INTRODUCTION TO THE RESEARCH AREA

Ricci-flat Kähler Metrics on Non-compact Spaces

1. Kähler manifold and Ricci curvature tensor

Kähler manifold and its Ricci curvature are central concepts of this memoire.

Definition 1.1. Let M be a smooth manifold. A Kähler structure on M is a triple (g, I, ω), where g is an I-invariant Riemannian metric (i.e. $g(Iu, Iv) = g(u, v)$), I is an integrable almost complex structure and ω is a symplectic form (i.e. a closed non-degenerate 2-form), satisfying

$$g(u, v) = \omega(u, Iv), \tag{1.1}$$

here u, v are tangent vectors of M. A smooth manifold with such a structure is called a Kähler manifold. The metric g of such a manifold is called the Kähler metric, and the form ω is called the Kähler form.

So a Kähler manifold is at the same time a Riemannian, complex and symplectic manifold. Denote its complex dimension by n. A classical example of Kähler manifold is the complex projective space $\mathbb{CP}^n$ together with the Fubini-Study metric. From the definition it is clear that any complex submanifold of a Kähler manifold inherits a Kähler structure. In particular any smooth projective variety over $\mathbb{C}$ is a Kähler manifold.

In local holomorphic coordinates (z_j), we may write $\omega = i \sum_{j,k=1}^{n} \omega_{j\bar{k}} dz_j \wedge d\bar{z}_k$. Recall that the Riemannian curvature tensor is defined by $R(u, v)w = \nabla_u \nabla_v w - \nabla_v \nabla_u w - \nabla_{[u,v]} w$. Define $R(u, v, w, x) = g(R(u, v)x, w)$, and under local coordinates write $R_{i\bar{j}k\bar{l}} = R(\frac{\partial}{\partial z_i}, \frac{\partial}{\partial \bar{z}_j}, \frac{\partial}{\partial z_k}, \frac{\partial}{\partial \bar{z}_l})$. Direct calculations show that (see [Tia00, Chapter 1.2])

$$R_{i\bar{j}k\bar{l}} = -\frac{\partial^2 \omega_{i\bar{j}}}{\partial z_k \partial \bar{z}_l} + \omega^{s\bar{t}} \frac{\partial \omega_{s\bar{j}}}{\partial z_k} \frac{\partial \omega_{i\bar{t}}}{\partial \bar{z}_l}, \tag{1.2}$$

where $(\omega^{s\bar{t}}) = ((\omega_{j\bar{k}})^t)^{-1}$.

The Ricci curvature tensor is defined by $\mathrm{Ric}_{k\bar{l}} = R_{k\bar{l}} = \omega^{i\bar{j}} R_{i\bar{j}k\bar{l}}$, and it follows that

$$\mathrm{Ric}_{k\bar{l}} = -\frac{\partial^2}{\partial z_k \partial \bar{z}_l} \log \det(\omega_{i\bar{j}}). \tag{1.3}$$

We then define $\mathrm{Ric}(\omega) = \frac{i}{2\pi} \sum_{j,k} \mathrm{Ric}_{j\bar{k}}\, dz_j \wedge d\bar{z}_k = -\frac{i}{2\pi} \partial\bar{\partial} \log \det(\omega_{i\bar{j}})$. It can be shown that the cohomology class of $\mathrm{Ric}(\omega)$ is $c_1(M)$, the first Chern class of M.

Definition 1.2. A Kähler metric ω is called a Kähler-Einstein metric, if there exists $\lambda \in \mathbb{R}$ such that $\mathrm{Ric}(\omega) = \lambda\omega$. If $\lambda = 0$, then we say that the Kähler metric ω is Ricci-flat.

A central problem in Kähler geometry is to look for special metrics, such as Kähler-Einstein metrics, constant scalar curvature metrics or extremal metrics (see for example [Szfrm[o]–4]). In this article we are concerned with Ricci-flat metrics, which is a special case of Kähler-Einstein metrics.

2. Ricci-flat metrics on compact Kähler manifold

A result closely related to the existence of Ricci-flat metrics is the Calabi conjecture, which was conjectured by E.Calabi and confirmed by S.T.Yau ([Yau78]).

