

Introduction au domaine de recherche

Keyu WANG

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Chapter 1

Introduction

This introductory report is a review of what I have learnt during the passed three years at ENS Paris and what I am planning to study in the following years.

During the first year, I finished a math-physics double license. Under the supervision of professor Denis Bernard and professor David Hernandez, I completed an internship about the topic 'TQ relations in quantum affine algebras'. The so-called TQ relations are certain relations in some physical solvable models, for example the six vertex model. I learned the expression of these relations in terms of representations of quantum affine algebras, and also how they are corresponded to certain second order ordinary differential equations. This unexpected correspondence is called the 'ODE/IM correspondence'. In the part of physics, I learned the formulation of Knizhnik-Zamolodchikov equations in Wess-Zumino-Witten models in the conformal field theory, which are compactly related to the six vertex model and also to the quantum affine algebras.

During the second year, I studied abroad at ETH Zurich in Switzerland for one semester. There I was under the supervision of professor Giovanni Felder. I learned Nakajima varieties, the geometric representation theory of quantization of Lie algebras, and the formulation of Knizhnik-Zamolodchikov equations in the geometric representation theory. This is a continue of the topic of my first year's internship. The main work is about solving elliptic KZ equations. However, the main theorem remains a conjecture till now. I will not talk about these works in this review because of page limit and because my PhD research is more closely related to the other two aspects above and below.

Last year was my final year at ENS, I finished my master mémoire and defended it at Paris Diderot. My supervisor is professor David Hernandez, who guided me for the license internship and is now my PhD adviser. The master mémoire titled as 'On the Bethe Ansatz conjecture for Gaudin model'. The Bethe Ansatz is a method of diagonalizing Gaudin models in physics. This method is formulated in pure mathematics as the problem of studying the spectrum of Gaudin algebras. We can describe their spectrum by a geometric object called opers. This mémoire is related to my first year's internship in the following sense. The transfer matrices in the six vertex model may be viewed as a quantization of the Gaudin Hamiltonians related to the affinization of the simplest Lie algebra \mathfrak{sl}_2 . The Bethe Ansatz method is also applied to the six vertex model. And the geometric object opers will provide the differential operators as in the ODE/IM correspondence. For example the Schrodinger operators corresponding the six vertex model can be derived from $\hat{\mathfrak{sl}}_2$ -opers. This mémoire is also related to my second year's project, as the Gaudin Hamiltonians are explicitly appeared in KZ equations. Also Bethe Ansatz provides solutions to KZ equations. And it is natural to consider this relation in their quantum analogue.

This review is organized as follows.

In chapter 2, we review the definitions and basic properties of affine Lie algebras and their quantization. Then we formulate the physical models we concern in chapter 3. And we recall the physical methods of solving these models. In chapter 4, we formulate the TQ relations in the six vertex model in mathematical language of representations of quantum affine \mathfrak{sl}_2 . And in general these relations can be formulated in the Grothendieck ring of representations of quantum affine algebras. We also see the similar relations in the theory of ordinary differential equations. In the last chapter, we will see how Bethe Ansatz equations are formulated in mathematics and how they are related to opers. The relations of the concepts in this review can be read from the table below.

Dictionary	
Classical	Quantum
$\hat{\mathfrak{g}}$ Gaudin model Gaudin Hamiltonians Bethe Ansatz \mathfrak{g} -opers Affine Harish-Chandra homomorphism Characters characters \xrightarrow{Spec} Cartan connections affine H-C homom. \uparrow Gaudin algebras \xrightarrow{Spec} opers Hitchin systems $\xleftrightarrow{Langlands}$ opers ...	$U_q(\hat{\mathfrak{g}})$ Six vertex model Transfer matrices Bethe Ansatz $\hat{\mathfrak{sl}}_2$ -opers Quantum affine Harish-Chandra homomorphism q -characters q-characters \longrightarrow Schrodinger operators q-affine H-C homom. \uparrow transfer matrices \longrightarrow affine opers quantum Hitchin systems $\xleftrightarrow{q\text{-Langlands}}$ affine opers ...
\downarrow Miura trans.	\downarrow Miura. trans.

Chapter 2

Algebraic concepts

2.1 Lie algebras

Let \mathfrak{g} be a finite-dimensional semisimple Lie algebra over \mathbb{C} with a Cartan subalgebra \mathfrak{h} . The Killing form on \mathfrak{g} restricting to \mathfrak{h} allows us to identify \mathfrak{h} and \mathfrak{h}^* and thus gives us a bilinear form $\langle \cdot, \cdot \rangle$ on \mathfrak{h}^* . Let $\Delta = \{\alpha_1, \dots, \alpha_l\}$ be a set of simple roots, the matrix $C_{ij} = \langle \alpha_i, \alpha_j \rangle$ is called the Cartan matrix of \mathfrak{g} .

Recall that the Cartan matrices classify simple Lie algebras and we can construct simple Lie algebras from Cartan matrices in the following way.

Theorem 2.1.1. *Let C_{ij} be a Cartan matrix in the classification table [Kac90, Section 4.8]. Define a Lie algebra \mathfrak{g} with generators $\{e_i, f_i, h_i | 1 \leq i \leq l\}$, satisfying:*

$$\begin{aligned} [h_i, h_j] &= 0, 1 \leq i, j \leq l, \\ [e_i, f_i] &= h_i, [e_i, f_j] = 0, i \neq j, \\ [h_i, e_j] &= C_{ij}e_j, [h_i, f_j] = -C_{ij}f_j, \\ (ade_i)^{-C_{ij}+1}(e_j) &= 0, i \neq j, \\ (adf_i)^{-C_{ij}+1}(f_j) &= 0, i \neq j. \end{aligned} \tag{2.1.1}$$

Then \mathfrak{g} is a finite-dimensional semisimple algebra, with a Cartan subalgebra \mathfrak{h} spanned by h_i , whose Cartan matrix is C_{ij} .

This construction is generalized to construct a family of infinite-dimensional Lie algebras, called Kac-Moody algebras, by V. Kac and R. Moody. The Cartan matrices will be replaced by generalized Cartan matrices.

Definition 2.1.2. An $n \times n$ matrix $C = (a_{ij})$ is called a generalized Cartan matrix of rank l if it satisfies

$$\begin{aligned} a_{ii} &= 2, \\ a_{ij} &\in \mathbb{Z}, \\ a_{ij} &\leq 0, i \neq j, \\ a_{ij} = 0 &\text{ implies } a_{ji} = 0. \end{aligned} \tag{2.1.2}$$

Definition 2.1.3. Given a generalized Cartan matrix $C = (a_{ij})_{n \times n}$ of rank l , we can define a Lie algebra $\mathfrak{g}(C)$ as follows: Let \mathfrak{h} be a complex vector space of dimension $2n - l$. Let $\{\alpha_1, \dots, \alpha_n\} \subset \mathfrak{h}^*$ be a linearly independent set, called the set of simple roots. Then there is a linearly independent set $\{\alpha_1^\vee, \dots, \alpha_n^\vee\}$ in \mathfrak{h} such that $\alpha_i(\alpha_j^\vee) = a_{ji}$. Define $\mathfrak{g}(C)$ to be the Lie algebra generated by $\{e_i, f_i, h | 1 \leq i \leq n, h \in \mathfrak{h}\}$

$$\begin{aligned} [h, h'] &= 0, h, h' \in \mathfrak{h}, \\ [e_i, f_i] &= \alpha_i^\vee, [e_i, f_j] = 0, i \neq j, \\ [h, e_i] &= \alpha_i(h)e_i, [h, f_i] = -\alpha_i(h)f_i, h \in \mathfrak{h}, \\ (ade_i)^{1-a_{ij}}(e_j) &= 0, i \neq j, \\ (adf_i)^{1-a_{ij}}(f_j) &= 0, i \neq j. \end{aligned} \tag{2.1.3}$$

In this review, we are interested in the untwisted affine Lie algebras, which are Kac-Moody algebras with generalized Cartan matrices of affine type from the table Aff1 in [Kac90, Section 4.8]. Note that the untwisted affine Lie algebras may be constructed by central extensions of loop algebras $\mathfrak{g}(C) = \mathfrak{g} \otimes \mathbb{C}((t)) \oplus \mathbb{C}K \oplus \mathbb{C}d$. Here \mathfrak{g} is the related simple Lie algebra. We will focus on the subalgebra

$$\hat{\mathfrak{g}} = \mathfrak{g} \otimes \mathbb{C}((t)) \oplus \mathbb{C}K,$$

defined by relations

$$[A \otimes f(t), B \otimes g(t)] = [A, B] \otimes f(t)g(t) + (A, B)\text{Res}_{t=0}fdgK, \quad \forall A, B \in \mathfrak{g}, \quad (2.1.4)$$

K is a central element.

2.2 Quantum universal enveloping algebras

Quantum groups first appeared in the theory of quantum integrable systems. V. Drinfeld and M. Jimbo formulate them as deformation of the universal enveloping algebras of semisimple Lie algebras.

Definition 2.2.1. Recall that a coalgebra over \mathbb{C} is a vector space A over \mathbb{C} together with linear maps comultiplication $\Delta : A \rightarrow A \otimes A$, and counit $\epsilon : A \rightarrow \mathbb{C}$, satisfying:

$$\begin{aligned} (id \otimes \Delta) \circ \Delta &= (\Delta \otimes id) \circ \Delta, \\ (id \otimes \epsilon) \circ \Delta &= id = (\epsilon \otimes id) \circ \Delta. \end{aligned}$$

A bialgebra is an algebra (A, m, ι) which is also a coalgebra (A, Δ, ϵ) , such that $\Delta : A \rightarrow A \otimes A$ and $\epsilon : A \rightarrow \mathbb{C}$ are homomorphisms of algebras.

A Hopf algebra is a bialgebra equipped with a linear invertible map $\gamma : A \rightarrow A$, called the antipode, satisfying:

$$m \circ (id \otimes \gamma) \circ \Delta = m \circ (\gamma \otimes id) \circ \Delta = \iota \circ \epsilon.$$

Example 2.2.2. The universal enveloping algebras of Lie algebras $\mathcal{U}(\mathfrak{g})$ admit the structure of Hopf algebras: $m(x, y) = xy$; $\Delta(1) = 1 \otimes 1$, $\Delta(x) = x \otimes 1 + 1 \otimes x$ for $x \in \mathfrak{g}$ determine Δ on $\mathcal{U}(\mathfrak{g})$; $\iota(1) = 1$; $\epsilon(1) = 1$, $\epsilon(x) = 0$ for $x \in \mathfrak{g}$ determine ϵ ; $\gamma(1) = 1$, $\gamma(x) = -x$ for $x \in \mathfrak{g}$ determine γ .

