Introduction to modulo p Langlands program and local-global compatibility

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1 Introduction

We give a brief introduction to the modulo p Langlands program which leads to the author's thesis project.

Let p be a prime number. Let L be a finite extension of $\mathbb{Q}_p$ with ring of integers $\mathcal{O}_L$. We fix a uniformizer ϖ_L of $\mathcal{O}_L$. We denote the residue field of $\mathcal{O}_L$ by $k_L = \mathbb{F}_q$, where $q = p^f$ with $f = [k_L : \mathbb{F}_p]$. Let $\overline{\mathbb{Q}_p}$ be an algebraic closure of $\mathbb{Q}_p$, and all finite extensions of $\mathbb{Q}_p$ are considered as subfields of $\overline{\mathbb{Q}_p}$. Let $G_L = \mathrm{Gal}(\overline{\mathbb{Q}_p}/L)$ be the absolute Galois group of L. Each element of G_L acts on the ring of integers $\overline{\mathbb{Z}_p}$ of $\overline{\mathbb{Q}_p}$, which is a local ring with maximal ideal $\mathfrak{m}_{\overline{\mathbb{Z}_p}}$, and hence acts on the residue field $\overline{\mathbb{Z}_p}/\mathfrak{m}_{\overline{\mathbb{Z}_p}} \cong \overline{\mathbb{F}_p}$, where $\overline{\mathbb{F}_p}$ is an algebraic closure of $\mathbb{F}_p$ containing k_L. This gives a surjective map $G_L \xrightarrow{\alpha} \mathrm{Gal}(\overline{\mathbb{F}_p}/k_L)$. Let I_L be its kernel, called the inertia subgroup of G_L. We get an exact sequence

$$0 \to I_L \to G_L \xrightarrow{\alpha} \mathrm{Gal}(\overline{\mathbb{F}_p}/k_L) \to 0.$$

The Galois group $\mathrm{Gal}(\overline{\mathbb{F}_p}/k_L)$ is isomorphic to $\hat{\mathbb{Z}}$, and has a specific topological generator Fr^f, where $\mathrm{Fr} : x \mapsto x^p$ is the Frobenius map. An element of G_L is called a geometric Frobenius element if its image under α is Fr^{-f}. Let W_L be the inverse image of $\langle \mathrm{Fr}^f \rangle \subset \mathrm{Gal}(\overline{\mathbb{F}_p}/k_L)$ under α, called the Weil group of L.

The subject of this mémoire is to look for a modulo p Langlands correspondence between two sides: One side consists of continuous representations of $\mathrm{GL}_2(L)$ over an algebraically closed field of characteristic p, usually called "the GL_2 side". The other side consists of continuous 2-dimensional representations of the Galois group G_L over the same field, usually called "the Galois side".

Let us first recall some known results about Langlands correspondence. We begin with the case of $\mathrm{GL}_1(L)$, which is essentially the class field theory.

Theorem 1.1. *There is a canonical isomorphism of topological groups*

$$\mathrm{Art}_L : L^\times \xrightarrow{\sim} W_L^{\mathrm{ab}},$$

normalized such that uniformizers of L are sent to geometric Frobenius elements of G_L. Here W_L^{ab} is the maximal abelian quotient of W_L.

Remark 1.2. *The topology on W_L is defined by giving I_L the subspace topology of G_L and imposing that I_L is an open subgroup of W_L. In particular, this is not the subspace topology of G_L.*

See [Ser62] for more details on class field theory.

Definition 1.3. *Let E be a field. Let ρ be a continuous representation of a topological group G on an E-vector space V.*

We say that (ρ, V) is a smooth representation of G if each vector $v \in V$ is fixed by an open subgroup of G.

We say that a smooth representation (ρ, V) of G is admissible if for each open subgroup H of G, the subspace V^H of elements fixed by H is finite-dimensional over E.

Remark 1.4. *We may regard E as given the discrete topology, so that smooth just means continuous.*

Theorem 1.1 has the following consequence, which is the local Langlands correspondence for $\mathrm{GL}_1(L)$.

Corollary 1.5. *Let E be an algebraically closed field. There is a canonical bijection between isomorphism classes of smooth irreducible 1-dimensional representations of W_L over E and isomorphism classes of smooth irreducible admissible representations of $\mathrm{GL}_1(L)$ over E.*

Remark 1.6. *Any 1-dimensional representation of W_L factors through its abelianization W_L^{ab}. Also, using that an endomorphism on a finite-dimensional E-vector space always has non-zero eigenvectors, one deduces that any smooth irreducible admissible representation of $\mathrm{GL}_1(L)$ over E is necessarily 1-dimensional.*

Now we turn to the case of $\mathrm{GL}_2(L)$. We first define what is a supercuspidal representation of $\mathrm{GL}_2(L)$. These are the building blocks of general smooth admissible representations of $\mathrm{GL}_n(L)$.

Let $B(L) \subset \mathrm{GL}_2(L)$ be the Borel subgroup consisting of upper triangular matrices. Let $\chi : B(L) \to E^\times$ be a smooth character, here smooth means locally constant, or equivalently, continuous with respect to the discrete topology on $E^\times$. Then we can form the induction

$$\mathrm{ind}_{B(L)}^{\mathrm{GL}_2(L)} \chi := \{f : \mathrm{GL}_2(L) \to E \text{ smooth}, \ f(bg) = \chi(b)f(g) \text{ for all } b \in B(L), \ g \in \mathrm{GL}_2(L)\},$$

with its left action of $\mathrm{GL}_2(L)$ given by $(g'f)(g) = f(gg')$. These representations are called parabolic inductions. They are smooth admissible, and are irreducible for "most" χ.

Definition 1.7. *A smooth irreducible admissible representation of $\mathrm{GL}_2(F)$ over E is called supercuspidal if it is not a subquotient of a parabolic induction.*

We can state the local Langlands correspondence for $\mathrm{GL}_2(L)$.

Theorem 1.8. *Let E be an algebraically closed field of characteristic 0. There is a canonical bijection between isomorphism classes of smooth irreducible 2-dimensional representations of W_L over E and isomorphism classes of smooth irreducible admissible supercuspidal representations of $\mathrm{GL}_2(L)$ over E.*

See [HT01] and [Hen00] for more details.

Remark 1.9. *In fact, [HT01] and [Hen00] extend theorem 1.8 to include all smooth irreducible admissible represenations on the GL_2 side, and they correspond to Frobenius-semisimple Weil-Deligne representations on the Galois side, which I'm not going to define. Also, this result can be extended to GL_n, called the local Langlands correspondence for GL_n.*

Remark 1.10. *The local Langlands correspondence is compatible with cohomology, in the sense that there exist towers of Shimura varieties $(S(U))_U$ over L of dimension d indexed by compact open subgroups U of $\mathrm{GL}_n(L)$, on which $\mathrm{GL}_n(L)$ acts on the right and such that the natural action of $\mathrm{GL}_n(L) \times G_L$ on the inductive limit of ℓ-adic étale cohomology groups*

$$\varinjlim_U H^d_{\acute{e}t}(S(U) \times_L \overline{\mathbb{Q}_p}, \overline{\mathbb{Q}_\ell})$$

makes it a direct sum of representations $\pi \otimes \rho$, where π matches ρ by the local Langlands correspondence. See [HT01] for more details and the notion of Shimura varieties. See [Mil80] for étale cohomology groups.