Theorem 2.1 (Calabi conjecture)**.** *If (M,ω) is a compact Kähler manifold, and R is any $(1,1)-$form representing $c_1(M)$, then there exists a unique Kähler metric $\tilde{\omega}$ such that ω and $\tilde{\omega}$ represents the same cohomology class in $H^2(M,\mathbb{R})$ and $\mathrm{Ric}(\tilde{\omega}) = R$.*

In fact, the searching of $\tilde{\omega}$ is equivalent to solving a non-linear partial differential equation, known as the complex Monge-Ampère equation. In his original proof, Yau showed the existence of a solution by continuity method and the uniqueness of the solution by maximum principle.

Since $\mathrm{Ric}(\omega)$ represents $c_1(M)$, a necessary condition for $\mathrm{Ric}(\omega) = 0$ is that $c_1(M) = 0$. If a M is a compact Kähler manifold with $c_1(M) = 0$, then it is called a Calabi-Yau manifold. As an easy consequence of the Calabi conjecture, if M a Calabi-Yau manifold, then there is a unique Ricci-flat Kähler metric in each cohomology class of Kähler metric. Such metrics are also called Calabi-Yau metrics. However, the proof does not tells us how to write down a Calabi-Yau metric. In fact, little is known about general Calabi-Yau metrics.

3. Kähler-Einstein metrics on compact Kähler manifold

We digress a little bit in this section to consider the Kähler-Einstein equation $\mathrm{Ric}(\omega) = \lambda\omega$ on compact manifolds. Note that by (1.3), one can rescale the metric by a positive constant, then there are only three cases to treat: $\lambda = 1$, $\lambda = 0$ and $\lambda = -1$. We have already discussed the case $\lambda = 0$ in the previous section.

When $\lambda = -1$, then a necessary condition is $c_1(M) < 0$. Conversely, Aubin [Aub78] and Yau [Yau78] showed that if $c_1(M) < 0$, then there exists a unique Kähler-Einstein metric solving $\mathrm{Ric}(\omega) = -\omega$.

Similarly, when $\lambda = 1$, then necessarily $c_1(M) > 0$, a compact Kähler manifold M with $c_1(M) > 0$ is called a Fano manifold. However there are obstructions to the existence of Kähler-Einstein metric on Fano manifold, the Futaki invariant [Fut83] being an example. It was suggested by Yau [Yau93] in the 1980's that the existence of Kähler-Einstein metric on Fano manifold should be equivalent to some notion of stability. There were many notions of stability proposed and the K-stability is the one that turned out to be right. While the formulation of K-stability by Tian is analytic, Donaldson [Don02] introduced an algebraic version of K-stability. Later it was shown by Li and Xu that the two notions are equivalent for Fano manifolds. A breakthrough is the following result proved by Chen, Donaldson and Sun [CDS15a], [CDS15b], [CDS15c]:

Theorem 3.1. *A Fano manifold M admits a Kähler-Einstein metric if and only if (M, K_M^{-1}) is K-stable.*

More generally, we have the following Yau-Tian-Donaldson conjecture which is still open:

Conjecture 3.2. *Let L be a positive holomorphic line bundle over M. Then there is a Kähler metric with constant scalar curvature in the class $c_1(L)$ if and only if (M, L) is K-stable.*

Note that Theorem 3.1 is a special case of the YTD conjecture when we take $L = K_M^{-1}$. For more discussion on the YTD conjecture, we refer to [Don18].

The subjects of Calabi-Yau manifold, Fano manifold and the Yau-Tian-Donaldson conjecture are huge. However, in this memoire, we will pay more attention to non-compact spaces.