Definition 2.2.3. Let \mathfrak{g} be a Kac-Moody algebra, $C = (a_{ij})$ be its generalized Cartan matrix, and $d_i = \frac{(\alpha_i, \alpha_i)}{2}$. Suppose $q \in \mathbb{C}^*$ is not a root of unity. The quantum universal enveloping algebra $\mathcal{U}_q(\mathfrak{g})$ is generated by the elements $\{e_i, f_i, q^h | 1 \leq i \leq r, h \in \mathfrak{h}\}$ satisfying:

$$\begin{aligned} q^0 &= 1, q^{a+b} = q^a q^b, a, b \in \mathfrak{h}, \\ q^h e_i q^{-h} &= q^{\alpha_i(h)} e_i, \\ q^h f_i q^{-h} &= q^{-\alpha_i(h)} f_i, \\ e_i f_j - f_j e_i &= \delta_{ij} \frac{q^{d_i h_i} - q^{-d_i h_i}}{q^{d_i} - q^{-d_i}}, \\ \sum_{n=0}^{1-a_{ij}} \frac{(-1)^n}{[n]_i! [1-a_{ij}-n]_i!} e_i^n e_j e_i^{1-a_{ij}-n} &= 0, \\ \sum_{n=0}^{1-a_{ij}} \frac{(-1)^n}{[n]_i! [1-a_{ij}-n]_i!} f_i^n f_j f_i^{1-a_{ij}-n} &= 0, \end{aligned}$$

where h_i such that $\alpha_j(h_i) = a_{ij}$, $[n]_i = \frac{q^{n d_i} - q^{-n d_i}}{q^{d_i} - q^{-d_i}}$, $[n]_i! = \prod_{p=1}^n [p]_i$.

Remark 2.2.4. In this review, we are only interested in the case when \mathfrak{g} is finite-dimensional or of untwisted affine type.

We will use the notion of quantum loop algebras which is the subalgebra of $\mathcal{U}_q(\mathfrak{g})$ generated by $\{e_i, f_i, q^{h_i} | 1 \leq i \leq r\}$. When the generalized Cartan matrix is of untwisted affine type, we use the notation $\mathcal{U}_q(\hat{\mathfrak{g}})$ for the quantum loop algebra, where now \mathfrak{g} denotes the related finite-dimensional simple Lie algebra.

Example 2.2.5. $\mathcal{U}_q(\mathfrak{g})$ are Hopf algebras with operations, for example,

$$\begin{aligned} \Delta(q^h) &= q^h \otimes q^h, \Delta(e_i) = e_i \otimes q^{d_i h_i} + 1 \otimes e_i, \Delta(f_i) = f_i \otimes 1 + q^{-d_i h_i} \otimes f_i, \\ \epsilon(q^h) &= 1, \epsilon(e_i) = \epsilon(f_i) = 0, \\ \gamma(q^h) &= q^{-h}, \gamma(e_i) = -e_i q^{-d_i h_i}, \gamma(f_i) = -q^{d_i h_i} f_i. \end{aligned}$$

It is known, due to V. Chari and A. Pressley, that finite-dimensional irreducible representations of $\mathcal{U}_q(\hat{\mathfrak{g}})$ are classified by highest l -weight representations. They are coded by all l -tuples of polynomials with constant term 1. We will state this theorem in section 4.2. The simplest example is called fundamental representations, where the l -tuple of polynomials is $(1, \dots, 1, 1 - \lambda z, 1, \dots, 1)$. In the case of $\mathcal{U}_q(\hat{\mathfrak{sl}}_2)$, the fundamental representations can be obtained by evaluation maps from $\mathcal{U}_q(\hat{\mathfrak{sl}}_2)$ to $\mathcal{U}_q(\mathfrak{sl}_2)$.

Definition 2.2.6. For any $\lambda \in \mathbb{C}^*$, we define the evaluation map $ev_\lambda : U_q(\hat{\mathfrak{sl}}_2) \rightarrow \mathcal{U}_q(\mathfrak{sl}_2)$ by

$$\begin{aligned} ev_\lambda(e_0) &= \lambda f, \quad ev_\lambda(f_0) = \lambda^{-1}e, \\ ev_\lambda(e_1) &= e, \quad ev_\lambda(f_1) = f, \\ ev_\lambda(q^{h_0}) &= q^{-h}, \quad ev_\lambda(q^{h_1}) = q^h. \end{aligned}$$

It is a homomorphism of algebras.

Definition 2.2.7. For any integer $n \geq 0$ and any $\lambda \in \mathbb{C}^*$, the evaluation representation $V_n(\lambda)$ of $U_q(\hat{\mathfrak{sl}}_2)$ is the pull-back of the representation π_n of $\mathcal{U}_q(\mathfrak{sl}_2)$ by the evaluation map ev_λ . We will denote it by $\pi_n(\lambda) = \pi_n \circ ev_\lambda : U_q(\hat{\mathfrak{sl}}_2) \rightarrow End(V_n(\lambda))$. Here π_n is the unique n dimensional highest weight representation of $\mathcal{U}_q(\mathfrak{sl}_2)$.

Definition 2.2.8. A Hopf algebra A is called quasi-triangular, if there is an invertible element $\mathcal{R} \in A$ such that

$$\begin{aligned} \mathcal{R}\Delta(x) &= T \circ \Delta(x)\mathcal{R} \\ (\Delta \otimes id)(\mathcal{R}) &= \mathcal{R}_{13}\mathcal{R}_{23} \\ (id \otimes \Delta)(\mathcal{R}) &= \mathcal{R}_{13}\mathcal{R}_{12}. \end{aligned}$$

where $T : A \otimes A \rightarrow A \otimes A, x \otimes y \mapsto y \otimes x$ and $\mathcal{R}_{13} = \phi_{13}(\mathcal{R})$ with $\phi_{13} : A \otimes A \rightarrow A \otimes A \otimes A, x \otimes y \mapsto x \otimes 1 \otimes y$, and $\mathcal{R}_{12}, \mathcal{R}_{23}$ are defined similarly. \mathcal{R} is called a universal R-matrix.

Theorem 2.2.9. A universal R-matrix \mathcal{R} satisfies the Yang-Baxter equation on $A \otimes A \otimes A$:

$$\mathcal{R}_{12}\mathcal{R}_{13}\mathcal{R}_{23} = \mathcal{R}_{23}\mathcal{R}_{13}\mathcal{R}_{12}. \tag{2.2.1}$$

Theorem 2.2.10. The quantum affine algebras $U_q(\hat{\mathfrak{g}})$ are quasi-triangular Hopf algebras.

Chapter 3

Integrable models

3.1 Six vertex model

There exist in nature numerous crystals with hydrogen bonding. The most familiar example is the ice, where the oxygen atoms form a lattice of coordination number four, and between each adjacent pair of atoms is an hydrogen ion. Each ion is located near one or the other end of the bond in which it lies. By the local electric neutrality, the ions should satisfy the ice rule: among the four ions surrounding each atom, there are exactly two of them are close to it, and the other two away from it. Therefore, there are six different ways of arranging the arrows. For this reason it is called a six-vertex model. Each of these six local arrangements will be associated a distinct energy.

The six vertex model is exactly solvable, which means its partition function can be calculated explicitly. We will introduce two methods to solve this model, namely the Bethe Ansatz and the TQ relations proposed by R. Baxter.

3.1.1 Definitions

Six vertex model is a $N \times P$ planar lattice model with P rows and N columns, where N and P are even numbers. An arrow is placed on each edge. The ice rule says among the four edges around each vertex of the lattice, there are two arrows pointing in, and two arrows pointing out. We impose cyclic boundary conditions on every rows and columns, which means the first arrow on each row or column has the direction as the last arrow on that row or column. Such an allowed lattice with arrows is called a state of the six vertex model.

Then the canonical partition function is defined by

$$Z = \sum_{s:\text{states}} e^{-\beta E_s} = \sum_{\text{states}} \prod_{\text{vertices}} W_{\alpha\beta}^{\alpha'\beta'}.$$

The Boltzmann weights $W_{\alpha\beta}^{\alpha'\beta'}$ on each vertex, depending on the direction of the 4 arrows around it, is defined by

$$\begin{aligned} W_{\rightarrow\rightarrow}^{\rightarrow\rightarrow}(\nu) &= W_{\leftarrow\leftarrow}^{\leftarrow\leftarrow}(\nu) = a(\nu, \eta) = \sin(\eta + i\nu), \\ W_{\leftarrow\rightarrow}^{\leftarrow\rightarrow}(\nu) &= W_{\rightarrow\leftarrow}^{\rightarrow\leftarrow}(\nu) = b(\nu, \eta) = \sin(\eta - i\nu), \\ W_{\leftarrow\leftarrow}^{\leftarrow\leftarrow}(\nu) &= W_{\rightarrow\rightarrow}^{\rightarrow\rightarrow}(\nu) = c(\nu, \eta) = \sin(2\eta). \end{aligned} \tag{3.1.1}$$

Here the index α and α' are the arrows under and above the vertex on the column, which are rotated clockwise by $\pi/2$ for convenience, and the the index β and β' are the arrows on the left and right of the vertex on the row.

Remark 3.1.1. The fact that the Boltzmann weights in this model are trigonometric functions corresponds to the fact that the underlying algebraic structure of this model is a quantum affine algebra. We also have other solvable models, for example XXX models (resp. XYZ models), where the parameters are rational functions (resp. elliptic functions). And these models will have underlying algebraic structure of Yangians (resp. elliptic quantum groups), which are other ways of quantization of classical Lie algebras.

Definition 3.1.2. The transfer matrix is defined by

$$\mathbb{T}_{\bar{\alpha}}^{\bar{\alpha}'} = \sum_{\{\beta_i\}} W_{\alpha_1\beta_1}^{\alpha'_1\beta_2} W_{\alpha_2\beta_2}^{\alpha'_2\beta_3} \dots W_{\alpha_N\beta_N}^{\alpha'_N\beta_1},$$

where the sum is taken for all sequence $\{\beta_i\}$ complying the ice rule. Here $\bar{\alpha} = (\alpha_1, \dots, \alpha_N)$ and so on.

