually independent of the choice of the Kähler metric, which can be proved using Bott-Chern cohomology. Hodge decomposition induces a filtration on each $H^k(X,\mathbb{C})$, called the Hodge filtration, by $F^pH^k(X,\mathbb{C}) = \oplus_{r\geq p} H^{r,k-r}(X)$. Hodge decomposition motives the definition of (pure) Hodge structure.

Definition 2 (pure $\mathbb{Q}$-Hodge structure)**.** *Let $n \in \mathbb{N}$. A pure $\mathbb{Q}$-Hodge structure of weight n consists of a finite dimensional $\mathbb{Q}$-vector space V, a finite decreasing filtration F^* on $V_{\mathbb{C}} = \mathbb{C} \otimes_{\mathbb{Q}} V$ such that for any $p \in \mathbb{Z}$, $F^pV \cap \overline{F^{n+1-p}V} = 0$ and $V_{\mathbb{C}} = F^pV \oplus \overline{F^{n+1-p}V}$.*

The existence of such a filtration is equivalent to that of a decomposition: to pass between theses two definitions, given the Hodge filtration F^*V, for $p \in \mathbb{Z}$, define $V^{p,n-p} = F^pV \cap \overline{F^{n-p}V}$, then $V_{\mathbb{C}} = \oplus_{p\in\mathbb{Z}} V^{p,n-p}$. The filtration is preferred as in the relative situation, the filtration varies holomorphically while the bigradings not and we have Proposition 3.

In the definition of Hodge structure, we may replace $\mathbb{Q}$ by other coefficient rings like $\mathbb{Z}, \mathbb{R}$.

Now that Hodge structure on $H^*(X,\mathbb{C})$ is determined by the complex structure of X, a natural question is the converse: Does the Hodge structure on cohomology determines the complex structure? The Global Torelli theorem is said to hold for a particular class of compact complex algebraic or Kähler manifolds if any two manifolds of the given type can be distinguished by their integral Hodge structures.

Theorem 2. *Two complex tori T and T' are biholomorphic if and only if there exists an isomorphism of weight one integral Hodge structures $H^1(T,\mathbb{Z}) \to H^1(T',\mathbb{Z})$.*

Hodge decomposition may fail for a general compact complex manifold, as shown by Example 1. This potential failure is encoded in the following spectral sequence.

Definition 3 (Frölicher spectral sequence)**.** *Let X be a complex manifold. Write A^k for the space of k-forms and $A^{p,q}$ for that of (p,q)-forms. The decreasing filtration $F^pA^k = \oplus_{i\geq p} A^{i,k-i}$ induces a spectral sequence $E_1^{p,q} = H^q(X,\Omega_X^p) \Rightarrow H^{p+q}(X,\mathbb{C})$.*

We call $h^{p,q} = \dim_{\mathbb{C}} H^q(X, \Omega_X^p)$ the Hodge numbers of the complex manifold X. The Hodge numbers of a compact complex manifold are finite.

Definition 4. *For a complex manifold X, the* putative Hodge filtration *on $H^k(X, \mathbb{C})$ is given by*

$$F^p H^k(X, \mathbb{C}) = \mathrm{Im}(\mathbb{H}^k(X, \Omega_X^{\geq p}) \to \mathbb{H}^k(X, \Omega_X^{\star})).$$

Corollary 1. *For a compact Kähler manifold X, its Frölicher spectral sequence degenerates at E_1. For any $p, q \in \mathbb{Z}$, $H^q(X, \Omega_X^p)$ is canonically isomorphic to $H^{p,q}(X)$. The putative Hodge filtration coincides with the actual Hodge filtration.*

Theorem 3. *[13]For a compact complex surface, the Frölicher spectral sequence degenerates at E_1.*

A consequence of the degeneration of the Frölicher spectral sequence is that any holomorphic global form is closed. Iwasawa manifold is a compact complex manifold of (complex) dimension 3 admitting a non-closed holomorphic 1-form. (See [8, Example VI.8.10])

3 Topological constraints

Hodge theory impose strong restriction on the topology of the manifold, as the following results show.