4. The flat metric on $\mathbb{C}^2$

As a first example of non-compact Ricci-flat Kähler metric, we have the flat metric on $\mathbb{C}^2$. Explicitly, let u_0, u_1, u_2, u_3 be the coordinates of $\mathbb{R}^4$, then the standard Euclidean metric is defined to be

$$g = du_0^2 + du_1^2 + du_2^2 + du_3^2.$$

Then we give a complex structure on $\mathbb{R}^4$ by identifying it with $\mathbb{C}^2$:

$$z_1 = u_0 + iu_1, z_2 = u_2 + iu_3.$$

Equivalently, the almost complex structure I is defined by $Ie_0 = e_1, Ie_2 = e_3$, where the vector $e_i = \frac{\partial}{\partial u_i}$ is the gradient of u_i with respect to g. Finally we define the Kähler form by

$$\omega = \frac{i}{2} dz_1 \wedge d\bar{z}_1 + \frac{i}{2} dz_2 \wedge d\bar{z}_2.$$

One checks easily that $d\omega = 0$ and the relation (1.1) holds. So the triple (g, I, ω) is a Kähler structure on $\mathbb{C}^2$. Using formulas (1.2) and (1.3), we see that the Riemannian curvature and Ricci curvature of g is zero. In the sequel, we will refer it to be the flat Kähler metric on $\mathbb{C}^2$, or simply the flat metric on $\mathbb{C}^2$. In the following, we will use it as an example to illustrate many ideas around constructing Ricci-flat Kähler metric.

5. Kähler cone metric and Sasakian geometry

A feature of the flat metric on $\mathbb{C}^2$ is that it is a Riemannian metric cone in the following sense:

Let $\rho = \sqrt{u_0^2 + u_1^2 + u_2^2 + u_3^2} = \sqrt{|z_1|^2 + |z_2|^2}$ to be the distance to the origin. Then the level set $\rho = 1$, equipped with the induced Riemannian metric, is the unit sphere $(\mathbb{S}^3, g_{\mathbb{S}^3})$. The map $\mathbb{S}^3 \times \mathbb{R}^+ \to \mathbb{C}^2 \setminus \{0\}$ given by $(x, \rho) \mapsto \rho x$ is clearly a diffeomorphism. Under this diffeomorphism, the flat metric on $\mathbb{C}^2$ can be expressed as $g = \rho^2 g_{\mathbb{S}^3} + d\rho^2$.

Definition 5.1. A compact Riemannian manifold (S, g_S) is Sasakian if and only if its metric cone $(C(S) = S \times \mathbb{R}^+, g_{C(S)} = \rho^2 g_S + d\rho^2)$ is a Kähler manifold. S is called the link of the cone. The vector field $\rho\frac{\partial}{\partial\rho}$ is called the Euler field, and $\xi = I(\rho\frac{\partial}{\partial\rho})$ is called the Reeb field. The restriction $\xi|_S$ of the Reeb field to the link is tangent to the link S, and we also call this restriction the Reeb field.

According to the above definition, we have already shown that the flat metric on $\mathbb{C}^2$ is the Riemannian metric cone of $(\mathbb{S}^3, g_{\mathbb{S}^3})$. Since the flat metric on $\mathbb{C}^2$ is Kähler,

$(\mathbb{S}^3, g_{\mathbb{S}^3})$ is Sasakian. Direct computation gives the Euler field and the Reeb field:

$$\rho\frac{\partial}{\partial\rho} = u_0\frac{\partial}{\partial u_0} + u_1\frac{\partial}{\partial u_1} + u_2\frac{\partial}{\partial u_2} + u_3\frac{\partial}{\partial u_3} \tag{5.1}$$

$$= \frac{1}{2}(z_1 d\bar{z}_1 + \bar{z}_1 dz_1) + \frac{1}{2}(z_2 d\bar{z}_2 + \bar{z}_2 dz_2); \tag{5.2}$$

$$\xi = u_0\frac{\partial}{\partial u_1} - u_1\frac{\partial}{\partial u_1} + u_2\frac{\partial}{\partial u_3} - u_3\frac{\partial}{\partial u_2} \tag{5.3}$$

$$= \frac{i}{2}(-z_1 d\bar{z}_1 + \bar{z}_1 dz_1) + \frac{i}{2}(-z_2 d\bar{z}_2 + \bar{z}_2 dz_2). \tag{5.4}$$