Then the partition function can be written as

$$Z = \sum_{\text{states}} \prod_{\text{vertices}} W_{\alpha\beta}^{\alpha'\beta'} = \sum_{\{\bar{\alpha}_i\}} \prod_{i=1}^P \mathbb{T}_{\bar{\alpha}_i}^{\bar{\alpha}_{i+1}} = \text{Tr}(\mathbb{T}^P). \quad (3.1.2)$$

Therefore, to solve the model, it is enough to compute the eigenvalues of \mathbb{T} .

The monodromy matrix is defined by

$$\mathcal{T}_{\bar{\alpha}}^{\bar{\alpha}'}(\nu)_a^b = \sum_{\{\beta_i\}} W_{\alpha_1 a}^{\alpha_1' \beta_1}(\nu) W_{\alpha_2 \beta_2}^{\alpha_2' \beta_2}(\nu) \cdots W_{\alpha_N \beta_N}^{\alpha_N' b}(\nu), \quad (3.1.3)$$

where we relax the restriction of cyclicity.

The elements of $\mathcal{T}_{\bar{\alpha}}^{\bar{\alpha}'}(\nu)$ are

$$\mathcal{T}_{\bar{\alpha}}^{\bar{\alpha}'}(\nu) = \begin{pmatrix} \mathcal{T}_{\bar{\alpha}}^{\bar{\alpha}'}(\nu)_{\rightarrow \rightarrow} & \mathcal{T}_{\bar{\alpha}}^{\bar{\alpha}'}(\nu)_{\rightarrow \leftarrow} \\ \mathcal{T}_{\bar{\alpha}}^{\bar{\alpha}'}(\nu)_{\leftarrow \rightarrow} & \mathcal{T}_{\bar{\alpha}}^{\bar{\alpha}'}(\nu)_{\leftarrow \leftarrow} \end{pmatrix} = \begin{pmatrix} A(\nu) & B(\nu) \\ C(\nu) & D(\nu) \end{pmatrix}.$$

Then $\mathbb{T}_{\bar{\alpha}}^{\bar{\alpha}'}(\nu) = \mathcal{T}_{\bar{\alpha}}^{\bar{\alpha}'}(\nu)_{\rightarrow \rightarrow} + \mathcal{T}_{\bar{\alpha}}^{\bar{\alpha}'}(\nu)_{\leftarrow \leftarrow} = A(\nu) + D(\nu)$.

Definition 3.1.3. Define the R matrix by $R_{\alpha\beta}^{\alpha'\beta'}(\nu) = W_{\alpha\beta}^{\alpha'\beta'}(\nu - i\eta)$.

One can calculate the R matrix explicitly under the ordered base $\{\rightarrow\rightarrow, \rightarrow\leftarrow, \leftarrow\rightarrow, \leftarrow\leftarrow\}$.

$$R(\nu) = \begin{pmatrix} \sin(2\eta + i\nu) & 0 & 0 & 0 \\ 0 & -\sin(i\nu) & \sin(2\eta) & 0 \\ 0 & \sin(2\eta) & -\sin(i\nu) & 0 \\ 0 & 0 & 0 & \sin(2\eta + i\nu) \end{pmatrix} \quad (3.1.4)$$

Under Einstein's summation convention, which an index appears in both superscripts and subscripts of a term, it implies a summation of that index over the index set. Here the index set is $\{\rightarrow, \leftarrow\}$.

Theorem 3.1.4. R-matrices defined above satisfy the Yang-Baxter relation.

$$R_{aa'}^{c'c}(\nu - \nu') R_{\alpha c}^{\delta b}(\nu + i\eta) R_{\delta c'}^{\alpha' b'}(\nu' + i\eta) = R_{\alpha\alpha'}^{\delta c}(\nu' + i\eta) R_{\delta a'}^{\alpha' c'}(\nu + i\eta) R_{cc'}^{b'b}(\nu - \nu'). \quad (3.1.5)$$

Corollary 3.1.5. Use Yang-Baxter relation (3.1.5) for N times, we have

$$R_{aa'}^{c'c}(\nu - \nu') (\mathcal{T}_{\bar{\alpha}}^{\bar{\delta}})_c^b(\nu) (\mathcal{T}_{\bar{\delta}}^{\bar{\alpha}'})_{c'}^{b'}(\nu') = (\mathcal{T}_{\bar{\alpha}}^{\bar{\delta}})_a^c(\nu') (\mathcal{T}_{\bar{\delta}}^{\bar{\alpha}'})_{a'}^{c'}(\nu) R_{cc'}^{b'b}(\nu - \nu'). \quad (3.1.6)$$

Theorem 3.1.6. The transfer matrices for different spectral parameters commute:

$$[\mathbb{T}(\nu), \mathbb{T}(\nu')] = 0 \quad (3.1.7)$$

3.1.2 Algebraic Bethe Ansatz

By taking different values of $\{a, a', b, b'\}$ in (3.1.6), we will get relations among $A(\nu), B(\nu), C(\nu)$ and $D(\nu)$.

$$D(\nu)B(\nu') = g(\nu - \nu')B(\nu')D(\nu) - h(\nu - \nu')B(\nu)D(\nu'),$$

$$[B(\nu), B(\nu')] = 0,$$

$$A(\nu)B(\nu') = g(\nu' - \nu)B(\nu')A(\nu) - h(\nu' - \nu)B(\nu)A(\nu'),$$

where $g(\nu) = \frac{R_{\leftarrow\leftarrow}^{\leftarrow\leftarrow}}{R_{\rightarrow\leftarrow}^{\leftarrow\leftarrow}} = -\frac{\sin(2\eta + i\nu)}{\sin(i\nu)}$ and $h(\nu) = \frac{R_{\rightarrow\leftarrow}^{\rightarrow\leftarrow}}{R_{\rightarrow\leftarrow}^{\leftarrow\leftarrow}} = -\frac{\sin(2\eta)}{\sin(i\nu)}$.

Since $[\mathbb{T}(\nu), \mathbb{T}(\nu')] = 0$, for any two ν, ν' . The $\mathbb{T}(\nu)$ have common eigenvectors. Bethe Ansatz is a method to find their common eigenvectors.

Recall that \mathbb{T} is a $2^N \times 2^N$ matrix, acts on the 2^N dimensional space with basis consists of N left or right arrows. Define the spin S of a base vector to be the total number of right arrows minus the total number of left arrows. A vector is called of spin S if it lies in the subspace spanned by spin S bases. Then one can prove easily that B maps a spin S vector to a spin $S - 2$, A and D maps a spin S to a spin S , and C maps a spin S to a spin $S + 2$.

Define $|\Omega\rangle = |\rightarrow \cdots \rightarrow\rangle$ the unique vector of spin N . Bethe Ansatz says the eigenvectors of transfer matrices are of the form $|\Psi\rangle = B(\nu_1)B(\nu_2)\cdots B(\nu_n)|\Omega\rangle$. Compute with the help of commutative relations among $A(\nu), B(\nu), C(\nu)$ and $D(\nu)$, we have

$$\begin{aligned} A(\nu)|\Psi\rangle &= A(\nu)B(\nu_1)\cdots B(\nu_n)|\Omega\rangle \\ &= a^N(\nu, \eta) \prod_{j=1}^n g(\nu_j - \nu)|\Psi\rangle \\ &+ \sum_{k=1}^n \{[-a^N(\nu_k, \eta)h(\nu_k - \nu) \prod_{j=1, j \neq k}^n g(\nu_j - \nu_k)]B(\nu) \prod_{j=1, j \neq k}^n B(\nu_j)|\Omega\rangle\} \end{aligned}$$

and

$$\begin{aligned} D(\nu)|\Psi\rangle &= D(\nu)B(\nu_1)\cdots B(\nu_n)|\Omega\rangle \\ &= b^N(\nu, \eta) \prod_{j=1}^n g(\nu - \nu_j)|\Psi\rangle \\ &+ \sum_{k=1}^n \{[-b^N(\nu_k, \eta)h(\nu - \nu_k) \prod_{j=1, j \neq k}^n g(\nu_k - \nu_j)]B(\nu) \prod_{j=1, j \neq k}^n B(\nu_j)|\Omega\rangle\}. \end{aligned}$$

Therefore $|\Psi\rangle$ is a eigenvector of \mathbb{T} if and only if $\{\nu_1, \nu_2, \cdots, \nu_n\}$ satisfies

$$\begin{aligned} &[-a^N(\nu_k, \eta)h(\nu_k - \nu) \prod_{j=1, j \neq k}^n g(\nu_j - \nu_k)] \\ &+ [-b^N(\nu_k, \eta)h(\nu - \nu_k) \prod_{j=1, j \neq k}^n g(\nu_k - \nu_j)] = 0, \quad \forall k. \end{aligned}$$

Or equivalently,

$$(-1)^n \prod_{j=1}^n \frac{\sinh(2i\eta - \nu_k + \nu_j)}{\sinh(2i\eta - \nu_j + \nu_k)} = -\frac{a^N(\nu_k, \eta)}{b^N(\nu_k, \eta)}, \quad \forall k. \quad (3.1.8)$$

And the corresponding eigenvalue of \mathbb{T} is

$$t(\nu) = a^N(\nu, \eta) \prod_{j=1}^n g(\nu_j - \nu) + b^N(\nu, \eta) \prod_{j=1}^n g(\nu - \nu_j). \quad (3.1.9)$$

(3.1.8) is called the Bethe Ansatz equations.

3.1.3 TQ relations

We proved in (3.1.7) that the transfer matrices commute for different values of spectral parameters $[\mathbb{T}(\nu), \mathbb{T}(\nu')] = 0$. Therefore $\mathbb{T}(\nu)$ can be diagonalized simultaneously, with diagonal elements the associated eigenvalues $t_0(\nu), t_1(\nu), \cdots$. Pay attention to the expression of matrix \mathbb{T} , we know that these functions are $2\pi i$ -periodic and entire on the complex plane.