Corollary 2. *The Betti numbers $b_{2k-1}(X)$ are even.*

Corollary 3. *If the fundamental group $\pi_1(X)$ is a free group, then it is trivial.*

Proof. Suppose that $\pi_1(X)$ is a free group on $m (\geq 1)$ generators. We can find a subgroup $H \leq \pi_1(X)$ of index 2. By [15, Theorem 2.10], H is a free group on $2m - 1$ generators. This corresponds to a two-sheeted cover $\pi : Y \to X$ where $\pi_1(Y)$ is isomorphic to H. The space Y with pullback structures is also compact Kähler. However, the betti number $b_1(Y) = 2m - 1$ is odd, which contradicts Corollary 2. □

The first example of compact complex surface with no Kähler metric is the following.

Example 1 (Hopf surface)**.** *Consider the action of* $\mathbb{Z}$ *on* $\mathbb{C}^2 - \{0\}$ *by* $(k, z) \mapsto 2^k z$*. The quotient space* X *is a compact complex manifold, which is diffeomorphic to* $S^1 \times S^3$*. In particular, the Bettic number* $b_1(X) = \dim_{\mathbb{C}} H^1(X, \mathbb{C}) = 1$ *is odd. Therefore,* $H^1(X, \mathbb{C})$ *admits no (strong) Hodge decomposition. By Corollary 2,* X *is not homeomorphic to any Kähler manifold. By Corollary 5,* X *cannot be algebraic. The Dolbeault cohomology group* $H^1(X, O_X) = H^{0,1}_{\overline{\partial}}(X)$ *is nonzero since the global form* $\alpha = \frac{z_1 d\bar{z}_1 + z_2 d\bar{z}_2}{|z_1|^2 + |z_2|^2}$ *is* $\overline{\partial}$*-closed but not* $\overline{\partial}$*-exact. By Theorem 3,* $b_1 = h^{1,0} + h^{0,1}$*, so we obtain the Hodge numbers* $h^{0,1} = 1$ *and* $h^{1,0} = 0$*.*

As Example 1 illustrates, the existence of Hodge decomposition is strictly stronger than the degeneration of Frölicher spectral sequence.

We turn to another aspect of classical Hodge theory. Define the Lefschetz operator $L : H^*_{dR}(X, \mathbb{R}) \to H^{*+2}_{dR}(X, \mathbb{R})$ by $[\eta] \mapsto [\eta \wedge \omega]$.

Theorem 4 (Hard Lefschetz)**.** *For* $0 \leq k \leq n$*,*

$$L^{n-k} : H^k_{dR}(X, \mathbb{R}) \to H^{2n-k}_{dR}(X, \mathbb{R})$$

is an isomorphism. For $k \leq j \leq n$*,*

$$L^{n-j} : H^k_{dR}(X, \mathbb{R}) \to H^{2n-2j+k}_{dR}(X, \mathbb{R})$$

is injective.

Corollary 4. *The even Betti numbers* $b_{2i}(X)$ *are increasing in the range* $2i \leq n$ *and similarly the odd Betti numbers* $b_{2i+1}(X)$ *are increasing in the range* $2i + 1 \leq n$ *.*

For $k \leq n$, define the primitive part of its cohomology as

$$H^k(X, \mathbb{R})_{prim} = \ker[L^{n-k+1} : H^k(X, \mathbb{R}) \to H^{2n-k+2}(X, \mathbb{R})].$$

When $p, q \geq 0$ and $p + q \leq n$, we can similarly define

$$H^{p,q}(X, \mathbb{C})_{prim} = \{[\alpha] \in H^{p,q}(X, \mathbb{C}) : L^{n-p-q+1}\alpha = 0\}.$$

Then Theorem 4 gives an isomorphism $L^{n-p-q} : H^{p,q}(X, \mathbb{C}) \to H^{n-q,n-p}(X, \mathbb{C})$ and another decomposition theorem of the cohomology groups.