ξ is the generator of the $\mathbb{S}^1$-action on $\mathbb{C}^2$ and $\mathbb{S}^3$ defined by $e^{it}.(z_1, z_2) = (e^{it}z_1, e^{it}z_2)$. This $\mathbb{S}^1$-action is free on $\mathbb{S}^3$ and the quotient space (orbit space) is $\mathbb{CP}^1$. (In fact, the quotient map $\mathbb{S}^3 \to \mathbb{CP}^1$ is called the Hopf fibration.) In this way, we think of $\mathbb{C}^2 \setminus \{0\}$ as the Riemannian metric cone of the total space of a $\mathbb{S}^1$-principal bundle over $\mathbb{CP}^1$. In fact, the associated line bundle of this $\mathbb{S}^1$-principal bundle is the tautological line bundle $O(-1)$. So we can identify $\mathbb{C}^2 \setminus \{0\}$ with the complement of the zero section in the total space of $O(-1)$.

Note that $(\mathbb{S}^3, g_{\mathbb{S}^3})$ is an Einstein manifold, (i.e. $\mathrm{Ric}(g_{\mathbb{S}^3}) = \lambda g_{\mathbb{S}^3}$ for some λ). And the Fubini-Study metric is a Kähler-Einstein metric on $\mathbb{CP}^1$. The situation described above is an example of the following proposition [MSY08, Proposition 1.9]:

Proposition 5.2. *Let (S, g_S) be a Sasakian manifold, then the followings are equivalent:*

(1) The Sasakian manifold (S, g_S) is Einstein;

(2) The Kähler cone $(C(S), g_{C(S)})$ is Ricci-flat;

Moreover, if the Reeb field generates a free $\mathbb{S}^1$-action on S, the the above two conditions are equivalent to

(3) The quotient space $S/\mathbb{S}^1$ is Fano and Kähler-Einstein.

In fact, given a Kähler-Einstein Fano manifold, there is an explicit procedure to reconstruct S and $C(S)$, known as the Calabi ansatz. (See [MSY08, Theorem 2.1] and [Cal79].)

According to the theorem, finding Ricci-flat Kähler cone metric is equivalent to finding Sasakian-Einstein metric. And when the Reeb field generates a free $\mathbb{S}^1$-action on S (such a Sasakian manifold is called a regular one) this is equivalent to finding Kähler-Einstein metrics on Fano manifolds, which is discussed in Section 3.

We refer to [BG08] for a detailed discussion of Sasakian geometry and especially of constructing Sasakian-Einstein metric.

6. Hyperkähler metric and hyperkähler structure

The flat metric on $\mathbb{C}^2$ is in fact a hyperkähler metric. We shall explain it in this section.

We denote $\mathbb{H}$ the (skew) field of quaternions. One can identify $\mathbb{H}$ with $\mathbb{R}^4$ by writing each quaternion u in the form $u = u_0 + u_1 i + u_2 j + u_3 k$, $(u_0, u_1, u_2, u_3) \in \mathbb{R}^4$. So we regard $\mathbb{H}$ as a smooth manifold diffeomorphic to $\mathbb{R}^4$. The multiplication in $\mathbb{H}$ is determined by $i^2 = j^2 = k^2 = -1, ijk = -1$. In particular, left multiplication by i, j, k in $\mathbb{H}$ defines three integrable almost complex structures on $\mathbb{R}^4$, denoted by I_1, I_2, I_3. One checks easily that I_1 is the same as the almost complex structure

I defined in Section 4. Explicitly, we have $I_1e_0 = e_1, I_1e_2 = e_3, I_2e_0 = e_2, I_2e_3 = e_1, I_3e_0 = e_3, I_3e_1 = e_2$, and $I_1I_2I_3 = -1$.

Recall the flat metric g on $\mathbb{C}^2$ is defined by $g = du_0^2 + du_1^2 + du_2^2 + du_3^2$. Then the corresponding Kähler forms, $\omega_j = g(I_j\cdot, \cdot)$ are:

(6.1)

$$\omega_1 = du_0 \wedge du_1 + du_2 \wedge du_3, \omega_2 = du_0 \wedge du_2 + du_3 \wedge du_1, \omega_3 = du_0 \wedge du_3 + du_1 \wedge du_2.$$

It is clear that ω_j is closed, $j = 1, 2, 3$. Hence we have three Kähler structures $(g, I_1, \omega_1), (g, I_2, \omega_2), (g, I_3, \omega_3)$.