The essential property of these eigenvalues is that there exist $2\pi i$ -periodic entire functions $q_i(\nu)$ such that

$$t_i(\nu)q_i(\nu) = a^N(\nu, \eta)q_i(\nu + 2i\eta) + b^N(\nu, \eta)q_i(\nu - 2i\eta). \quad (3.1.10)$$

This relation recovers the Bethe ansatz equation as follows: Suppose the real zeros of $q(\nu)$ are ν_1, \cdots, ν_n , then $q(\nu) = \prod_{l=1}^n \sinh(\nu - \nu_l)$ up to a constant. By setting $\nu = \nu_l$ in (3.1.10), we have $\frac{q(\nu_l - 2i\eta)}{q(\nu_l + 2i\eta)} = -\frac{a^N(\nu_l, \eta)}{b^N(\nu_l, \eta)}$, which is exactly the Bethe ansatz equations with ν_l the roots. Therefore, the Bethe Ansatz equations may be understood as pole cancellation equations. We will see this point of view again in the context of opers.

In fact, we can construct a matrix $\mathbb{Q}(\nu)$ such that $[\mathbb{Q}(\nu), \mathbb{T}(\mu)] = [\mathbb{Q}(\nu), \mathbb{Q}(\mu)] = 0$ for any ν, μ and

$$\mathbb{T}(\nu)\mathbb{Q}(\nu) = a^N(\nu, \eta)\mathbb{Q}(\nu + 2i\eta) + b^N(\nu, \eta)\mathbb{Q}(\nu - 2i\eta).$$

Then we can diagonalize \mathbb{T} and \mathbb{Q} simultaneously, and the corresponding diagonal elements $t_i(\nu), q_i(\nu)$ satisfy (3.1.10).

3.2 Gaudin models

Let \mathfrak{g} be a simple complex Lie algebra. Let V_λ be the finite-dimensional simple module over \mathfrak{g} of highest weight λ . Denote $\{J_a\}_{a=1}^d$ to be a basis of \mathfrak{g} , $\{J^a\}$ the dual basis with respect to the renormalized Killing form, and

$$\Delta = \frac{1}{2} \sum_{a=1}^d J_a J^a$$

the Casimir element. In Gaudin's model of spin chains, the Hamiltonians were constructed as operators acting on $V_{\lambda_1} \otimes \cdots \otimes V_{\lambda_N}$ formulated by

$$\Xi_i = \sum_{j \neq i} \sum_{a=1}^d \frac{J_a^{(i)} J_a^{(j)}}{z_i - z_j}. \quad (3.2.1)$$

The problem in Gaudin models is to find eigenvectors and associated eigenvalues of Gaudin Hamiltonians.

When solving \mathfrak{sl}_2 Gaudin models, Gaudin proposed the following procedure, known as Bethe Ansatz, to find common eigenvectors of Gaudin Hamiltonians.

Let $|0\rangle$ be a tensor product of highest weight vectors in $V_{\lambda_1} \otimes \cdots \otimes V_{\lambda_N}$. Note that $|0\rangle$ is automatically an eigenvector of Gaudin Hamiltonians. Let e, f, h be the standard basis of \mathfrak{sl}_2 . Gaudin made a hypothesis that all eigenvectors can be obtained from $|0\rangle$ by applying the operator

$$F(w) := \varphi_{w-z_1, \dots, w-z_N}(f \otimes t^{-1}) = \sum_{i=1}^N \frac{f^{(i)}}{w - z_i}.$$

Denote $|w_1, \dots, w_m\rangle = F(w_1) \cdots F(w_m)|0\rangle$, then it is calculated that

$$S(u)|w_1, \dots, w_m\rangle = s_m(u)|w_1, \dots, w_m\rangle + \sum_{j=1}^m \frac{f_j}{u - w_j} |w_1, \dots, w_{j-1}, u, w_{j+1}, \dots, w_m\rangle, \quad (3.2.2)$$

where $s_m(u) \in \mathbb{C}$ and

$$f_j = \sum_{i=1}^N \frac{\lambda_i}{w_j - z_i} - \sum_{s \neq j} \frac{2}{w_j - w_s}.$$

Thus $|w_1, \dots, w_m\rangle$ is an eigenvector if and only if the Bethe ansatz equations

$$\sum_{i=1}^N \frac{\lambda_i}{w_j - z_i} - \sum_{s \neq j} \frac{2}{w_j - w_s} = 0, \quad j = 1, \dots, m, \quad (3.2.3)$$

are satisfied.

A natural question to ask is when the vector $|w_1, \dots, w_m\rangle$ associated to a solution of Bethe ansatz equations is non-zero. From which we can deduce, under Gaudin's hypothesis, a correspondence between eigenvectors of Gaudin Hamiltonians and a subset of solutions to Bethe ansatz equations.

For general simple Lie algebra \mathfrak{g} , it was introduced by Babujian and Flume [BF94] vectors $|w_1^{i_1}, \dots, w_m^{i_m}\rangle = F_{i_1}(w_1) \cdots F_{i_m}(w_m)|0\rangle$ analogous to $|w_1, \dots, w_m\rangle$ for \mathfrak{sl}_2 . Here $i_j \in I$ are indices of simple roots. If $|w_1^{i_1}, \dots, w_m^{i_m}\rangle$ is an eigenvector of Gaudin Hamiltonians, then the Bethe ansatz equations

$$\sum_{i=1}^N \frac{(\lambda_i, \alpha_{i_j})}{w_j - z_i} - \sum_{s \neq j} \frac{(\alpha_{i_s}, \alpha_{i_j})}{w_j - w_s} = 0, \quad j = 1, \dots, m \quad (3.2.4)$$

are satisfied. Moreover, if $|w_1^{i_1}, \dots, w_m^{i_m}\rangle$ is an eigenvector, then it is of highest weight $\sum_{i=1}^N \lambda_i - \sum_{i=1}^m \alpha_{i_j}$.

Chapter 4

TQ relations

4.1 TQ relations as operators

The T-operator and Q-operator can be defined as operators in representations of $U_q(\widehat{\mathfrak{sl}}_2)$.

Definition 4.1.1. We define the monodromy operator $M_{m,n}(\lambda, \nu) = (\pi_m(\lambda) \otimes \pi_n(\nu))(\mathcal{R})$. It is an element of $End(V_m) \otimes End(V_n)$.

Definition 4.1.2. Then define

$$M_{m,n}(\lambda; \nu_1, \dots, \nu_N) = M_{m,n}^{0,1}(\lambda, \nu_1) \cdots M_{m,n}^{0,N}(\lambda, \nu_N)$$

an operator acting on $V_m \otimes V_n^{\otimes N}$, where $M_{m,n}^{0,i}(\lambda, \nu_i)$ is the monodromy operator $M_{m,n}(\lambda, \nu_i)$ acts on V_m and the i th component of $V_n^{\otimes N}$.

Definition 4.1.3. Define the transfer operator

$$\mathbb{T}_{m,n}(\lambda; \nu_1, \dots, \nu_N) = \text{Tr}_m(M_{m,n}(\lambda; \nu_1, \dots, \nu_N)F_m(\lambda)),$$

where $F_m(\lambda) \in Mat_m(\mathbb{C})$ is the matrix of $\pi_m(\lambda)(q^{\phi h_1})$, $\phi \in \mathbb{C}$ plays as a twist parameter. It is an element of $End(V_n^{\otimes N})$. We will simply denote it by $\mathbb{T}_{m,n}(\lambda; \bar{\nu})$.

Theorem 4.1.4.

$$\mathbb{T}_{m,n}(\lambda; \bar{\nu})\mathbb{T}_{m',n}(\lambda'; \bar{\nu}) = \mathbb{T}_{m',n}(\lambda'; \bar{\nu})\mathbb{T}_{m,n}(\lambda; \bar{\nu}) \quad (4.1.1)$$

for all parameters $\lambda, \lambda' \in \mathbb{C}$ and $m, m' \in \mathbb{N}^*$.

As indicated by [BLZ99], to define the Q-operator, we need to use the q-oscillator algebras.

Definition 4.1.5. The q-oscillator algebra \mathcal{A}_q is the associative algebra generated by four elements a^+, a, q^N, q^{-N} subject to the relations

$$\begin{aligned} aa^+ - qa^+a &= q^{-N}, aa^+ - q^{-1}a^+a = q^N, \\ q^{-N}q^N &= q^Nq^{-N} = 1, q^Na^+ = qa^+q^N, q^Na = q^{-1}aq^N. \end{aligned}$$

We consider the representation T^+ of \mathcal{A}_q with basis $\{|m\rangle : m \in \mathbb{N}\}$ defined by

$$\begin{aligned} T^+(N)|m\rangle &= m|m\rangle, \\ T^+(a^+)|m\rangle &= |m+1\rangle, \\ T^+(a)|m\rangle &= [m]|m-1\rangle, \end{aligned}$$

where $|-1\rangle := 0$.

Similarly, the representation T^- with basis $\{|-1-m\rangle : m \in \mathbb{N}\}$ defined by

$$\begin{aligned} T^-(N)|-1-m\rangle &= (-1-m)|-1-m\rangle, \\ T^-(a)|-1-m\rangle &= |-m-2\rangle, \\ T^-(a^+)|-1-m\rangle &= [m]|-m\rangle, \end{aligned}$$

where $|0\rangle := 0$.

Definition 4.1.6. The Borel subalgebra $U_q(\mathfrak{b})$ of $U_q(\widehat{\mathfrak{sl}}_2)$ is the Hopf subalgebra generated by $q^{h_0}, q^{h_1}, e_0, e_1$. And $U_q(\mathfrak{b}^-)$ is the Hopf subalgebra generated by $q^{h_0}, q^{h_1}, f_0, f_1$. We know that the R -matrix is an element in $U_q(\mathfrak{b}) \otimes U_q(\mathfrak{b}^-)$.

We need a family of homomorphisms from $\mathcal{U}_q(\mathfrak{b})$ to \mathcal{A}_q .

For $\lambda \in \mathbb{C}^*$, define $\varphi(\lambda)(q^{h_1}) = q^{-2N}$, $\varphi(\lambda)(q^{h_0}) = q^{2N}$, $\varphi(\lambda)(e_1) = \frac{aq^{-N}}{q-q^{-1}}$, $\varphi(\lambda)(e_0) = \lambda a^+$. And also $\bar{\varphi}(\lambda)(q^{h_1}) = q^{2N}$, $\bar{\varphi}(\lambda)(q^{h_0}) = q^{-2N}$, $\bar{\varphi}(\lambda)(e_0) = \lambda \frac{aq^{-N}}{q-q^{-1}}$, $\bar{\varphi}(\lambda)(e_1) = a^+$.