Now we turn to the case that $\mathrm{char}\, E \neq 0$. When $\mathrm{char}\, E = \ell$ with $\ell \neq p$, by the result of [Vig01], theorem 1.8 goes through without change.

In the remaining of the mémoire, we will study the case that $\mathrm{char}\, E = p$, which is much more difficult.

First we fix the coefficient field. When $E = \overline{\mathbb{F}_p}$, all smooth finite-dimensional representations of W_L over E extends to G_L, so that we can forget about the Weil group. In the remaining of the mémoire, we let $\mathbb{F}$ be a large enough finite extension of $\mathbb{F}_p$ inside $\overline{\mathbb{F}_p}$. This is going to be the coefficient field of all the representations.

The first case of the modulo p Langlands correspondence is when $L = \mathbb{Q}_p$. In this case, by the work of Barthel-Livné ([BL95],[BL94]) and Breuil ([Bre03]), there is a classification of smooth irreducible admissible representations of $\mathrm{GL}_2(\mathbb{Q}_p)$, and theorem 1.8 still holds. We give this correspondence in section 2.

In case $L \neq \mathbb{Q}_p$, the representation theory of $\mathrm{GL}_2(L)$ is much more difficult. We don't even have a classification of smooth irreducible admissible representations of $\mathrm{GL}_2(L)$. In fact, there are much more representations of $\mathrm{GL}_2(L)$ than those of G_L, so it is not possible to look for a one-to-one correspondence as in the case of $\mathrm{GL}_2(\mathbb{Q}_p)$. Instead, inspired by remark 1.10, we try to study the represenations of $\mathrm{GL}_2(L)$ that come from cohomology, and expect that this gives a well-behaved correspondence. We turn to the global settings in section 3.

The representations of $\mathrm{GL}_2(L)$ that come from cohomology are constructed globally. In section 4, we give the local constructions by Breuil and Paškūnas ([BP12]) of some representations of $\mathrm{GL}_2(L)$ which are the possible candidates for those coming from cohomology.

Finally, in section 5, we talk about some recent development on the understanding of the representations of $\mathrm{GL}_2(L)$ that come from cohomology, and talk about the author's thesis project.

2 Modulo p Langlands correspondence for $\mathrm{GL}_2(\mathbb{Q}_p)$

We keep the notation of section 1 and start with an arbitrary L. First we recall 2-dimensional Galois representations of G_L. Then we turn to smooth irreducible admissible representations of $\mathrm{GL}_2(L)$ following [BL95] and [BL94]. Then we specialize to the case $L = \mathbb{Q}_p$, and give an explicit modulo p Langlands correspondence for $\mathrm{GL}_2(\mathbb{Q}_p)$ following [Bre03]. Finally we give some complements of this correspondence.

We begin with the Galois side. Fix an embedding $\iota : \mathbb{F}_{q^2} \hookrightarrow \mathbb{F}$ (recall that we assumed $\mathbb{F}$ is large enough), and the composite $\mathbb{F}_q \hookrightarrow \mathbb{F}_{q^2} \overset{\iota}{\hookrightarrow} \mathbb{F}$ is still denoted by ι. Let $m \in \{f, 2f\}$. We define Serre's fundamental character of level m.

Definition 2.1. *Let $\omega_m : I_L \to \mathbb{F}^\times$ be the composite*

$$I_L \to \mu_{p^m-1}(\overline{\mathbb{Z}_p}) \xrightarrow{\sim} \mathbb{F}_{p^m}^\times \overset{\iota}{\hookrightarrow} \mathbb{F}^\times,$$

where $\mu_{p^m-1}(\overline{\mathbb{Z}_p})$ is the group of (p^m-1)-th roots of unity in $\overline{\mathbb{Z}_p}$. The first map is given by $g \mapsto \frac{g(\sqrt[p^m-1]{\varpi_L})}{\sqrt[p^m-1]{\varpi_L}}$, where $\sqrt[p^m-1]{\varpi_L} \in \overline{\mathbb{Z}_p}$ is any choice of (p^m-1)-th root of ϖ_L. The second map is reduction modulo the maximal ideal $\mathfrak{m}_{\overline{\mathbb{Z}_p}}$ of $\overline{\mathbb{Z}_p}$. We call ω_m the Serre's fundamental character of level m.

This definition is independent of the choice of ϖ_L and $\sqrt[p^m-1]{\varpi_L}$. We have $\omega_{2f}^{q+1} = \omega_f$, and ω_m has order $p^m - 1$. Moreover, the character ω_f can be extended to a character of G_L to be the composite

$$G_L \to \mu_{p^f-1}(\overline{\mathbb{Z}_p}) \xrightarrow{\sim} \mathbb{F}_{p^f}^\times \overset{\iota}{\hookrightarrow} \mathbb{F}^\times,$$

where the first map is given by the same formula $g \mapsto \frac{g(\sqrt[p^f-1]{\varpi_L})}{\sqrt[p^f-1]{\varpi_L}}$. This definition is independent of the choice of $\sqrt[p^m-1]{\varpi_L}$ but depend on the choice of ϖ_L. We still denote this character by ω_f.

It is elementary to have the following classification of 2-dimensional representations of G_L.

Proposition 2.2. *Let $\overline{\rho} : G_L \to \mathrm{GL}_2(\mathbb{F})$ be a continuous representation. Then $\overline{\rho}$ is one of the following forms:*

(1) $\overline{\rho}$ is reducible, and

$$\overline{\rho}\big|_{I_L} \cong \begin{pmatrix} \omega_f^{\sum_{i=0}^{f-1}(r_i+1)p^i} & * \\ 0 & 1 \end{pmatrix} \otimes \eta,$$

where η is a character that extends to G_L, and r_i are integers such that $-1 \le r_i \le p-2$ and $(r_0, \dots, r_{f-1}) \neq (p-2, \dots, p-2)$.