Definition 6.1. Let M be a $4n-$manifold. A hyperkähler structure on M is a set of data $(g, I_1, I_2, I_3, \omega_1, \omega_2, \omega_3)$ where each (g, I_j, ω_j) is a Kähler structure and $I_1I_2I_3 = -1$. In this case, we call M a hyperkähler manifold, g a hyperkähler metric. We refer to the Kähler forms $\omega_1, \omega_2, \omega_3$ of I_1, I_2, I_3 as the hyperkähler 2-forms of M.

Thus, we have shown that $\mathbb{C}^2 = \mathbb{H}$ is a hyperkähler manifold and the standard Euclidean metric g is a hyperkähler metric. Similarly $\mathbb{H}^n$ is also hyperkähler. We shall see more examples of non-compact hyperkähler manifold later. As for examples of compact hyperkähler manifold, we have K3 surfaces in dimension 4, and the Hilbert schemes of points on K3 surfaces in higher dimension [N+99].

From the point of view concerning the Ricci curvature, we are interested in hyperkähler manifold mainly because of the following proposition [Joy00]:

Proposition 6.2. *All hyperkähler metrics are Ricci-flat.*

It should be noted that the converse is far from being true. However, if M is four dimensional and simply connected, then any Ricci-flat metric is hyperkähler (see [Joy00] for an explanation by Riemannian holonomy).

By Proposition 6.2, one can construct Ricci-flat metric by constructing hyperkähler manifold. One powerful method of producing new hyperkähler manifold out of known ones is the hyperkähler quotient. It is an analogue of the symplectic quotient, also known as symplectic reduction. The twistor method is another way of producing hyperkähler manifold. See [Hit92] for these two methods and an introduction to hyperkähler geometry.

7. Hyperkähler 4-manifold with an $\mathbb{S}^1$-symmetry

Recall that the almost complex structures I_1, I_2, I_3 of the hyperkähler structure of $\mathbb{C}^2 = \mathbb{H}$ is given by left multiplication by i, j, k, hence the hyperkähler structure is preserved by right multiplication of quaternions of norm 1. In particular we have a (right) $\mathbb{S}^1$-action which preserves the hyperkähler structure:

$$\begin{aligned} ue^{it} &= (u_0 \cos t - u_1 \sin t) + (u_1 \cos t + u_0 \sin t)i \\ &\quad + (u_2 \cos t + u_3 \sin t)j + (u_3 \cos t - u_2 \sin t)k. \end{aligned}$$

Equivalently, the action can be expressed by

$$(z_1, z_2)e^{it} = (z_1e^{it}, z_2e^{-it}).$$

Note that this $\mathbb{S}^1$-action is different from the action discussed in Section 5, but they are conjugate by sending z_2 to $\bar{z}_2$. Let T denote the infinitesimal generator of the action, then

$$T = -u_1e_0 + u_0e_1 + u_3e_2 - u_2e_3.$$

Note that $g(T,T) = \sum_{i=0}^3 u_i^2 = \rho^2$. Denote by ι_T the interior multiplication by T, then direct computation shows that

$$\begin{aligned} -\iota_T\omega_1 &= u_0du_0 + u_1du_1 - u_2du_2 - u_3du_3 = dx_1, \\ -\iota_T\omega_2 &= u_3du_0 + u_2du_1 + u_1du_2 + u_0du_3 = dx_2, \\ -\iota_T\omega_3 &= -u_2du_0 + u_3du_1 - u_0du_2 + u_1du_3 = dx_3, \end{aligned}$$

where the momenta x_1, x_2, x_3 are given by

$$x_1 = \frac{1}{2}(u_0^2 + u_1^2 - u_2^2 - u_3^2), x_2 = u_1u_2 + u_0u_3, x_3 = u_1u_3 - u_0u_2.$$

Observe that if we set $r^2 = x_1^2 + x_2^2 + x_3^2$, then $\rho^2 = 2r$. Define θ to be the 1-form dual to T with respect to g, i.e. $\theta(X) = g(T,X)$, then we have

$$\theta = -u_1du_0 + u_0du_1 + u_3du_2 - u_2du_3.$$

Define $\eta = \frac{\theta}{\rho^2}$, then $\eta(T) = 1$.