The resulting representations of the quantum Borel subalgebra turns out to be the prefundamental representations of quantum Borel subalgebras [HJ12].

Definition 4.1.7. The L-operator is defined by

$$\begin{aligned}\mathcal{L}(\lambda) &= (1 \otimes q^{\frac{uh_1}{4}})(\varphi(\lambda) \otimes id)(\mathcal{R}), \\ \bar{\mathcal{L}}(\lambda) &= (1 \otimes q^{-\frac{uh_1}{4}})(\bar{\varphi}(\lambda) \otimes id)(\mathcal{R}).\end{aligned}$$

Here $q^u = \lambda$. They are elements in $\mathcal{A}_q \otimes \mathcal{U}_q(\mathfrak{b}^-)$.

Then let $L_n(\lambda, \nu) = (1 \otimes \pi_n(\nu))\mathcal{L}(\lambda)$ and $\bar{L}_n(\lambda, \nu) = (1 \otimes \pi_n(\nu))\bar{\mathcal{L}}(\lambda)$. They are elements of $\mathcal{A}_q \otimes End(V_n)$. Generally L-operator are defined by

$$\begin{aligned}L_n(\lambda; \nu_1, \dots, \nu_N) &= L_n(\lambda, \nu_1)^{0,1} \dots L_n(\lambda, \nu_N)^{0,N}. \\ \bar{L}_n(\lambda; \nu_1, \dots, \nu_N) &= \bar{L}_n(\lambda, \nu_1)^{0,1} \dots \bar{L}_n(\lambda, \nu_N)^{0,N}.\end{aligned}$$

They are elements in $\mathcal{A}_q \otimes End(V_n)^{\otimes N}$. Where $L_n(\lambda, \nu_i)^{0,i}$ (resp. $\bar{L}_n(\lambda, \nu_i)^{0,i}$) is the L-operator $L_n(\lambda, \nu_i)$ (resp. $\bar{L}_n(\lambda, \nu_i)$) lying in \mathcal{A}_q and the i th component of $V_n^{\otimes N}$.

The Q-operator is defined by

$$Q_{+,n}(\lambda; \nu_1, \dots, \nu_N) = tr_+(L_n(\lambda; \nu_1, \dots, \nu_N)(q^{-2\phi N} \otimes 1 \otimes \dots \otimes 1)),$$

and

$$Q_{-,n}(\lambda; \nu_1, \dots, \nu_N) = tr_-(\bar{L}_n(\lambda; \nu_1, \dots, \nu_N)(q^{2\phi N} \otimes 1 \otimes \dots \otimes 1)).$$

Where the traces tr_{\pm} are taken in representations T_0^+ and T_{-1}^- respectively. We will denote them by $\mathbb{Q}_{\pm,n}(\lambda; \bar{\nu})$.

Remark The Q-operator is an element in $End(V^{\otimes N})$, we can write them in matrix form under the same basis as we chose for \mathbb{T} as:

$$\mathbb{Q}_{\pm,n}(\lambda; \nu_1, \dots, \nu_N) = (Q_{i_1, \dots, i_N; j_1, \dots, j_N}(\lambda; \nu_1, \dots, \nu_N)).$$

It is an $(n+1)^N$ by $(n+1)^N$ matrix with complex entries.

Theorem 4.1.8. The operators T, Q satisfy the commutation relations

$$\begin{aligned}\mathbb{Q}_{\pm,n}(\lambda; \bar{\nu})\mathbb{T}_{m,n}(\lambda'; \bar{\nu}) &= \mathbb{T}_{m,n}(\lambda'; \bar{\nu})\mathbb{Q}_{\pm,n}(\lambda; \bar{\nu}), \\ \mathbb{Q}_{\pm,n}(\lambda; \bar{\nu})\mathbb{Q}_{\pm,n}(\lambda'; \bar{\nu}) &= \mathbb{Q}_{\pm,n}(\lambda'; \bar{\nu})\mathbb{Q}_{\pm,n}(\lambda; \bar{\nu}), \\ \mathbb{Q}_{\pm,n}(\lambda; \bar{\nu})\mathbb{Q}_{\mp,n}(\lambda'; \bar{\nu}) &= \mathbb{Q}_{\mp,n}(\lambda'; \bar{\nu})\mathbb{Q}_{\pm,n}(\lambda; \bar{\nu}), \quad \forall \lambda, \lambda' \in \mathbb{C}, \bar{\nu} \in \mathbb{C}^N,\end{aligned}$$

and the TQ relations

$$\mathbb{T}_{1,n}(\lambda; \bar{\nu})\mathbb{Q}_{\pm,n}(\lambda; \bar{\nu}) = q^{N\phi}\mathbb{Q}_{\pm,n}(q^2\lambda; \bar{\nu}) + q^{-N\phi}\mathbb{Q}_{\pm,n}(q^{-2}\lambda; \bar{\nu}). \quad (4.1.2)$$

4.2 TQ relations in the Grothendieck ring

To study the representations of quantum affine algebras, another set of generators described by Drinfeld is useful.

Theorem 4.2.1. The algebra $U_q(\widehat{\mathfrak{g}})$ is generated by $x_{i,n}^{\pm}$ ($i \in I, n \in \mathbb{Z}$), k_i^{\pm} ($i \in I$), $h_{i,n}$ ($i \in I, n \in \mathbb{Z} \setminus \{0\}$) and central elements $c^{\pm 1/2}$, with the Drinfeld's relations.

We introduce the elements $\phi_{i,n}^{\pm}$ determined by the formal power series

$$\sum_{n=0}^{\infty} \phi_{i,\pm n}^{\pm} u^{\pm n} = k_i^{\pm} \exp(\pm(q - q^{-1}) \sum_{m=1}^{\infty} h_{i,\pm m} u^{\pm m}).$$

A presentation of $U_q(\widehat{\mathfrak{g}})$ is called of type 1 if the central elements $c^{\pm 1/2}$ acts as identity, and it can be decomposed into direct sum of weight spaces of k_i^{\pm} when viewed as representations of $U_q(\mathfrak{g})$.

Definition 4.2.2. Let V be a type 1 representation. A vector $v \in V$ is called a highest l -weight vector if

$$x_{i,r}^+ v = 0, \quad \phi_{i,r}^\pm v = \gamma_{i,r}^\pm v,$$

for some complex numbers $\gamma_{i,r}^\pm$. A type 1 representation V is a highest l -weight representation if $V = U_q(\hat{\mathfrak{g}})v$, for some highest l -weight vector v . In that case, the set $(\gamma_{i,r}^\pm)_{i \in I, r \in \mathbb{Z}}$ is called the highest l -weight of V .

An important theorem of Chari and Pressley [CP95] says

Theorem 4.2.3. *Every finite-dimensional irreducible representation of $U_q(\hat{\mathfrak{g}})$ of type 1 is a highest l -weight representation. Moreover, there exists a set of polynomials $(P_i)_{i \in I}$ such that the highest l -weight satisfies*

$$\sum_{m \geq 0} \gamma_{i,\pm m}^\pm u^{\pm m} = q_i^{\deg(P_i)} \frac{P_i(uq_i^{-1})}{P_i(uq_i)}.$$

Conversely, for each set of polynomials $(P_i)_{i \in I}$ with constant terms 1, the corresponding representation is finite-dimensional.

Example 4.2.4. When $(P_i)_{i \in I} = (1, \dots, 1, 1 - \lambda u, 1, \dots, 1)$, we called the corresponding representations the fundamental representations, denoted by $V_{i,\lambda}$.

The similar arguments also apply to the quantum Borel subalgebra $U_q(\mathfrak{b})$ [FH15]. The Borel subalgebra $U_q(\mathfrak{b})$ contains Drinfeld generators $x_{i,m}^+, x_{i,r}^-, k_i^\pm, \phi_{i,r}^\pm$, where $i \in I, m \geq 0, r > 0$.

Example 4.2.5. For the highest l -weight

$$\left(\sum_{m \geq 0} \gamma_{i,m}^+ u^m \right)_{i \in I} = (1, \dots, 1, (1 - au)^{\pm 1}, 1, \dots, 1),$$

the corresponding highest l -weight representations of $U_q(\mathfrak{b})$ are called the prefundamental representations, denoted by $L_{i,a}^\pm$. They have infinite dimension.

For the highest l -weight

$$\left(\sum_{m \geq 0} \gamma_{i,m}^+ u^m \right)_{i \in I} = (1, \dots, 1, q_i, 1, \dots, 1),$$

the corresponding highest l -weight representations of $U_q(\mathfrak{b})$ are 1-dimensional representations, denoted by $L(\omega_i)$.

Remark 4.2.6. The evaluation representations of $U_q(\hat{\mathfrak{sl}}_2)$ turn out to be the fundamental representations. And the representations of the Borel subalgebra constructed from q -oscillator algebras turn out to be the prefundamental representations. Therefore, we expect to write the TQ relations in terms of fundamental representations and prefundamental representations.

Theorem 4.2.7. [FH15] *In the case of $U_q(\hat{\mathfrak{sl}}_2)$, in the Grothendieck ring of a certain category \mathcal{O} of representations of $U_q(\mathfrak{b})$, we have*

$$[V_{1,a}][L_{1,aq}^+] = [L_{1,aq^{-1}}^+][L(\omega_1)] + [L_{1,aq^3}^+][L(-\omega_1)]. \quad (4.2.1)$$

The next question is to reconstruct the operator TQ relations from the above relation in the Grothendieck ring.

Recall that the universal R -matrix of $U_q(\hat{\mathfrak{g}})$, \mathcal{R} , lies in $U_q(\mathfrak{b}) \hat{\otimes} U_q(\mathfrak{b}^-)$. Let V be a representation of $U_q(\mathfrak{b})$ from the category \mathcal{O} , we can define the transfer matrix

$$t_V(z, u) = (\mathrm{Tr}_{V,u} \circ \pi_{V(z)} \otimes \mathrm{Id})(\mathcal{R}) \in U_q(\mathfrak{b}^-)[[z, u_i^\pm]].$$

Here $\mathrm{Tr}_{V,u}(g) = \sum_\lambda \mathrm{Tr}_{V_\lambda}(\pi_V(g))(\prod_{i \in I} u_i^{\lambda_i})$, $V(z)$ is the twisting of V by the action $x_{i,r}^\pm = z^r x_{i,r}^\pm$, $\phi_{i,r}^\pm = z^r \phi_{i,r}^\pm$.