Theorem 5 (Lefschetz decomposition theorem)**.**

$$H^{p,q}(X,\mathbb{C}) = H^{p,q}(X,\mathbb{C})_{prim} \oplus L(H^{p-1,q-1}(X,\mathbb{C})) \tag{1}$$

$$H^k(X,\mathbb{C}) = \oplus_{k-2r\leq n} L^r H^{k-2r}(X,\mathbb{C})_{prim} \tag{2}$$

Since the Kähler form $\omega \in A^{1,1}$ is real, the above theorem remains true with real coefficients. We mention a striking example in passing. For a compact simply connected Kähler manifold, its real homotopy type is determined by its cohomology ring (cf.[7]). This is later improved to rational homotopy type in [20].

4 Polarized Hodge structure

Note that any two compact Riemann surfaces of genus g share isomorphic integral Hodge structure $H^1(-,\mathbb{Z})$. We need an extra structure to distinguish them. This motivates the definition of polarization. Again let X be a compact Kähle manifold. For $k \leq n$, define a bilinear form on $H^k(X,\mathbb{R})$ by

$$Q(\alpha,\beta) = \int_X \omega^{n-k} \wedge \alpha \wedge \beta.$$

Then Q is $(-1)^k$-symmetric. This form is called the intersection form.

Definition 5. *A polarized $\mathbb{R}$-Hodge structure of weight k is a $\mathbb{R}$-Hodge structure $(V_{\mathbb{R}}, V^{p,q})$ of weight k, with a $(-1)^k$-symmetric bilinear form $Q : V_{\mathbb{R}} \times V_{\mathbb{R}} \to \mathbb{R}$, such that its extension to $V_{\mathbb{C}}$ satisfies Hodge-Riemann bilibear relations:*

1. *the Hodge decomposition is orthogonal with respect to Q*

2. *for any $\alpha \in H^{p,q}(X,\mathbb{C})$ nonzero,*

$$i^{p-q}(-1)^{k(k-1)/2} Q(\alpha,\bar{\alpha}) > 0.$$

Proposition 1. *$(H^k(X,\mathbb{R})_{prim}, Q)$ is a polarized real Hodge structure of weight k.*

Theorem 6 (Riemann)**.** *Let $L \subseteq \mathbb{C}^n$ be a lattice. Then the complex torus $X = \mathbb{C}^n/L$ is algebraic if and only if the Hodge structure of $H^1(X,\mathbb{Z})$ admits a polarization.*

With polarized Hodge structure we can state the following two Torelli type results.

Theorem 7. *Two compact Riemann surfaces M and N are isomorphic if and only if there exists an isomorphism of weight one Hodge structures $H^1(M,\mathbb{Z}) \to H^1(N,\mathbb{Z})$ that respects the intersection pairing.*

Definition 6 ((analytic) K3 surface)**.** *A compact complex surface X with trivial canonical bundle and $h^{0,1}(X) = 0$ is called a K3 surface.*

Every K3 surface is Kähler (cf.[18]).

Theorem 8. *Two complex K3 surfaces S and S' are isomorphic if and only if there exists an isomorphism of Hodge structures $H^2(S,\mathbb{Z}) \to H^2(S',\mathbb{Z})$ respecting the intersection pairing.*

5 Mixed Hodge structure

Given a field k, by k-variety we mean a finite type, separated k-scheme. Recall Chow's theorem: a projective manifold is algebraic. We are thus led to the following question: can Hodge theory be extended to complex algebraic varieties?

For a general complex variety X, its analytification X^{an} may not be compact nor smooth. For example, consider the punctured line $X = \mathrm{Spec}\mathbb{C}[\mathrm{t},\mathrm{t}^{-1}]$, then $X^{an} = \mathbb{C}^*$ and $H^1(X^{an},\mathbb{C}) = \mathbb{C}$ is of odd dimension. Here X is smooth but not proper. On the other hand, take C to be the plane projective curve $Y^2Z = X^2(X-Z)$. Then still $\dim_{\mathbb{C}} H^1(C,\mathbb{C}) = 1$. In this case, C has a nodal singularity. The cohomology groups cannot be expected to have a Hodge decomposition in both cases. We are forced to seek a weaker notion than Hodge structure.