(2) $\overline{\rho}$ is irreducible, and

$$\overline{\rho}\big|_{I_L} \cong \begin{pmatrix} \omega_{2f}^{\sum_{i=0}^{f-1}(r_i+1)p^i} & 0 \\ 0 & \omega_{2f}^{q\sum_{i=0}^{f-1}(r_i+1)p^i} \end{pmatrix} \otimes \eta,$$

where η is a character that extends to G_L, and r_i are integers such that $0 \le r_0 \le p-1$, $-1 \le r_i \le p-2$ for $i > 0$, and $(r_0, \dots, r_{f-1}) \neq (p-1, p-2, \dots, p-2)$. Moreover, we have $\overline{\rho} \cong \mathrm{ind}_{G_{L'}}^{G_L} \omega_{2f}^{\sum_{i=0}^{f-1}(r_i+1)p^i} \otimes \eta$, where L' is the unramified extension of L of degree 2, and ω_{2f} canonically extends to $G_{L'}$ once we fix our uniformizer ϖ_L.

Remark 2.3. *The digits r_i are not uniquely determined. Also, we use $(r_i + 1)$ in the p-adic expansion in order to make it fit well into the Serre's conjecture to be discussed in section 3.*

Now we turn to the GL_2 side. We first consider the maximal compact subgroup $K := \mathrm{GL}_2(\mathcal{O}_L)$ inside $\mathrm{GL}_2(L)$. Also, we define $K_1 = \mathrm{Ker}(\mathrm{GL}_2(\mathcal{O}_L) \twoheadrightarrow \mathrm{GL}_2(\mathbb{F}_q))$ to be the first congruence subgroup of K.

Definition 2.4. *A Serre weight of K is a continuous irreducible represenation of K over $\mathbb{F}$.*

For any integer $r \in \mathbb{N}$, consider the $(r+1)$-dimensional representation of $\mathrm{GL}_2(\mathbb{F}_q)$: $\mathrm{Sym}^r \mathbb{F}^2$, where $\mathrm{GL}_2(\mathbb{F}_q)$ acts naturally on $\mathbb{F}^2$ via the embedding $\iota : \mathbb{F}_q \hookrightarrow \mathbb{F}$ using the canonical $\mathbb{F}$-basis of $\mathbb{F}^2$. This representation can be identified with $\bigoplus_{i=0}^{r} \mathbb{F} x^{r-1} y^i$, where the action of $\mathrm{GL}_2(\mathbb{F}_q)$ is given by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} x^{r-1} y^i = (ax+cy)^{r-i}(bx+dy)^i$$

via the embedding $\iota : \mathbb{F}_q \hookrightarrow \mathbb{F}$. For $0 \le j \le f-1$, we denote by $(\mathrm{Sym}^r \mathbb{F}^2)^{\mathrm{Fr}^j}$ the $(r+1)$-dimensional representation $\bigoplus_{i=0}^{r} \mathbb{F}x^{r-1}y^i$ with the action of $\mathrm{GL}_2(\mathbb{F}_q)$ given by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} x^{r-1}y^i = (a^{p^j}x + c^{p^j}y)^{r-i}(b^{p^j}x + d^{p^j}y)^i.$$

Since K_1 is a pro-p normal subgroup of K, any continuous irreducible representation of K factor through the quotient $K/K_1 \cong \mathrm{GL}_2(\mathbb{F}_q)$. By the modular representation theory due to Brauer (see [Ser77] for more details), we get the following classification of Serre weights.

Proposition 2.5. *The isomorphism classes of irreducible representations of* $\mathrm{GL}_2(\mathbb{F}_q)$ *over* $\mathbb{F}$ *are the following:*

$$(\mathrm{Sym}^{r_0} \mathbb{F}^2) \otimes_{\mathbb{F}} (\mathrm{Sym}^{r_1} \mathbb{F}^2)^{\mathrm{Fr}} \otimes_{\mathbb{F}} \cdots \otimes_{\mathbb{F}} (\mathrm{Sym}^{r_{f-1}} \mathbb{F}^2)^{\mathrm{Fr}^{f-1}} \otimes_{\mathbb{F}} \det{}^m,$$

where $0 \le r_i \le p-1$ *and* $0 \le m < q-1$.

For simplicity, we denote by $(r_0, \ldots, r_{f-1}) \otimes \det^m$ the representations in Proposition 2.5. When $m = 0$, we also denote it by $\mathrm{Sym}^{\vec{r}} \mathbb{F}^2$.

Now we can use compact induction to pass to representations of $\mathrm{GL}_2(L)$.

Definition 2.6. *Let H be a closed subgroup of $\mathrm{GL}_2(L)$. Let σ be a smooth representation of H on a finite-dimensional $\mathbb{F}$-vector space V_σ. We define $\mathrm{ind}_H^{\mathrm{GL}_2(L)} \sigma$ to be the $\mathbb{F}$-vector space of functions $f : \mathrm{GL}_2(L) \to V_\sigma$ which are locally constant, compact support modulo H, and satisfy $f(hg) = \sigma(h)f(g)$ for all $h \in H$, $g \in \mathrm{GL}_2(L)$. The action of $\mathrm{GL}_2(L)$ is defined by $(g'f)(g) = f(gg')$.*

Let Z be the center of $\mathrm{GL}_2(L)$, which is identified with the scalar matrices in $\mathrm{GL}_2(L)$, and is isomorphic to $L^\times$. For each Serre weight σ of K, we view it as a representation of KZ by letting ϖ_L act trivially. Then we can consider the representation $\mathrm{ind}_{KZ}^{\mathrm{GL}_2(L)} \sigma$.

Definition 2.7. *We define the Hecke algebra of $\mathrm{ind}_{KZ}^{\mathrm{GL}_2(L)} \sigma$ to be*

$$\mathcal{H}(KZ, \sigma) := \mathrm{End}(\mathrm{ind}_{KZ}^{\mathrm{GL}_2(L)} \sigma).$$

Remark 2.8. *By Frobenius reciprocity (see [Ser77] for more details), $\mathcal{H}(KZ, \sigma)$ is identified with the space of functions $\varphi : \mathrm{GL}_2(L) \to \mathrm{End}_{\mathbb{F}} V_\sigma$ which have compact support modulo Z, and satisfy $\varphi(h_1 g h_2) = \sigma(h_1) \circ \varphi(g) \circ \sigma(h_2)$ for all $h_1, h_2 \in KZ$, $g \in \mathrm{GL}_2(L)$.*

For each Serre weight $\sigma = (r_0, \ldots, r_{f-1}) \otimes \det^m$, its representation space can be identified with

$$V_\sigma = \bigoplus_{i_0=0}^{r_0} \bigoplus_{i_1=0}^{r_1} \cdots \bigoplus_{i_{f-1}=0}^{r_{f-1}} \mathbb{F} x^{\sum_{j=0}^{f-1}(r_j - i_j)p^j} y^{\sum_{j=0}^{f-1} i_j p^j}.$$

By Cartan decomposition (see [BZ76] for the definition), we deduce the structure of the Hecke algebras.

Proposition 2.9. *We have $\mathcal{H}(KZ, \sigma) \cong \mathbb{F}[T]$, where T corresponds by Remark 2.8 to the function $\varphi : \mathrm{GL}_2(L) \to \mathrm{End}_{\mathbb{F}} V_\sigma$ defined as follows:*

(1) φ is supported on the double coset $KZ \begin{pmatrix} 1 & 0 \\ 0 & \varpi_L^{-1} \end{pmatrix} K$.