Proposition 4.2.8. *If $[W] = [V] + [V']$ in the Grothendieck ring of the category \mathcal{O} , then $t_W(z, u) = t_V(z, u) + t_{V'}(z, u)$;*

If $[W] = [V \otimes V']$ in the Grothendieck ring of the category \mathcal{O} , then $t_W(z, u) = t_V(z, u)t_{V'}(z, u)$.

Then the operator TQ relations are nothing but the result of applying transfer matrices to the TQ relations in the Grothendieck ring.

Remark 4.2.9. We define the q -character of a representation V of $U_q(\mathfrak{b})$ by

$$\chi_q(V) = \sum_\gamma \dim(V_\gamma) e^\gamma.$$

Here γ are l -weights of V , $V_\gamma = \{v | \exists p, (\phi_{i,m}^+ - \gamma_{i,m})^p v = 0\}$. Then the q -characters can be calculated by applying the quantum affine Harish-Chandra homomorphisms to the transfer matrices. We will see, in the classical case, the relation between affine Harish-Chandra homomorphisms and Miura transformations in the next chapter.

4.3 Schrodinger operators and CD relations

The TQ relations appear in the theory of second order ordinary differential equations [DDT07] when we study structures of solutions of the following time-independent Schrodinger equation:

$$\frac{d^2}{dx^2}\Psi(x) = [x^{2M} + \frac{l(l+1)}{x^2} - E]\Psi(x), \quad (4.3.1)$$

where $M > 1$ and $l > -\frac{1}{2}$ are real numbers.

The following result is obtained by WKB approximation.

Theorem 4.3.1. *For the case $M \in \mathbb{N}$, we write $x = \rho e^{i\theta}$ with ρ real and $-\pi < \theta \leq \pi$. Then for $\theta \notin \{\pm \frac{(2k+1)\pi}{2M+2} | k \in \mathbb{Z}\}$, Ψ has the asymptotic approximation*

$$\Psi_{\pm}(x) \sim P^{-\frac{1}{4}}(x) \exp(\pm \frac{1}{M+1} e^{i\theta(1+M)} \rho^{1+M}), \quad \rho = |x| \rightarrow \infty. \quad (4.3.2)$$

Where $P(x) = x^{2M} + \frac{l(l+1)}{x^2} - E$.

Remark The rays $\theta = \pm \frac{(2k+1)\pi}{2M+2}$ for $k \in \mathbb{Z}$ are called "anti-Stokes lines", and a wedge-like region between two neighboring anti-Stokes lines is called a Stokes sector, denoted by $S_k = \{x : |\arg(x) - \frac{2\pi k}{2M+2}| < \frac{\pi}{2M+2}\}$. We remark that in a Stokes sector, one of $\Psi_{\pm}(x)$ is exponentially growing and the other exponentially decaying. The exponentially growing solution is called dominant in that sector, and the decaying solution is called subdominant.

Then for each pair of Stokes sectors, we can state an eigenvalue problem: for which values of E , there exist solutions of (4.3.1) subdominant in the two sectors? They are called lateral connection problems.

Another type of eigenvalue problems, called radical connection problems, can be stated as follows: we are finding for solutions of (4.3.1) which decay exponentially in one of Stokes sectors, and we require $\Psi(x) \sim ax^{l+1} + bx^{-l}$ as $x \rightarrow 0$ in this sector, $a, b \in \mathbb{C}$.

The problem gets complicated when M takes real values, since x^{2M} is not globally defined. We can solve the problem in $\mathbb{C} \setminus \{\arg(x) = \pi\}$. Then S_k is defined for k such that $|k| < \frac{2M+1}{2}$ and the asymptotic relation (4.3.2) still holds.

For solutions of (4.3.1), we have the following theorem.

Theorem 4.3.2. *There is a unique solution $y(x, E, l)$ of (4.3.1) such that*

$$y(x, E, l) \sim \frac{x^{-M/2}}{\sqrt{2i}} \exp(-\frac{x^{M+1}}{M+1}), \quad (4.3.3)$$

$$y'(x, E, l) \sim -\frac{x^{M/2}}{\sqrt{2i}} \exp(-\frac{x^{M+1}}{M+1}), \quad (4.3.4)$$

as $|x| \rightarrow \infty$ with $|\arg x| < \frac{3\pi}{2M+2}$. It is called the basic solution.

With the help of the basic solution, we can define a set of functions

$$y_k(x, E, l) = \omega^{k/2} y(\omega^{-k} x, \omega^{2k} E, l), \quad (4.3.5)$$

where $\omega = e^{\frac{2\pi i}{2M+2}}$, $k \in \mathbb{Z}$.

Theorem 4.3.3. *For all $k \in \mathbb{Z}$,*

- 1) y_k solves (4.3.1).
- 2) y_k is subdominant in S_k , and dominant in S_{k-1} and S_{k+1} .
- 3) y_k and y_{k+1} are linearly independent and form a basis of solutions of (4.3.1).

Express y_{-1} under the basis $\{y_0, y_1\}$ as $y_{-1} = C(E, l)y_0 + \tilde{C}(E, l)y_1$. Recall that the Wronskian of two functions f and g is defined to be $W[f, g] = fg' - f'g$. For two solutions of a second order ODE whose first order term vanishes, their Wronskian $W[f, g]$ is a constant, and $W[f, g]$ vanishes if and only if f and g are proportional.

If we write $W_{k_1, k_2}(E, l) = W[y_{k_1}(x, E, l), y_{k_2}(x, E, l)]$, then it satisfies

Lemma 4.3.4.

$$W_{k_1, k_2}(E, l) = W_{k_1+1, k_2+1}(\omega^{-2} E, l), \quad W_{0,1}(E, l) = 1.$$

Taking Wronskian with y_1 and y_0 on both sides of $y_{-1} = C(E, l)y_0 + \tilde{C}(E, l)y_1$ shows that $C(E, l) = \frac{W_{-1,1}(E, l)}{W_{0,1}(E, l)} = W_{-1,1}(E, l)$, $\tilde{C}(E, l) = -\frac{W_{-1,0}(E, l)}{W_{0,1}(E, l)} = -1$. Thus

$$C(E, l)y_0(x, E, l) = y_{-1}(x, E, l) + y_1(x, E, l).$$

We will simply admit that, for the radical problem, there is a solution $\psi(x, E, l)$ such that $\psi(x, E, l) \sim x^{l+1}$ as $x \rightarrow 0$. Since the equation is invariant when replacing l by $-1-l$, $\psi(x, E, -1-l)$ is also a solution of the equation. If we denote $\psi_+(x, E) = \psi(x, E, l)$ and $\psi_-(x, E) = \psi(x, E, -1-l)$, then $\psi_-(x, E) \sim x^{-l}$ as $x \rightarrow 0$.

Analogously, define

$$\psi_k(x, E, l) = \omega^{k/2}\psi(\omega^{-k}x, \omega^{2k}E, l), \quad k \in \mathbb{Z}. \quad (4.3.6)$$

Then ψ_k also solves (4.3.1) and the behavior at $x \rightarrow 0$ shows that

$$\psi_k(x, E, l) = \omega^{-(l+1/2)k}\psi(x, E, l).$$

In addition,

$$W[y_k, \psi_k](E, l) = W[y, \psi](\omega^{2k}E, l).$$

Thus we conclude

$$W[y_k, \psi](E, l) = \omega^{(l+1/2)k}W[y, \psi](E, l).$$

Taking Wronskian with $\psi(x, E, l)$ on both sides of $y_{-1} = C(E, l)y_0 + \tilde{C}(E, l)y_1$,

$$C(E, l)W[y_0, \psi](E, l) = W[y_{-1}, \psi](E, l) + W[y_1, \psi](E, l),$$

if we denote $D(E, l) = W[y, \psi](E, l)$, then

$$C(E, l)D(E, l) = \omega^{-(l+1/2)}D(\omega^{-2}E, l) + \omega^{(l+1/2)}D(\omega^2E, l), \quad (4.3.7)$$

Theorem 4.3.5. Define $D_{\mp}(E) = W[y, \psi_{\pm}](E, l)$. Then by taking l to be l and $-l-1$ in (4.3.7), it becomes

$$C(E, l)D_{\mp}(E) = \omega^{\mp(l+1/2)}D_{\mp}(\omega^{-2}E, l) + \omega^{\pm(l+1/2)}D_{\mp}(\omega^2E, l). \quad (4.3.8)$$

This is the CD relation. It has the same form as the TQ relation we have seen before.

Chapter 5

Bethe Ansatz and opers

We have seen that the Bethe Ansatz is a possible method to solve Gaudin models. In this chapter, we review another approach to Bethe Ansatz equations, the singularities cancellation of opers. Therefore a correspondence between the spectrum of Gaudin Hamiltonians and the space of opers is supposed. The opers concerning in Gaudin models are the opers associated to finite-dimensional simple Lie algebras. Recall in the six vertex model, we also have Bethe Ansatz. We note that the opers associated to affine Lie algebras are related to quantum KdV systems and, in particular, the six vertex model.

5.1 Gaudin algebras

Gaudin algebras are larger commutative sub-algebras in $U(\mathfrak{g})^{\otimes N}$ containing the Gaudin Hamiltonians Ξ_i . Furthermore, we give a system of generators of Gaudin algebras including the Gaudin Hamiltonians. For general \mathfrak{g} , the Gaudin algebras are generated by Gaudin Hamiltonians together with so-called higher Gaudin Hamiltonians. Roughly speaking, Gaudin algebra is constructed as follows. Consider the space

$$(U(\hat{\mathfrak{g}})/U(\hat{\mathfrak{g}})(\mathfrak{g}[[t]] + \mathbb{C}(K + h^\vee)))^{\mathfrak{g}[[t]]}.$$

Each element in this space has a unique representative in $U(t^{-1}\mathfrak{g}[t^{-1}])$. The resulting subalgebra in $U(t^{-1}\mathfrak{g}[t^{-1}])$ is called the universal Gaudin algebra, denoted by $\mathfrak{z}(\hat{\mathfrak{g}})$.