The heuristic evidence for the existence of such a weaker notion comes from the properties of étale cohomology of varieties over fields with positive characteristic. A dictionary between l-adic cohomology and Hodge theory is in [4]. Deligne was also the first to give an affirmative answer to the question. More precisely, he established the existence of mixed Hodge structure on the cohomology of complex algebraic varieties.

Definition 7 (mixed Hodge structure)**.** *Let H be a finite $\mathbb{Z}$-module. A mixed Hodge structure on H consists of*

1. *an increasing filtration* W_* *on* $H_{\mathbb{Q}} = \mathbb{Q} \otimes_{\mathbb{Z}} H$*, called the* weight filtration*;*

2. *a decreasing filtration* F^* *on* $H_{\mathbb{C}} = \mathbb{C} \otimes_{\mathbb{Z}} H$*, called the Hodge filtration*

with the property that F^* *induces a pure* $\mathbb{Q}$*-Hodge structure of weight* k *on the graded piece*

$$Gr_k^W(H_{\mathbb{Q}}) = W_k/W_{k-1}.$$

Theorem 9. *[5, Theorem 3.2.5] Let* $X/\mathbb{C}$ *be a smooth algebraic variety. Then* $H^k(X, \mathbb{C})$ *has a canonical mixed Hodge structure. This structure is fuctorial. If* X *is further more complete (proper), then this structure is pure of weight* k*.*

Sketch of the proof of Hodge II: a smooth variety $X/\mathbb{C}$ has a smooth compactification $j : X \to \bar{X}$ such that $D = \bar{X} - X$ is a divisor of normal crossing by Nagata's theorem [19, Tag 0F41] and Hironaka's resolution of singularity. Then Deligne used differential forms with logarithmic singularities along the boundary and the residue maps between them to show Theorem 9. In Hodge III [6], the result is generalized to proper varieties and finally to any complex algebraic variety. In the singular case, varieties are replaced by simplicial schemes, leading to more complicated homological algebra. Using the theory of motives, it is possible to refine the weight filtration on the cohomology with rational coefficients to one with integral coefficients.

Remark 1. *The weight filtration on* $H^k(X)$ *satisfies the following properties:*

1. $W_{-1} = 0$ *and* $W_{2k} = H^k$*.*

2. *If* $X/\mathbb{C}$ *is proper, then* $W_k = H^k(X)$ *and for any resolution of singularity* $\tilde{X} \to X$*, we have* $W_{k-1}H^k(X) = \ker(H^k(X) \to H^k(\tilde{X})$*.*

3. *If* $X/\mathbb{C}$ *is smooth, then* $W_{k-1}H^k(X) = 0$ *and for any smooth compactification* $i : X \to \bar{X}$*, then* $i^*H^k(\bar{X}) = W_k H^k(X)$*.*

In particular, when $X/\mathbb{C}$ *is a smooth projective variety, the weight filtration is trivial* $0 = W_{k-1} \subset W^k = H^k(X)$*, that is, we recover the pure Hodge structure given by Theorem 1.*

Corollary 5. *Let* $X/\mathbb{C}$ *be a proper smooth variety, then* X *admits a strong Hodge decomposition.*

Example 2. *[9, Section 4] Let* $X_1, X_2 \subset \mathbb{C}P^N$ *be two complex submanifolds intersecting transversally. Let* $X = X_1 \cup X_2$*. The Mayer-Vietoris sequence is*

$$\to H^{m-1}(X_1 \cap X_2) \xrightarrow{\delta} H^m(X) \to H^m(X_1) \oplus H^m(X_2) \to H^m(X_1 \cap X_2) \to$$

The weight filtration of $H^m(X)$ *is* $W_{m-2} = 0$*,* $W_{m-1} = \mathrm{Im}\delta$ *and* $W_m = H^m(X)$*. (Note that the category of mixed Hodge structures is abelian and any morphism is strict. See [16, Corollary 3.6])*

Example 3. *For the projective curve* $C : Y^2Z = X^2(X - Z)$*, then* $0 = W_{-1} \subset W_0 H^1(C) = H^1(C)$*.*

Example 4. $X = A^1_{\mathbb{C}} - 0$*, then* $0 = W_1 H^1 \subset W_2 H^1 = H^1$

We present one application.