Now we notice that $\{\frac{1}{\rho}dx_1, \frac{1}{\rho}dx_2, \frac{1}{\rho}dx_3, \rho\eta\}$ is an orthonormal coframe of g, so we infer that

$$g = \frac{1}{\rho^2}\sum_{j=1}^3 dx_j \otimes dx_j + \rho^2\eta\otimes\eta. \tag{7.1}$$

Thus, with the help of this $\mathbb{S}^1$-symmetry, we get a new expression of the flat metric on $\mathbb{C}^2$.

The following observation is crucial: Denote by $\pi : \mathbb{R}^4 \to \mathbb{R}^3$ the moment map defined by $\pi(u) = (x_1, x_2, x_3)$. Then in restriction to $\mathbb{R}^4 \setminus \{0\}$, π is a $\mathbb{S}^1$-principal bundle whose fibers are orbits of the $\mathbb{S}^1$-action. η is a connection 1-form of this principal bundle, so the curvature $d\eta$ of this connection can be viewed as a 2-form defined on the total space $\mathbb{R}^4 \setminus \{0\}$ or on the base $\mathbb{R}^3 \setminus \{0\}$. By calculation, we get:

$$d\eta = -\frac{x_1dx_2\wedge dx_3 + x_2dx_3\wedge dx_1 + x_3dx_1\wedge dx_2}{2r^3} = *_3 d\left(\frac{1}{2r}\right), \tag{7.2}$$

where $*_3$ denotes the Hodge operator on $\mathbb{R}^3$ oriented by $dx_1 \wedge dx_2 \wedge dx_3$. Consequently, since $dd\eta = 0$, we have $\Delta_{\mathbb{R}^3}\frac{1}{2r} = d *_3 d\frac{1}{2r} = 0$, which means that $\frac{1}{\rho^2} = \frac{1}{2r}$ is a positive harmonic function on $\mathbb{R}^3 \setminus \{0\}$.

The Gibbons-Hawking ansatz is a procedure to produce hyperkähler 4-manifold with an $\mathbb{S}^1$-symmetry: If V is a positive harmonic function defined a connected open subset U of $\mathbb{R}^3$ such that $\frac{1}{2\pi}(*_3dV) \in H^2(U,\mathbb{Z})$ (more precisely, the image of $H^2(U,\mathbb{Z})$ in $H^2(U,\mathbb{R})$), then $*_3dV$ can be realized as the curvature form of some $\mathbb{S}^1$-principal bundle $\pi : M \to U$, equipped with a connection form η such that $d\eta = \pi^*(*_3dV)$. Then the Riemannian metric g defined on M by

$$g = V\sum_{i=1}^3 dx_i \otimes dx_i + \frac{1}{V}\eta\otimes\eta, \tag{7.3}$$

is a hyperkähler metric. So the flat metric on $\mathbb{C}^2$ can be regarded as the result of the Gibbons-Hawking ansatz applied to the harmonic function $\frac{1}{2r}$.

Using the Gibbons-Hawking ansatz, one can construct hyperkähler metric easily. Let r_1, r_2 be the distances in $\mathbb{R}^3$ to two different points, then $V = \frac{1}{2r_1} + \frac{1}{2r_2}$ is a positive harmonic function. The resulting hyperkähler metric is known as the Eguchi-Hanson metric. If one takes $V = \frac{1}{2r_1} + \cdots + \frac{1}{2r_k}$, then the resulting hyperkähler metric is called the ALE gravitational instanton of type A_{k-1}. If one take $V = \frac{1}{2r} + a^2$ where $a > 0$, then the resulting hyperkähler metric is called the Taub-NUT metric. Finally starting with $V = \frac{1}{2r_1} + \cdots + \frac{1}{2r_k} + a^2$, the corresponding metric is called the ALF gravitational instanton of type A_{k-1}, or multi-Taub-NUT metric. We will explain these strange terminologies later, here we just remark that the harmonic functions corresponding to the ALF metrics are bounded below by a positive number a^2. By formula (7.3), this means that the length of the $\mathbb{S}^1$-orbit is bounded from above by $\frac{2\pi}{a}$. While this is not the case for ALE metrics.