Theorem 5.1.1. $\mathfrak{z}(\hat{\mathfrak{g}})$ is a commutative subalgebra of $U(t^{-1}\mathfrak{g}[t^{-1}])$. And it is stable under ∂_t .

Definition 5.1.2. Denote by \mathfrak{J} the left ideal of $U(t^{-1}\mathfrak{g}[t^{-1}])$ generated by $U(t^{-1}\mathfrak{n}_-[t^{-1}])$. Note that we have a direct sum

$$U(t^{-1}\mathfrak{g}[t^{-1}])^{\mathfrak{h}} = U(t^{-1}\mathfrak{h}[t^{-1}]) \oplus (U(t^{-1}\mathfrak{g}[t^{-1}])^{\mathfrak{h}} \cap \mathfrak{J}).$$

The affine Harish-Chandra homomorphism $\bar{\iota}_{HC}$ can be obtained as the restriction of the projection

$$U(t^{-1}\mathfrak{g}[t^{-1}])^{\mathfrak{h}} \rightarrow U(t^{-1}\mathfrak{h}[t^{-1}])$$

on $\mathfrak{z}(\hat{\mathfrak{g}})$.

Example 5.1.3. When $\mathfrak{g} = \mathfrak{sl}_2$, $(\text{Fun } \mathfrak{g})^{\mathfrak{g}} = \mathbb{C}[P]$, with $P = ef + \frac{1}{4}h^2$. The affine analogue $(\text{Fun } \mathfrak{g}[[t]])^{\mathfrak{g}[[t]]}$ is the polynomial algebra on coefficients of

$$P(z) = e(z)f(z) + \frac{1}{4}h(z)^2 = (e \otimes t^{-1})(f \otimes t^{-1}) + \frac{1}{4}(h \otimes t^{-1})^2 + z(\dots).$$

Denote by $P_0 = (e \otimes t^{-1})(f \otimes t^{-1}) + \frac{1}{4}(h \otimes t^{-1})^2$, then $\mathfrak{z}(\hat{\mathfrak{sl}}_2) = \mathbb{C}[T^n S]_{n \geq 0}$ with

$$S = \frac{1}{2}[(e \otimes t^{-1})(f \otimes t^{-1}) + (f \otimes t^{-1})(e \otimes t^{-1}) + \frac{1}{2}(h \otimes t^{-1})^2]v_0$$

the Segal-Sugawara vector.

$\bar{\iota}_{HC} : \mathfrak{z}(\hat{\mathfrak{sl}}_2) \rightarrow U(t^{-1}\mathfrak{h}[t^{-1}])$ maps the corresponding field $S(z) = \frac{1}{2} : e(z)f(z) + f(z)e(z) + \frac{1}{2}h(z)^2 :$ to the generating series $\frac{1}{4}b(z)^2 - \frac{1}{2}\partial_z b(z)$. Here $b(z) = \sum_{k \geq 0} (h \otimes t^{-k-1})z^k$.

To define a subalgebra of $U(\mathfrak{g})^{\otimes N}$, we apply the morphism of algebras

$$\varphi_{w-z_1, \dots, w-z_N} : U(t^{-1}\mathfrak{g}[t^{-1}]) \rightarrow U(\mathfrak{g})^{\otimes N}$$

which is defined on generators

$$A \otimes t^{-n} \mapsto \frac{A^{(1)}}{(w - z_1)^n} + \cdots + \frac{A^{(N)}}{(w - z_N)^n}$$

and extended to $U(t^{-1}\mathfrak{g}[t^{-1}])$. Here $A^{(i)}$ denotes the element $1 \otimes \cdots \otimes A \otimes \cdots \otimes 1 \in U(\mathfrak{g})^{\otimes N}$ with A appearing on the i -th component.

Definition 5.1.4. $\mathcal{A}(z_1, \dots, z_N) := \varphi_{w-z_1, \dots, w-z_N}(\mathfrak{z}(\hat{\mathfrak{g}}))$ is a commutative subalgebra of $U(\mathfrak{g})^{\otimes N}$, called the Gaudin algebra. It is independent of the choice of $w \in \mathbb{C} \setminus \{z_1, \dots, z_N\}$.

Theorem 5.1.5. [FFR94] $\mathcal{A}(z_1, \dots, z_N)$ contains the Gaudin Hamiltonians Ξ_i . In fact,

$$\varphi_{w-z_1, \dots, w-z_N}(S) = \sum_{i=1}^N \frac{\Delta^{(i)}}{(w - z_i)^2} + \sum_{i=1}^N \frac{\Xi_i}{w - z_i},$$

where S is the Segal-Sugawara vector.

Example 5.1.6. When $\mathfrak{g} = \mathfrak{sl}_2$, by the above discussion, the Gaudin algebra $\mathcal{A}(z_1, \dots, z_N)$ is generated by Gaudin Hamiltonians Ξ_i and the central elements $\Delta^{(i)}$. The only relation among these generators is $\sum_{i=1}^N \Xi_i = 0$.

5.2 Opers

In this chapter we will review the definition of opers. And we explain its relation with Gaudin algebras. The notion of opers was defined by Beilinson, Drinfeld and Sokolov. For simplicity, we will follow the notation of Frenkel.

Let G be the simple Lie group of adjoint type associated to the simple Lie algebra \mathfrak{g} . For example, when $\mathfrak{g} = \mathfrak{sl}_n$, $G = PGL(n)$.

Definition 5.2.1. A G -oper on a smooth curve X is the following data

- a principal G -bundle \mathcal{F} on X
- a connection ∇ on \mathcal{F}
- a B -reduction \mathcal{F}_B of \mathcal{F} , i.e. a principal B -bundle \mathcal{F}_B together with an isomorphism of G -bundles

$$G \times_B \mathcal{F}_B \rightarrow \mathcal{F}.$$

such that under local trivialization of \mathcal{F}_B and induced local trivialization of \mathcal{F} ,

$$\nabla = \partial_t + \sum_{i=1}^l \psi_i(t) f_i + \mathbf{v}(t),$$

where $\psi_i(t)$ are nowhere vanishing functions and $\mathbf{v}(t)$ is a \mathfrak{b} -valued function.

Example 5.2.2. Let $X = D = \text{Spec } \mathbb{C}[[t]]$ be the formal disc with coordinate t . Then under a trivialization of \mathcal{F}_B on D , an oper is an operator $\nabla = \partial_t + \sum_{i=1}^l \psi_i(t) f_i + \mathbf{v}(t)$. For different choices of trivialization, the operators differ by a gauge transformation by $B[[t]]$:

$$g \cdot (\partial_t + A(t)) = \partial_t + gA(t)g^{-1} - g^{-1}\partial_t g. \quad (5.2.1)$$

So the space of G -opers on the unit disc is the space

$$\text{Op}_G(D) = \{\partial_t + \sum_{i=1}^l \psi_i(t) f_i + \mathbf{v}(t)\} / B[[t]] \simeq \{\partial_t + p_{-1} + \mathbf{v}(t)\} / N[[t]], \quad (5.2.2)$$

where $p_{-1} = \sum_{i=1}^l f_i$. The later is obtained by renormalizing ψ_i by applying $H[[t]]$ actions.

If we choose a different coordinate $t = \varphi(s)$, then the operator becomes $\partial_s + \varphi'(s)p_{-1} + \varphi'(s)\mathbf{v}(\varphi(s))$. Denote by $\check{\rho} : \mathbb{C}^\times \rightarrow H$ the sum of fundamental coweights of G . We also use the symbol $\check{\rho}$ for its corresponding element in \mathfrak{h} by a little abuse of language. The gauge transformation by $\check{\rho}(\varphi'(s))$ makes this operator to the form

$$\partial_s + p_{-1} + \varphi'(s)\check{\rho}(\varphi'(s))\mathbf{v}(\varphi(s))\check{\rho}(\varphi'(s))^{-1} - \check{\rho} \frac{\varphi''(s)}{\varphi'(s)}. \quad (5.2.3)$$

This formula exhibits the action of $\text{Der } \mathbb{C}[[t]] = \mathbb{C}[[t]]\partial_t$ on $\text{Op}_G(D)$.

E. Frenkel found the following non-trivial theorem.

Theorem 5.2.3. [Fre02, Theorem 11.2.] *We have canonical isomorphisms of algebras*

$$\mathfrak{z}(\hat{\mathfrak{g}}) \simeq \text{Fun Op}_{L_G}(D).$$

Here ${}^L G$ is the adjoint type Lie group associated to the Lie algebra ${}^L \mathfrak{g}$. ${}^L \mathfrak{g}$ is the Kac-Moody algebra whose Cartan matrix is the transpose of the Cartan matrix of \mathfrak{g} .

Example 5.2.4. When $\mathfrak{g} = \mathfrak{sl}_2$, $p_{-1} = f$, $2\check{\rho} = h$, $p_1 = e$. A PGL_2 -oper is the spaces of operators of the form

$$\partial_t + \begin{pmatrix} 0 & v(t) \\ 1 & 0 \end{pmatrix}, \quad v(t) \in \mathbb{C}[[t]].$$

Therefore, $\text{Fun Op}_{PGL_2}(D) = \mathbb{C}[v_n]_{n \geq 0}$ if we write $v(t) = \sum_{n \geq 0} v_n t^n$. Under a change a coordinate $t = \varphi(s)$, the oper becomes

$$\partial_s + \begin{pmatrix} 0 & \varphi'(s)v(\varphi(s)) \\ \varphi'(s) & 0 \end{pmatrix}.$$

We make it into the form $\partial_s + p_{-1} + \check{\mathbf{v}}(s)$, $\check{\mathbf{v}}(s) \in \mathfrak{b}[[t]]$ by gauge transformation by $g = \begin{pmatrix} \varphi'(s)^{1/2} & 0 \\ 0 & \varphi'(s)^{-1/2} \end{pmatrix}$,

and the resulting oper is

$$\partial_s + \begin{pmatrix} -\frac{1}{2} \frac{\varphi''(s)}{\varphi'(s)} & \varphi'(s)^2 v(\varphi(s)) \\ 1 & \frac{1}{2} \frac{\varphi''(s)}{\varphi'(s)} \end{pmatrix}.$$

It can be further made into the canonical form by the gauge transformation by $\begin{pmatrix} 1 & \frac{1}{2} \frac{\varphi''(s)}{\varphi'(s)} \\ 0 & 1 \end{pmatrix}$.