(2) $\varphi(\begin{pmatrix} 1 & 0 \\ 0 & \varpi_L^{-1} \end{pmatrix})(x^{\sum_{j=0}^{f-1}(r_j - i_j)p^j} y^{\sum_{j=0}^{f-1} i_j p^j}) = 0$ if $(i_j) \neq (r_j)$.

(3) $\varphi(\begin{pmatrix} 1 & 0 \\ 0 & \varpi_L^{-1} \end{pmatrix})(y^{\sum_{j=0}^{f-1} r_j p^j}) = y^{\sum_{j=0}^{f-1} r_j p^j}$.

Now we can state the classification of smooth irreducible admissible representations of $\mathrm{GL}_2(L)$ by Barthel-Livné ([BL95] and [BL94]). For simplicity, we denote

$$\pi(\vec{r}, \lambda, \eta) := \frac{\mathrm{ind}_{KZ}^{\mathrm{GL}_2(L)} \mathrm{Sym}^{\vec{r}} \mathbb{F}^2}{(T - \lambda)} \otimes (\eta \circ \det)$$

for $0 \leq r_i \leq p-1$, $\lambda \in \mathbb{F}$, and $\eta : L^\times \to \mathbb{F}^\times$ a continuous character. We denote the character $\eta \circ \mathrm{Art}_L^{-1} : G_L \to \mathbb{F}^\times$ also by η.

Theorem 2.10. *The smooth irreducible admissible representations of $\mathrm{GL}_2(L)$ are one of the following:*

(1) Characters: $\eta \circ \det$.

(2) Principal series: $\pi(\vec{r}, \lambda, \eta)$, where $\lambda \neq 0$ and $(\vec{r}, \lambda) \notin \{(\vec{0}, \pm 1), (\vec{p} - \vec{1}, \pm 1)\}$.

(3) Special series: $\mathrm{Ker}(\pi(0, 1, \eta) \twoheadrightarrow \eta \circ \det)$.

(4) Supersingular representations: The irreducible subquotients of $\pi(\vec{r}, 0, \eta)$.

Remark 2.11. *For each $x \in \mathbb{F}^\times$, let $\mu_x : L \to \mathbb{F}^\times$ be the unramified character sending ϖ_L to x. In case (2) of Theorem 2.10, we have*

$$\pi(\vec{r}, \lambda, \eta) \cong \mathrm{ind}_{B(L)}^{\mathrm{GL}_2(L)} (\mu_{\lambda^{-1}} \otimes \omega_f^{\vec{r}} \mu_\lambda),$$

where $\omega_f^{\vec{r}}$ means $\omega_f^{\sum_{j=0}^{f-1} r_j p^j}$. This is why we call them principal series.

Now we speclize to the case that Ł $= \mathbb{Q}_p$. In this case, by a result of Breuil ([Bre03]), the representations $\pi(r, 0, \eta)$ are irreducible. Combine Proposition 2.2 and Theorem 2.10, we get the modulo p Langlands correspondence for $\mathrm{GL}_2(\mathbb{Q}_p)$.

Theorem 2.12. *There is a canonical bijection between isomorphism classes of smooth irreducible 2-dimensional representations of $G_{\mathbb{Q}_p}$ over $\mathbb{F}$ and isomorphism classes of smooth irreducible admissible supersingular representations of $\mathrm{GL}_2(\mathbb{Q}_p)$ over $\mathbb{F}$, given by*

$$\mathrm{ind}_{G_{\mathbb{Q}_{p^2}}}^{G_{\mathbb{Q}_p}} \omega_2^{r+1} \otimes \eta \leftrightarrow \pi(r, 0, \eta),$$

where $G_{\mathbb{Q}_{p^2}}$ is the unramified extension of $\mathbb{Q}_p$ of degree 2.

Let $\omega : G_{\mathbb{Q}_p} \to \mathbb{F}^\times$ be the fundamental character ω_1 determined by taking $\varpi_{\mathbb{Q}_p} = -p$, so that ω is the mod p cyclotomic character (see [BC09] for the definition). Theorem 2.12 can be immediately generalized to a "semisimple modulo p correspondence" between semisimple 2-dimensional representations of $G_{\mathbb{Q}_p}$ and certain semisimple smooth admissible representations of $\mathrm{GL}_2(\mathbb{Q}_p)$, given by

$$\begin{pmatrix} \omega^{r+1}\mu_\lambda & 0 \\ 0 & \mu_{\lambda^{-1}} \end{pmatrix} \otimes \eta \leftrightarrow \pi(r, \lambda, \eta)^{\mathrm{ss}} \oplus \pi([p-3-r], \lambda^{-1}, \omega^{r+1}\eta)^{\mathrm{ss}},$$

where $\lambda \in \mathbb{F}^\times$, "ss" means semisimplification, and $[p-3-r]$ is the unique integer in $[0, p-2]$ which is congruent to $p-3-r$ modulo $p-1$.

This correspondence is compatible with the weight part of Serre's conjecture to be introduced in section 3.

Definition 2.13. *The socle of a representation of a group G is the maximal semisimple subrepresentations.*

Example 2.14. *We have*

$$\mathrm{soc}_{\mathrm{GL}_2(\mathbb{Z}_p)}(\pi(r, 0, 1)) = \mathrm{Sym}^r \mathbb{F}^2 \oplus (\mathrm{Sym}^{p-1-r} \mathbb{F}^2 \otimes \det{}^r).$$

Also, we have

$$\mathrm{soc}_{\mathrm{GL}_2(\mathbb{Z}_p)}(\pi(r, \lambda, \eta)^{\mathrm{ss}} \oplus \pi([p-3-r], \lambda^{-1}, \omega^{r+1}\eta)^{\mathrm{ss}}) = \mathrm{Sym}^r \mathbb{F}^2 \oplus (\mathrm{Sym}^{p-3-r} \mathbb{F}^2 \otimes \det{}^{r+1})$$

if $r \leq p-3$.

To end the story of $\mathrm{GL}_2(\mathbb{Q}_p)$, we remark that the above modulo p correspondence for $\mathrm{GL}_2(\mathbb{Q}_p)$ has the following extension:

Theorem 2.15. *There exists an exact covariant functor from the category of smooth admissible finite length representations of $\mathrm{GL}_2(\mathbb{Q}_p)$ over $\mathbb{F}$ to the category of finite-dimensional continuous representations of $G_{\mathbb{Q}_p}$ over $\mathbb{F}$.*

The construction of this functor is due to Colmez ([Col10]), and uses (φ, Γ)-modules (see [Fon90] for the definition) in p-adic Hodge theory. Moreover, this functor is proved to be an equivalence of categories ([Paš13]).