Theorem 10 (Weight principle)**.** *Let* $Z \subset U \subset X$ *be inclusions, where* $X/\mathbb{C}$ *is a proper smooth variety,* $U \subset X$ *is a Zariski dense open and* $Z \subset X$ *is a closed subvariety, for each* l*,*

$$H^l(X, Q) \xrightarrow{a} H^l(U, Q) \xrightarrow{b} H^l(Z, Q)$$

have $\mathrm{Im}(ba) = \mathrm{Im}(b)$

Example 5 shows that the real counterpart is false.

Example 5. *Let* $X = \mathbb{C}P^1$*,* $U = \mathbb{C}^*$*,* $Z = S^1$*, then* $H^1(X, \mathbb{Q}) \to H^1(Z, \mathbb{Q})$ *has trivial image while* $H^1(U, \mathbb{Q}) \to H^1(Z, \mathbb{Q})$ *is an isomorphism.*

6 Variation of Hodge structure

We turn to a relative version of Hodge theory.

Let $\pi : \mathcal{X} \to S$ be a smooth proper analytic morphism of relative dimension n, such that the Frölicher spectral sequence degenerates at each fiber X_s. (This happens if all the fibers admit a Kähler metric. Another example is that $\pi : \mathcal{X} \to S$ is algebraic, then the degeneration is guaranteed by [3, Theorem 5.5].) We can ask how the Hodge structures (if exist) on $H^*(X_s, \mathbb{C})$ vary. The variation of Hodge structures is closely related to monodromy action and has important application in arithmetic.

Proposition 2. *The Hodge numbers of X_s are locally constant.*

Corollary 6. $R^i\pi_*\Omega^{\geq k}_{\mathcal{X}/S}$ *is a holomorphic vector bundle on S for all i and k.*

Define the i-th relative de Rham cohomology to be the O_S-module $R^i\pi_*\Omega^{\cdot}_{\mathcal{X}/S}$, denoted by $\mathcal{H}^i_{dR}(\mathcal{X}/S)$. A relative version of Definition 4 is

$$F^p\mathcal{H}^i_{dR}(\mathcal{X}/S) = \mathrm{Im}(R^i\pi_*\Omega^{\geq p}_{\mathcal{X}/S} \to \mathcal{H}^i_{dR}(\mathcal{X}/S)).$$

Consider the local system of complex vector spaces $R^i\pi_*\underline{\mathbb{C}}$. We have $\mathcal{H}^i_{dR}(\mathcal{X}/S) = O_S \otimes R^i\pi_*\underline{\mathbb{C}}$, which induces a flat connection $\nabla : \mathcal{H}^i(\mathcal{X}/S) \to \mathcal{H}^i(\mathcal{X}/S) \otimes \Omega^1_S$ by Riemann-Hilbert correspondence (cf.[16, Corollary 10.4]). This connection is known as Gauss-Manin connection. In terms of parallel transport, we may identify nearby fibers of $\mathcal{H}^i(\mathcal{X}/S)$.

Proposition 3 (Griffiths' transversality)**.** *[16, Corollary 10.31]*

$$\nabla(F^p\mathcal{H}^i_{dR}(\mathcal{X}/S)) \subset F^{p-1}\mathcal{H}^i_{dR}(\mathcal{X}/S) \otimes \Omega^1_S.$$

The properties above lead to an abstract definition.