8. Kleinian singularities and their crepant resolutions

As said in the beginning of the previous section, the hyperkähler structure on $\mathbb{C}^2$ is preserved by by right multiplication of quaternions of norm 1, i.e. the right action of $\mathrm{Sp}(1) = \{u \in \mathbb{H} | |u|^2 = 1\}$. It is well-known that $\mathrm{Sp}(1) = \mathrm{SU}(2)$. In Section 7 we take a subgroup $\mathbb{S}^1$ of it. In this section we take $G \subset \mathrm{SU}(2)$ to be a finite subgroup. Denote by $X = \mathbb{C}^2/G$ the quotient space. The quotient space X is not smooth, since it has 0 as a singularity. Such singularities were first classified by Klein and are called Kleinian singularities, they are also called Du Val surface singularities, or rational double points. They were very well understood, see for instance [Slo80].

Definition 8.1. Let Y be a complex algebraic variety, and suppose that K_Y is an invertible sheaf, then a resolution $\pi : \tilde{Y} \to Y$ is called a crepant resoulution or minimal resolution if $K_{\tilde{Y}} = \pi^*(K_Y)$.

As $G \subset \mathrm{SU}(2)$, K_X is an invertible sheaf, and in fact each Kleinian singularity admits a unique crepant resolution $(\tilde{X}, \pi)$.

There is a 1-1 correspondence between nontrivial finite subgroups $G \subset \mathrm{SU}(2)$ and the Dynkin diagrams Γ of type $A_r (r \geq 1), D_r (r \geq 4), E_6, E_7, E_8$, which are all simply laced. Each of these diagrams appear in the classification of Lie groups, corresponding to a unique compact simple Lie group. The preimage $\pi^{-1}(0)$ of the singular point is a union of finite number of rational curves in $\tilde{X}$. These curves correspond to points of Γ and their intersection number corresponds to the Cartan matrix of Γ. This correspondence between the Kleinian singularities, Dynkin diagrams, their crepant resolutions and other objects is known as the McKay correspondence, which was first pointed by John McKay.

Since $\mathbb{C}^2$ is hyperkähler, and the hyperkähler structure is invariant by G, it is natural to intuitively think of $X = \mathbb{C}^2/G$ as a "singular hyperkähler manifold" (in fact orbifold). To resolve the singularity, one hopes to pull back the hyperkähler metric on the base. Of course the pulled back metric $\pi^* g$ is not defined on $\pi^{-1}(0)$, but it gives us a model when the radius is large enough. This heuristic thought will be made precise in the next section.

9. Gravitational instantons

Hyperkähler ALE spaces are called ALE gravitational instantons by physicists.

Definition 9.1. Let G be a finite subgroup of $\mathrm{SU}(2)$, $(g, I_1, I_2, I_3, \omega_1, \omega_2, \omega_3)$ the hyperkähler structure of the flat metric on $\mathbb{C}^2$, $r : \mathbb{C}^2/G \to [0, +\infty)$ the radius function on $\mathbb{C}^2/G$. We say that a hyperkähler 4-manifold $(M, \tilde{g}, \tilde{I}_1, \tilde{I}_2, \tilde{I}_3, \tilde{\omega}_1, \tilde{\omega}_2, \tilde{\omega}_3)$ is Asymptotically Locally Euclidean, or ALE, and asymptotic to $\mathbb{C}^2/G$, if there exists a compact subset $S \subset M$ and a map $\pi : M \setminus S \to \mathbb{C}^2/G$ that is a diffeomorphism between $X \setminus S$ and $\{u \in \mathbb{C}^2/G | r(u) > R\}$ for some $R > 0$, such that

$$\nabla^k(\pi_*(\tilde{g}) - g) = O(r^{-4-k}), \nabla^k(\pi_*(\tilde{I}_j) - I_j) = O(r^{-4-k}) \tag{9.1}$$

as $r \to +\infty$ for $j = 1, 2, 3$ and $k \geq 0$, where ∇ is the Levi-Civita connection of g.