When $L \neq \mathbb{Q}_p$, the representation $\pi(\vec{r}, 0, \eta)$ is much more difficult to understand. It has infinite length, and we don't have a complete classification of its irreducible subquotients. In particular, it is not possible to look for a one-to-one correspondence as in the case of $\mathrm{GL}_2(\mathbb{Q}_p)$. Instead, inspired by remark 1.10, we hope that the represenations of $\mathrm{GL}_2(L)$ that come from cohomology can give a well-behaved correspondence. We study this point of view and give partiel results in the remaining sections.

3 Global setup

In the remaining of the mémoire, we assume that L is unramified over $\mathbb{Q}_p$. We turn to the modulo p Langlands correspondence for $\mathrm{GL}_2(L)$ with $L \neq \mathbb{Q}_p$, and hope to find a nice correspondence via geometry. First we define the global geometric objects. Then we construct the representations of $\mathrm{GL}_2(L)$ which are expected to give a nice correspondence. Finally we state some conjectures and first properties on these representations.

We consider the global setting of Shimura curves. See [BD14] and the references therein for more details on Shimura curves. Let F be a totally real number field which is unramified at places above p. Let D be a quaternion algebra with center F which is split at places above p and at exactly one infinite place.

Let $\mathbb{A}_F^\infty$ be the ring of finite adèles. For each compact open subgroup U of $(D \otimes_F \mathbb{A}_F^\infty)^\times$, let X_U be the associated smooth projective Shimura curve over F.

Fix $\overline{r} : G_F \to \mathrm{GL}_2(\mathbb{F})$ a continuous (absolutely) irreducible totally odd representation, where totally odd means all complex conjugates have determinant -1.

We define a smooth admissible representation of $(D \otimes_F \mathbb{A}_F^\infty)^\times$ over $\mathbb{F}$ of the form

$$\pi := \varinjlim_{U} \mathrm{Hom}_{\mathrm{Gal}(\overline{F}/F)}(\overline{r}, H^1_{\text{ét}}(X_U \times_F \overline{F}, \mathbb{F})),$$

where the inductive limit runs over compact open subgroups U of $(D \otimes_F \mathbb{A}_F^\infty)^\times$, and $\overline{r}$ is chosen so that $\pi \neq 0$.

For each finite place w of F, let F_w be the completion of F with respect to w. Fix an embedding $\overline{F} \hookrightarrow \overline{F_w}$, which gives an embedding $G_{F_w} \hookrightarrow G_F$. Let $\overline{r}_w := \overline{r}\big|_{G_{F_w}}$. We have the following conjecture on π due to Buzzard-Diamond-Jarvis ([BDJ10]).

Conjecture 3.1. *There is a decomposition*

$$\pi = \otimes_w' \pi_w(\overline{r}),$$

where the restricted tensor product is taken over all finite places of F, and each $\pi_w(\overline{r})$ is a smooth admissible representation of $(D \otimes_F F_w)^\times$ over $\mathbb{F}$.

Remark 3.2. *(1) If $w \nmid p$, then the local factor $\pi_w(\overline{r})$ is predicted by [BDJ10].*

(2) If $w \mid p$, [BD14] gives a construction of $\pi_w(\overline{r})$. However, this construction is global, and until now there is no local construction of $\pi_w(\overline{r})$.

(3) If $F = \mathbb{Q}$, by the work of Emerton ([Eme11]), the conjecture is true. Moreover, the assignment $\overline{r}_p \mapsto \pi_p(\overline{r})$ is the modulo p Langlands correspondence for $\mathrm{GL}_2(\mathbb{Q}_p)$ stated in section 2.

Now we fix a place v over p. Fix an isomorphism $(D \otimes_F F_v)^\times \cong \mathrm{GL}_2(F_v)$. We denote $K_v = \mathrm{GL}_2(\mathcal{O}_{F_v})$, and $K_v(1)$ the first congruence subgroup of K_v.

One of the aims of the modulo p Langlands program is to understand the $\mathrm{GL}_2(F_v)$-representation $\pi_v(\overline{r})$. It is expected that $\pi_v(\overline{r})$ is local, meaning that it only depends on $\overline{r}_v$, not on any global settings. Moreover, it is expected that the assignment $\overline{r}_v \mapsto \pi_v(\overline{r})$ realizes a modulo p Langlands correspondence for $\mathrm{GL}_2(F_v)$, as in Remark 3.2(3).

The rest of the mémoire focuses on the $\mathrm{GL}_2(F_v)$-representation $\pi_v(\overline{r})$. It is very difficult to understand completely this representation. Instead, we try to understand certain pieces of this representation. We start with its K_v-socle, which is a direct sum of Serre weights of K_v. These Serre weights are predicted by [BDJ10]. This is thought of as a generalization of the Serre's conjecture ([Ser87]).

Although [BDJ10] makes the prediction of these Serre weights for all possible $\overline{r}_v$, we make the following additional assumption on $\overline{r}_v$ so that each Serre weight in $\mathrm{soc}_{K_v}\pi_v(\overline{r})$ has multiplicity one and we can express them in a nice way.

Let $\overline{\rho} : G_L \to \mathrm{GL}_2(\mathbb{F})$ be a continuous representation as in Proposition 2.2. Recall that we assume L to be unramified over $\mathbb{Q}_p$, for example F_v.

Definition 3.3. *We say that $\overline{\rho} : G_L \to \mathrm{GL}_2(\mathbb{F})$ is generic if one of the following holds:*

(1) $\overline{\rho}$ is reducible, and

$$\overline{\rho}\big|_{I_L} \cong \begin{pmatrix} \omega_f^{\sum_{i=0}^{f-1}(r_i+1)p^i} & * \\ 0 & 1 \end{pmatrix} \otimes \eta,$$

with $0 \le r_i \le p-3$, and $\vec{r} \neq \vec{0}, \vec{p}-\vec{3}$.

(2) $\overline{\rho}$ is irreducible, and

$$\overline{\rho}\big|_{I_L} \cong \begin{pmatrix} \omega_{2f}^{\sum_{i=0}^{f-1}(r_i+1)p^i} & 0 \\ 0 & \omega_{2f}^{q\sum_{i=0}^{f-1}(r_i+1)p^i} \end{pmatrix} \otimes \eta,$$

with $1 \le r_0 \le p-2$, and $0 \le r_i \le p-3$ for $i > 0$.

Now we define the set of Serre weights associated to $\overline{\rho}$.