Definition 8 (variation of Hodge structure)**.** *Let S be a complex manifold. A variation of Hodge structure of weight k on S consists of the following data:*

1. *a local system $\mathbb{V}$ of finite dimensional $\mathbb{Q}$-vector spaces on S;*
2. *the Hodge filtration: a finite decreasing filtration F^* of the holomorphic vector bundle $\mathcal{V} = \mathbb{V} \otimes_{\mathbb{Q}} O_S$ by holomorphic subbundles.*

These data must satisfy the following conditions:

- *for any $s \in S$, $F^*(s)$ of $\mathbb{V}_s \otimes_{\mathbb{Q}} \mathbb{C}$ defines a pure Hodge structure of weight k on $\mathbb{V}_s$.*
- *The induced connection satisfies the Griffith transversality: $\nabla(F^p) \subset F^{p-1} \otimes \Omega^1_S$*

We

Proposition 4. *[16, Proposition 1.38]For a smooth projective morphism $\pi : \mathcal{X} \to S$, the restriction maps*

$$H^m(\mathcal{X}, \mathbb{Q}) \to H^0(S, R^m\pi_*\mathbb{Q})$$

are surjective.

The right hand side is exactly the invariants under the monodromy action. Therefore, the proposition has the interpretationt: for any $s \in S$, an invariant of $H^m(X_S, \mathbb{Q})$ come from a global class.

In Hodge II, Deligne considered variation of polarized Hodge structure, and showed the semi-simplicity of monodromy representation for a wide class of morphisms. The underlying variation of Hodge structure of a local system can be used to show that the monodromy group is big in suitable sense. For an example, see [16, Theorem 10.22].

Theorem 11. *[17, Theorem 7.22] Let H be a complex polarized variation of Hodge structure over a quasi-projective base S. Let e be a global flat section of H, and write $e = \sum e^{p,q}$ for $e^{p,q} \in H^{p,q}$, then each $e^{p,q}$ is again flat.*

The variation of Hodge structure is encoded in the period mapping, introduced by Griffiths, which can be defined using the Gauss-Manin connection introduced above. (Consult [1] for details.) The interaction of period mappings and monodromy finds application in number theory, for example, the proof of Mordell conjecture given in [14].

7 Open problems

Let X be a projective manifold with $\dim_{\mathbb{C}} X = n$. Define the Hodge class of $H^{2k}(X)$ to be $Hdg^{2k}(X, \mathbb{Q}) = H^{2k}(X, \mathbb{Q}) \cap H^{k,k}(X)$. If $j : C \to X$ is a complex submanifold of codimension k, consider $j_* : H_{2n-2k}(C, \mathbb{Z}) \to H_{2n-2k}(X, \mathbb{Z})$ and Poincaré duality $H_{2n-2k}(X, \mathbb{Z}) \to H^{2k}(X, \mathbb{Z})$. The fundamental class of C is mapped to $[C] \in H^{2k}(X, \mathbb{Z})$. By passing to desingulariztion, for any algebraic subvariety C (not necessarily smooth), we may also define $[C] \in H^{2k}(X, \mathbb{Z})$. In fact, $[C] \in Hdg^{k,k}(X, \mathbb{Z})$. This construction extends to a cycle class map starting from the Chow group

$$cl : CH^k(X) \to H^{2k}(X, \mathbb{Z}). \tag{3}$$

For a complex variety Y, let $Z^k(Y)$ be the group of cycles of codimension k on Y.

Conjecture 1 (Hodge conjecture). *Let X be a projective complex manifold, is $Hdg^k(X, \mathbb{Q}) = \{[Z]_B : Z \in Z^k(X) \otimes_{\mathbb{Z}} \mathbb{Q}\}$? Or equivalently, the cycle class map $cl : CH^k(X)_{\mathbb{Q}} \to Hdg^{2k}(X, \mathbb{Q})$ is it surjective?*

Informally, the conjecture is that every Hodge class is algebraic. The theory of Mixed Hodge structures was used by Cattani, Deligne and Kaplan to prove an algebraicity theorem that provides strong evidence for the Hodge conjecture (cf.[2]). Hodge conjecture is related to generalized Bloch conjecture, cf.[21].

Conjecture 2 (generalized Bloch conjecture). *If the Hodge numbers $h^{p,q} = 0$ for $p \neq q$ and $p < c$ or $q < c$, then for any $i < c - 1$, then cycle class map*

$$cl : CH_i(X) \otimes \mathbb{Q} \to H^{2n-2i}(X, \mathbb{Q})$$

is injective

Complex subvarieties are rather rigid, making it difficult to construct them.

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