5.3 Miura transformation

Definition 5.3.1. A G -oper on $D = \text{Spec } \mathbb{C}[[t]]$ with regular singularity at 0 is by definition an element in

$$\left\{ \partial_t + \frac{1}{t}(p_{-1} + \mathbf{v}(t)) \right\} / N[[t]], \quad \mathbf{v}(t) \in \mathfrak{b}[[t]].$$

Denote by $\text{Op}_G^{RS}(D)$ the space of G -opers on D with regular singularity. In particular, it is a subspace of $\text{Op}_G(D^\times)$.

The residue of an oper in $\text{Op}_G^{RS}(D)$ is defined to be the projection of $p_{-1} + \mathbf{v}(0)$ onto $\mathfrak{g}/G := \text{Spec Fun}(\mathfrak{g})^G$. Note that under the gauge transformation by $g(t) \in N[[t]]$, $p_{-1} + \mathbf{v}(0)$ is conjugated by $g(0)$. So the residue in \mathfrak{g}/G is well-defined. We identify $\text{Fun}(\mathfrak{g})^G \simeq \text{Fun}(\mathfrak{h})^W$ and denote $\varpi : \mathfrak{h} \rightarrow \mathfrak{h}/W$ the projection. For $\check{\lambda} \in \mathfrak{h}$, define $\text{Op}_G^{RS}(D)_{\check{\lambda}}$ to be the space of G -opers with regular singularities with residue $\varpi(-\check{\lambda} - \check{\rho})$.

Generally, we define an G -oper on \mathbb{P}^1 with multiple regular singularities at z_1, \dots, z_n to be an oper on $\mathbb{P}^1 \setminus \{z_1, \dots, z_n\}$ whose restriction to neighborhood of each z_i is an oper with regular singularity. Denote by $\text{Op}_G^{RS}(\mathbb{P}^1)_{z_1, \dots, z_n}$ the space of G -opers on \mathbb{P}^1 with regular singularities. When $\{z_1, \dots, z_n\} = \{z_1, \dots, z_N, \infty\}$, we simplify the notation by $\text{Op}_G^{RS}(\mathbb{P}^1)_{(z)}$. We also denote by $\text{Op}_G^{RS}(\mathbb{P}^1)_{(z), (\check{\lambda})}$ to be the space of G -oper on \mathbb{P}^1 with regular singularities at z_1, \dots, z_N, ∞ with residues $\varpi(-\check{\lambda}_1 - \check{\rho}), \dots, \varpi(-\check{\lambda}_N - \check{\rho}), \varpi(-\check{\lambda}_\infty - \check{\rho})$.

Define the H -bundle $\Omega^{\check{\rho}}$ on X determined by the following property: for any character $\lambda : H \rightarrow \mathbb{C}^\times$, the line bundle $\Omega^{\check{\rho}} \times_H \mathbb{C}_\lambda = \Omega^{\langle \lambda, \check{\rho} \rangle}$, i.e. it is the H -bundle so that in local coordinate, its sections are $\mathbf{s}(t) = \sum_{i=1}^l \check{\omega}_i s_i(t)$ with $s_i(t)$ transforming like a one-form.

Definition 5.3.2. Denote by $\text{Conn}(\Omega^{\check{\rho}}, X)$, or by $\text{Conn}(X)$ for short, the space of connections on $\Omega^{\check{\rho}}$ over X . In particular, when X is the formal disc $D = \text{Spec } \mathbb{C}[[t]]$ with coordinate t , a connection in $\text{Conn}(D)$ is presented by a connection form $\partial_t + \mathbf{u}(t)$, where $\mathbf{u}(t) \in \mathfrak{h}[[t]]$. We also define the space $\text{Conn}(D)^{RS}$ of Cartan connections on D with regular singularity to be the connections whose operator has the form $\check{\nabla} = \partial_t + \frac{\check{\lambda}}{t} + \mathbf{u}(t)$. $\check{\lambda} \in \mathfrak{h}$ is called the residue of the connection.

For each connection $\check{\nabla} \in \text{Conn}(X)$, we associate to it an oper as follows: Choose a splitting of $B \rightarrow H$ and set $\mathcal{F} = \Omega^{\check{\rho}} \times_H G$, $\mathcal{F}_B = \Omega^{\check{\rho}} \times_H B$ and locally $\nabla = \check{\nabla} + p_{-1}$. We note that under the gauge transformation on ∇ by $B[[t]]$, this map is independent of the choice of the splitting.

Definition 5.3.3. The Miura transformation $\text{Conn}(D) \rightarrow \text{Op}_G(D)$ maps a connection to the associated oper as above. Similarly, we have $\text{Conn}(D)^{RS} \rightarrow \text{Op}_G^{RS}(D)$.

Remark 5.3.4. Notice that under a change of coordinate $t = \varphi(s)$, the Cartan connection $\partial_t + \mathbf{u}(t)$ becomes

$$\partial_s + \varphi'(s)\mathbf{u}(\varphi(s)) - \check{\rho} \frac{\varphi''}{\varphi'}.$$

Because $\mathbf{u}(t) \in \mathfrak{h}$, this formula is the same as the coordinate change ofopers in (5.2.3). So, the Miura transformation is well-defined.

It immediately follows that the Miura transformation maps a connection with residue $\check{\lambda} \in \mathfrak{h}$ to an oper with residue $\varpi(\check{\lambda} - \check{\rho}) \in \mathfrak{h}/W$.

Example 5.3.5. When $\mathfrak{g} = \mathfrak{sl}_2$, connections in $\text{Conn}(D)$ have the form

$$\partial_t + \begin{pmatrix} \frac{1}{2}u(t) & 0 \\ 0 & -\frac{1}{2}u(t) \end{pmatrix}, \quad u(t) \in \mathbb{C}[[t]].$$

The Miura transformation maps this connection to an oper in $\text{Op}_{\mathfrak{sl}_2}(D)$ which is the gauge conjugation class of

$$\partial_t + \begin{pmatrix} \frac{1}{2}u(t) & 0 \\ 1 & -\frac{1}{2}u(t) \end{pmatrix}.$$

This oper can be presented by the operator

$$\partial_t + \begin{pmatrix} 0 & \frac{1}{4}u(t)^2 + \frac{1}{2}\partial_t u(t) \\ 1 & 0 \end{pmatrix}.$$

We will also use connections on the bundle $\Omega^{-\check{\rho}}$. Note that $\text{Conn}(\Omega^{\check{\rho}}, D) \simeq \text{Conn}(\Omega^{-\check{\rho}}, D)$ maps $\partial_t + \mathbf{u}(t)$ to $\partial_t - \mathbf{u}(t)$.

Remark 5.3.6. Remark that the Miura transformation is the spectrum of the affine Harish-Chandra homomorphism in the following sense: $\text{Conn}(\Omega^{-\check{\rho}}, D) = \{\partial_t + \mathbf{u}(t)\}$, write $\sum_{n < 0} u_{i,n} z^{-n-1} = u_i(z) = \langle \mathbf{u}(z), h_i \rangle$. Thus $\text{Fun Conn}(\Omega^{-\check{\rho}}, D) = \mathbb{C}[u_{i,n}]$ is identified with $U(t^{-1}\mathfrak{h}[t^{-1}])$, mapping $u_{i,n}$ to the generator $b_{i,n}$. Taking spectrum of the affine Harish-Chandra homomorphism $\mathfrak{z}(\hat{\mathfrak{g}}) \rightarrow U(t^{-1}\mathfrak{h}[t^{-1}])$ gives the homomorphism $\text{Conn}(\Omega^{-\check{\rho}}, D) \rightarrow \text{Spec } \mathfrak{z}(\hat{\mathfrak{g}}) \simeq \text{Op}_{L_G}(D)$. This is the Miura transformation under the identification $\text{Conn}(\Omega^{\check{\rho}}, D) \simeq \text{Conn}(\Omega^{-\check{\rho}}, D)$.

We consider the Langlands dual Lie algebra ${}^L\mathfrak{g}$ and the adjoint type Lie group ${}^L G$ associated to ${}^L\mathfrak{g}$. Fix $\lambda_i \in {}^L\mathfrak{h} \simeq \mathfrak{h}^*$ weights and $\alpha_{i_j} \in {}^L\mathfrak{h} \simeq \mathfrak{h}^*$ simple roots. Consider connections on \mathbb{P}^1 on $\Omega^{-\rho}$ with regular singularities whose restriction to $\mathbb{P}^1 \setminus \infty$ has the form

$$\bar{\nabla} = \partial_t + \chi(t) = \partial_t + \sum_{i=1}^N \frac{\lambda_i}{t - z_i} - \sum_{j=1}^m \frac{\alpha_{i_j}}{t - w_j} \quad (5.3.1)$$

Under the change of coordinate $z = \frac{1}{t}$, we have

$$\bar{\nabla} = \partial_z - z^{-2}\chi(z^{-1}) - 2\rho z^{-1}.$$

Thus $\chi(t)$ has an expansion at infinity

$$\chi_\infty(t) = (-\sum \lambda_i + \sum \alpha_{i_j} - 2\rho)t^{-1} + \text{regular terms}.$$

Denote $\lambda_\infty = -\sum \lambda_i + \sum \alpha_{i_j} - 2\rho$.

Definition 5.3.7. Define $\text{Conn}(\mathbb{P}^1)_{(z);(\lambda)}^{gen}$ to be the space of connections of the form (5.3.1) whose Miura transformation lies in the space $\text{Op}_{L_G}^{RS}(\mathbb{P}^1)_{(z);(\lambda)}$.

This means that the possible singularities w_j , $j = 1, \dots, m$, of its Miura transformation are regular points.

Theorem 5.3.8. [Fre04, Proposition 4.10.] *There is a bijection between the set of solutions of the Bethe ansatz equations (3.2.4) and the set $\text{Conn}(\mathbb{P}^1)_{(z);(\lambda)}^{gen}$.*

Gaudin algebras can be described by opers according to the following important theorem.

Theorem 5.3.9. [Fre04, Theorem 2.7.] *The algebra $\mathcal{A}(z_1, \dots, z_N)$ is isomorphic to the algebra $\text{Fun Op}_{L_G}^{RS}(\mathbb{P}^1)_{(z)}$.*

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