Definition 3.4. *Assume that $\overline{\rho} : G_L \to \mathrm{GL}_2(\mathbb{F})$ is generic.*

If $\overline{\rho}$ is reducible, we define

$$W(\overline{\rho}) = \left\{ (s_0,\ldots,s_{f-1}) \otimes \theta \;\middle|\; \begin{array}{l} \text{there exists } J \subset \{0,\ldots,f-1\} \text{ with} \\ \overline{\rho}\big|_{I_L} \cong \begin{pmatrix} \omega_f^{\sum_{j\notin J}(s_j+1)p^j} & * \\ 0 & \omega_f^{\sum_{j\in J}(s_j+1)p^j} \end{pmatrix} \otimes \theta, \\ \text{where the extension ``*'' is Fontaine-Laffaille} \\ \text{in the sense of [FL82]} \end{array} \right\}.$$

If $\overline{\rho}$ is irreducible, we define

$$W(\overline{\rho}) = \left\{ (s_0,\ldots,s_{f-1}) \otimes \theta \;\middle|\; \begin{array}{l} \text{there exists } J \subset \{0,\ldots,f-1\} \text{ with} \\ \overline{\rho}\big|_{I_L} \cong \begin{pmatrix} \omega_{2f}^{\sum_{j\notin J}(s_j+1)p^j + q\sum_{j\in J}(s_j+1)p^j} & 0 \\ 0 & \omega_{2f}^{q(same)} \end{pmatrix} \otimes \theta \end{array} \right\}.$$

Example 3.5. *Let $f = 1$, and $\overline{\rho} : G_{\mathbb{Q}_p} \to \mathrm{GL}_2(\mathbb{F})$ generic.*

If $\overline{\rho}\big|_{I_L} \cong \begin{pmatrix} \omega^{r+1} & * \\ 0 & 1 \end{pmatrix}$ with $1 \leq r \leq p-4$, then

$$W(\overline{\rho}) = \begin{cases} \{\mathrm{Sym}^r \mathbb{F}^2, \mathrm{Sym}^{p-3-r} \mathbb{F}^2 \otimes \det^{r+1}\} & \text{if } * = 0, \\ \{\mathrm{Sym}^r \mathbb{F}^2\} & \text{if } * \neq 0. \end{cases}$$

If $\overline{\rho}\big|_{I_L} \cong \begin{pmatrix} \omega_2^{r+1} & 0 \\ 0 & \omega_2^{p(r+1)} \end{pmatrix}$ with $1 \leq r \leq p-2$, then

$$W(\overline{\rho}) = \{\mathrm{Sym}^r, \mathrm{Sym}^{p-1-r} \otimes \det^r\}.$$

Example 3.6. *We also give an example for $f = 2$, and $\overline{\rho}$ is split reducible and generic.*

If $\overline{\rho}\big|_{I_L} \cong \begin{pmatrix} \omega_2^{r_0+1+p(r_1+1)} & 0 \\ 0 & 1 \end{pmatrix}$, then

$$W(\overline{\rho}) = \left\{ \begin{matrix} (r_0, r_1), (r_0+1, p-2-r_1) \otimes \det^{p-1+pr_1}, (p-2-r_0, r_1+1) \otimes \det^{r_0+p(p-1)}, \\ (p-3-r_0, p-3-r_1) \otimes \det^{r_0+1+p(r_1+1)} \end{matrix} \right\}.$$

Remark 3.7. *In general, the set of Serre weights of $\overline{\rho}$ can be expressed nicely by combinatorics (see [BP12]), or by extension graph (see [LLHLM20],[LMS20]).*

Now we can state the following conjecture made by [BDJ10], usually called the weight part of Serre's conjecture.

Conjecture 3.8. *Let σ be a Serre weight of K_v. We have*

$$\mathrm{Hom}_K(\sigma, \pi_v(\overline{r})) \neq 0 \text{ if and only if } \sigma \in W(\overline{r}_v).$$

Remark 3.9. *When $\overline{r}_v$ is generic in the sense of Definition 3.3, we have the refined conjecture, which states that each Serre weight of $\overline{r}_v$ appears only once in the K_v-socle of $\pi_v(\overline{r})$, that is, $\mathrm{soc}_{K_v} \pi_v(\overline{r}) = \bigoplus_{\sigma \in W(\overline{r}_v)} \sigma$. In particular, it is local and multiplicity-free.*

Example 3.10. *Let $F = \mathbb{Q}$. Let $\overline{r} : G_{\mathbb{Q}} \to \mathrm{GL}_2(\mathbb{F})$ has the additional assumption that $\overline{r}_p : G_{\mathbb{Q}_p} \to \mathrm{GL}_2(\mathbb{F})$ is generic.*

If $\overline{r}_p \cong \mathrm{ind}_{G_{\mathbb{Q}_{p^2}}}^{G_{\mathbb{Q}_p}} \omega_2^{r+1}$ with $1 \leq r \leq p-2$, then by Example 3.5, we have

$$W(\overline{r}_p) = \{\mathrm{Sym}^r, \mathrm{Sym}^{p-1-r} \otimes \det^r\}.$$

By Remark 3.2(3) and Theorem 2.12, $\pi_p(\overline{r}) = \pi(r, 0, 1)$, and by Example 2.14, we have

$$\mathrm{soc}_{\mathrm{GL}_2(\mathbb{Z}_p)}(\pi_p(\overline{r})) = \mathrm{Sym}^r \mathbb{F}^2 \oplus (\mathrm{Sym}^{p-1-r} \mathbb{F}^2 \otimes \det^r),$$

which is exactly the direct sum of the Serre weights of $\overline{r}_p$.

If $\overline{r}_p \cong \begin{pmatrix} \omega^{r+1}\mu_\lambda & 0 \\ 0 & \mu_{\lambda^{-1}} \end{pmatrix}$ *with* $1 \leq r \leq p-4$ *and* $\lambda \in \mathbb{F}^\times$*, then by Example 3.5, we have*

$$W(\overline{r}_p) = \{\mathrm{Sym}^r, \mathrm{Sym}^{p-3-r} \otimes \det{}^{r+1}\}.$$

By Remark 3.2(3) and Theorem 2.12, $\pi_p(\overline{r}) = \pi(r, \lambda, 1) \oplus \pi(p-3-r, \lambda^{-1}, \omega^{r+1})$*, and by Example 2.14, we have*

$$\mathrm{soc}_{\mathrm{GL}_2(\mathbb{Z}_p)}(\pi_p(\overline{r})) = \mathrm{Sym}^r\, \mathbb{F}^2 \oplus (\mathrm{Sym}^{p-3-r}\, \mathbb{F}^2 \otimes \det{}^{r+1}),$$

which is exactly the direct sum of the Serre weights of $\overline{r}_p$*.*

4 The local construction by Breuil and Paškūnas

We keep the notation of the previous sections. In particular, L is an unramified extension of $\mathbb{Q}_p$. The local factor $\pi_v(\overline{r})$ introduced in section 3 is only constructed globally by [BD14], and is expected to be local. In another direction, Breuil and Paškūnas ([BP12]) give the local constructions of many representations of $\mathrm{GL}_2(L)$, which are the possible candidates for the local factor $\pi_v(\overline{r})$. We give their constructions in this section.

We give some more notations. Let $I \subset K$ be the subgroup of upper triangular matrices modulo p. $I_1 \subset I$ be its maximal pro-p subgroup, that is the subgroup of upper unipotent matrices modulo p. Let $N \subset \mathrm{GL}_2(L)$ be the normalizer of I in $\mathrm{GL}_2(L)$. This is the subgroup of $\mathrm{GL}_2(L)$ generated by K, Z and the matrix $\Pi = \begin{pmatrix} 0 & 1 \\ p & 0 \end{pmatrix}$. If $\chi : I \to \mathbb{F}$ is a continuous character, define $\chi^s = \chi(\Pi \cdot \Pi^{-1}) : I \to \mathbb{F}$.

We begin with the group theoretic constructions called the basic diagrams introduced by Paškūnas ([Paš04]) which are used to build smooth admissible representations of $\mathrm{GL}_2(L)$ with given K-socle.

Definition 4.1. *A diagram is a triple* (D_0, D_1, r)*, such that*

(1) D_0 *is a smooth representation of* KZ *over* $\mathbb{F}$ *on which* Z *acts trivially.*

(2) D_1 *is a smooth representation of* N *over* $\mathbb{F}$ *on which* Z *acts trivially.*

(3) $r : D_1 \to D_0$ *is* IZ*-equivariant, which induces an isomorphism* $D_1 \xrightarrow{\sim} D_0^{I_1}$*.*

Example 4.2. *Let* π *be a smooth representation of* $\mathrm{GL}_2(L)$ *on which* Z *acts trivially, then* $(\pi^{K_1}, \pi^{I_1}, \mathrm{can})$ *is a basic diagram, where* can *is the canonical injection* $\pi^{I_1} \hookrightarrow \pi^{K_1}$*.*

The following theorem gives the constructions of smooth admissible representations of $\mathrm{GL}_2(L)$ with given K-socle from a basic diagram.

Theorem 4.3. *Let* $p > 2$*. Let* (D_0, D_1, r) *be a basic diagram such that* $D_0 = D_0^{K_1}$ *is finite-dimensional over* $\mathbb{F}$*. Then there exists at least one smooth admissible representation* π *of* $\mathrm{GL}_2(L)$*, such that*

(1) $\mathrm{soc}_K\,\pi = \mathrm{soc}_K\,D_0$.

(2) $(\pi^{K_1}, \pi^{I_1}, \mathrm{can})$ *contains* (D_0, D_1, r) *in the obvious sense.*

(3) π *is generated by* D_0 *over* $\mathrm{GL}_2(L)$.

Remark 4.4. *The construction of* π *is not unique. There could be infinitely many representations* π *of* $\mathrm{GL}_2(L)$ *satisfying the three properties of Theorem 4.3 ([Hu10]).*

The study of the local factor $\pi_v(\overline{r})$, for example, the technique of weight cycling first discovered by Buzzard and first written in [Tay06], motivates the following constructions of basic diagrams.

Theorem 4.5. *Let* $\overline{\rho} : G_L \to \mathrm{GL}_2(\mathbb{F})$ *be a generic Galois representation in the sense of Definition 3.3.*

(1) There exists a unique finite-dimensional representation $D_0(\overline{\rho})$ *of* $\mathrm{GL}_2(\mathbb{F}_q)$ *over* $\mathbb{F}$*, such that*

  (a) $\mathrm{soc}_{\mathrm{GL}_2(\mathbb{F}_q)}\,D_0(\overline{\rho}) = \bigoplus_{\sigma \in W(\overline{\rho})} \sigma$.

  (b) each $\sigma \in W(\overline{\rho})$ *only occurs once as a Jordan-Hölder factor of* $D_0(\overline{\rho})$ *(hence in the socle).*

  (c) $D_0(\overline{\rho})$ *is maximal (for inclusion) for properties (a) and (b).*

(2) Each Jordan-Hölder factor of $D_0(\overline{\rho})$ *appears only once in* $D_0(\overline{\rho})$.

(3) As an I*-representations,*

$$D_0(\overline{\rho})^{I_1} \cong \bigoplus_{\substack{\textit{certain } (\chi, \chi^s) \\ \chi \neq \chi^s}} (\chi \oplus \chi^s)$$

In particular, $D_0(\overline{\rho})^{I_1}$ *is stable under* $\chi \mapsto \chi^s$.

Using (2) and (3) of Theorem 4.5, one can uniquely extend the action of I on $D_0(\overline{\rho})$ to an action of N. We denote $D_1(\overline{\rho})$ the resulting representation of N. Then for any IZ-equivariant injection $r : D_1(\overline{\rho}) \hookrightarrow D_0(\overline{\rho})$, we get a basic diagram $D(\overline{\rho}, r) := (D_0(\overline{\rho}), D_1(\overline{\rho}), r)$.

Combine Theorem 4.3 and Theorem 4.5, we get the following corollary.

Corollary 4.6. *Fix a basic diagram* $D(\overline{\rho}, r)$*. There exists a smooth admissible representation* π *of* $\mathrm{GL}_2(L)$ *over* $\mathbb{F}$*, such that*

(1) $\mathrm{soc}_K\,\pi = \bigoplus_{\sigma \in W(\overline{\rho})} \sigma$.

(2) $(\pi^{K_1}, \pi^{I_1}, \mathrm{can})$ *contains* $D(\overline{\rho}, r)$.

(3) π is generated by $D_0(\overline{\rho})$ over $\mathrm{GL}_2(L)$.

Remark 4.7. *Moreover, one can show that if $D(\overline{\rho}, r)$, $D(\overline{\rho}, r')$ are non-isomorphic, and π, π' are as in Corollary 4.6, then π and π' are non-isomorphic.*

Example 4.8. *Let $L = \mathbb{Q}_p$, and $\overline{\rho} = \mathrm{ind}_{G_{\mathbb{Q}_{p^2}}}^{G_{\mathbb{Q}_p}} \omega_2^{r+1}$ for $1 \leq r \leq p-2$. In this case, $D_0(\overline{\rho}) \cong E_r \oplus (E_{p-1-r} \otimes \det^r)$, where E_s is the unique nontrivial extension*

$$0 \to \mathrm{Sym}^s \mathbb{F}^2 \to E_s \to \mathrm{Sym}^{p-3-s} \otimes \det^{s+1} \to 0$$

for $s \leq p-3$, and $E_s = \mathrm{Sym}^s \mathbb{F}^2$ for $s = p-2$. In this case, there a unique choice of the injection $r : D_1(\overline{\rho}) \hookrightarrow D_0(\overline{\rho})$ up to isomorphism, hence a unique choice of diagrams $D(\overline{\rho}, r)$ up to isomorphism. Moreover, one can show that the $\mathrm{GL}_2(\mathbb{Q}_p)$-representation π in Corollary 4.6 must be $\pi(r, 0, 1)$ defined in section 2. In particular, we have

$$(\pi(r,0,1)^{K_1}, \pi(r,0,1)^{I_1}, \mathrm{can}) \cong D(\overline{\rho}, r)$$

Remark 4.9. *If $f > 1$, there are infinitely many choices of the injection r, hence infinitely many choices of diagrams $D(\overline{\rho}, r)$. Also, for each diagram $D(\overline{\rho}, r)$, there could be infinitely many choices of π in Corollary 4.6. We expect that the local factor $\pi_v(\overline{r})$ defined in section 3 is one of the representations π constructed in this way.*

5 Recent development

Keep the notation of the previous sections. In this section, we give some recent development of the study of the local factor $\pi_v(\overline{r})$ defined in section 3 which leads to the author's thesis project. Recall that we expect it to be local and realize a modulo p Langlands correspondence.

By remark 3.2, when $F = \mathbb{Q}$, the local factor is completely known by the work of Emerton ([Eme11]). More generally, as soon as $F_v = \mathbb{Q}_p$, it is reasonable to expect an analogous description ([CEG+18]).

In general, there are still some results on certain spaces of invariants attached to the representation $\pi_v(\overline{r})$ and some local-global compatibility related to these spaces. Recall that $K = \mathrm{GL}_2(\mathcal{O}_{F_v})$, and $K_1 = 1 + p\,\mathrm{M}_2(\mathcal{O}_{F_v})$. Let Z_1 be the center of K_1. Let $\mathbb{F}[[K_1/Z_1]]$ be the Iwasawa algebra of K_1/Z_1 (see [Laz65] for the definition), and $\mathfrak{m}_K$ be its maximal ideal. Under some mild assumptions, it is known that

1. The weight part of Serre's conjecture (Conjecture 3.8) is true (see [GK14], [GLS15]). Moreover, the refined conjecture (Remark 3.9) is also true (see [EGS15]), that is,

$$\mathrm{soc}_K\, \pi_v(\overline{r}) = \bigoplus_{\sigma \in W(\overline{r}_v)} \sigma,$$

   where $W(\overline{r}_v)$ is the set of Serre weights associated to $\overline{r}_v$ defined in section 3.

2. The local factor $\pi_v(\overline{r})$ contains one of the representations of $\mathrm{GL}_2(F_v)$ constructed locally by Breuil and Paskunas in section 4 (see [Bre14],[EGS15]).

3. We have
$$\pi_v(\overline{r})[\mathfrak{m}_K] = \pi_v(\overline{r})^{K_1} \cong D_0(\overline{r}_v),$$
where $D_0(\overline{r}_v)$ is the representation of $\mathrm{GL}_2(\mathbb{F}_q)$ constructed in section 4. See [LMS20], [HW18] for $\overline{r}_v$ semisimple, and [Le19] for $\overline{r}_v$ non-semisimple. In particular, it is local and multiplicity-free.

We go one step further. We consider the $\mathrm{GL}_2(F_v)$-representation $\pi_v(\overline{r})[\mathfrak{m}_K^2]$. Under some mild assumptions, we show that it is multiplicity-free, from this we deduce that it is local, and get a complete description of this representation.

The goal is to prove the following theorem:

Theorem 5.1. *Under some mild assumptions, the smooth finite-dimensional K-representation $\pi_v(\overline{r})[\mathfrak{m}_K^2]$ is explicitly known. In particular, it is local and multiplicity-free.*

The main tool of the proof is the patching functor defined by Emerton-Gee-Savitt ([EGS15]) building on the work of Wiles ([Wil95]) and Taylor-Wiles ([TW95]) and of Kisin ([Kis09]). It is an exact functor from continuous representations of K over finite type $W(\mathbb{F})$-modules to finite type R_∞-modules with additional properties, where R_∞ is the patched deformation ring. See [EGS15] for a precise statement. This functor enables us to transfer statements from the GL_2 side to the Galois side, where it turns out they can be proven.

To give the statements on the Galois side, we need to introduce various Galois deformation rings with certain Hodge-Tate weights and types. We only give references for the definition of these rings. See [Maz89] for Galois deformation rings. See [Fon94b] for crystalline representations and Hodge–Tate weights. See [Fon94a],[BM02] for inertial types.

Let $R_{\overline{r}_v}$ be the framed deformation ring of $\overline{r}_v$. Let τ be a tame inertial type. Let $R_{\overline{r}_v}^{(1,0),\tau}$, resp. $R_{\overline{r}_v}^{(2,-1)_j,\tau}$ for $j \in \{0,\dots,f-1\}$, be the reduced p-torsion free quotient of $R_{\overline{r}_v}$ parametrizing those deformations which are potentially crystalline of inertial type τ and parallel Hodge–Tate weights $(1,0)$, resp. Hodge–Tate weights $(2,-1)$ in the embedding $F_v \hookrightarrow W(\mathbb{F})[1/p]$ induced by $\iota \circ \varphi^j$ and $(1,0)$ elsewhere. Let $\mathfrak{p}_\tau^{(1,0)} = \ker(R_{\overline{r}_v} \twoheadrightarrow R_{\overline{r}_v}^{(1,0),\tau})$, and $\mathfrak{p}_\tau^{(2,-1)_j} = \ker(R_{\overline{r}_v} \twoheadrightarrow R_{\overline{r}_v}^{(2,-1)_j,\tau})$. Under the mild assumptions on $\overline{r}$, the patching argument reduces to the proof of subtle (non-)congruence properties, for instance one has to prove (among other statements) that
$$p \in \cap_\tau \mathfrak{p}_\tau^{(1,0)} + \cap_\tau \mathfrak{p}_\tau^{(2,-1)_j}, \text{ for each } j \in \{0,\dots,f-1\},$$
where τ runs over the tame inertial types such that σ is a Jordan–Hölder factor in the mod p semisimplification of $\sigma(\tau)$ (here $\sigma(\tau)$ is the irreducible smooth representation of K associated by Henniart to τ in the appendix to [BM02]) for a fixed Serre weight $\sigma \in W(\overline{r}_v)$.

When $\overline{r}_v$ is semisimple and sufficiently generic, the above congruence relation (hence the theorem) is proved by [BHH+20] by an explicit computation of potentially crystalline deformation rings using the machinery of Kisin modules discovered by Kisin ([Kis06]) and developed in [LLHLM18] and [LLHL19]. On the other hand, [HW20] is able to prove Theorem 5.1 for $\overline{r}_v$ non-semisimple and sufficiently generic by a different method which only works for $\overline{r}_v$ non-semisimple.

The first step of the author's thesis project is to compute explicitly the potentially crystalline deformation rings as above for $\overline{r}_v$ non-semisimple and sufficiently generic, and then generalize the proof of [BHH$^+$20] for Theorem 5.1 to all cases.

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