The definition means that a hyperkähler ALE space is a non-compact hyperkähler 4-manifold which at infinity looks like $\mathbb{C}^2/G$, with a prescribed rate of decay of their difference.

We will not give explicit formula as example. Instead, some historical remarks are in order. The first examples of ALE gravitational instantons were written down explicitly by Eguchi and Hanson [EH79] and are called Eguchi-Hanson spaces. They are asymptotic to $\mathbb{C}^2/\{\pm 1\}$. They are soon generalized by Gibbons and Hawking [GH78] [GH79] who constructed explicit examples of ALE gravitational instantons asymptotic to $\mathbb{C}^2/\mathbb{Z}_k$, $k \geq 2$, which is discussed in Section 7. Using twistor methods, Hitchin [Hit79] constructed the same spaces. Finally Kronheimer [Kro89a] [Kro89b] gave a complete construction and classification of all ALE gravitational instantons. His construction makes uses of the McKay correspondence and hyperkähler quotient, while the classification relies on the twistor method and facts about the deformations of $\mathbb{C}^2/G$. In his classification, all the ALE gravitational instantons asymptotic to $\mathbb{C}^2/G$ are hyperkähler manifolds whose underlying space is the crepant resolution of $\mathbb{C}^2/G$.

Besides the ALE gravitational instantons, there are more general gravitational instantons. They are hyperkähler manifolds of real dimension 4 satisfying a decaying condition of its Riemannian curvature. Such an object plays an important role in theoretical physics. According to its 'dimension at infinity' m, we have ALE ($m = 4$), ALF ($m = 3$), ALG ($m = 2$) and ALH ($m = 1$) gravitational instantons (see also [CC15] for a different definition). Roughly speaking, it means that the metric is asymptotic to a $\mathbb{T}^{4-m}$-fibration over $\mathbb{R}^m$ at infinity. Here 'ALF' stands for 'asymptotically locally flat' and ALG, ALH are named by induction: E,F,G,H.

As we have seen, the ALE gravitational instantons are very well-understood. As for the ALF gravitational instanton, it is known that its 'fundamental group at infinity' is either cyclic or binary dihedral. The family of cyclic type is classified by [Min11], which corresponds to multi-Taub-NUT metrics as we have seen in Section 7, they are asymptotic to $\mathbb{R}^3$ at infinity, known as ALF-A_k. [CK98] [CK99] produced a family of dihedral type and it is conjectured that these two families are all possible ALF gravitational instantons. The conjecture is confirmed in [CC19] under the assumption of $O(r^{-2-\epsilon})$ decay of Riemannian curvature. Such instantons are asymptotic to $\mathbb{R}^3/\mathbb{Z}_2$ and known as ALF-D_k. Under the same assumption of Riemannian curvature decay, [CC20] classifies the ALG gravitational instantons and proves a Torelli-type theorem for ALH gravitational instantons. They are asymptotic to flat cones and $\mathbb{R}^+$ respectively.

It should be noted that there are many slightly different definitions of gravitational instantons so one must be careful when using any classification results. In fact, if one allows the decay of Riemannian curvature to be slower, then there exists examples of hyperkähler metrics with $r^{\frac{4}{3}}$, $r^{-\frac{1}{3}}$, r^{-2}, $(\log r)^{-\frac{1}{2}}$ rate of curvature decay [Hei12].

10. A glimpse to higher dimensions

So far, all the presented examples of non compact Ricci-flat metrics are of real dimension 4. As we have pointed out in Section 6, this is closely related to hyperkähler metrics. Taking account of many strong implication of hyperkähler structure, 4-dimensional case is very special. However, one can still try to generalize some concepts in dimension 4 to higher dimension, for example the condition of being ALE and ALF.

Contrary to dimension 4, less is known for higher dimensional case. [BKN89] proved that a decaying condition of Riemannian curvature and Euclidean volume growth implies ALE. [CL19] then considered the situation where the volume growth is not maximal (note that by Bishop-Gromov volume comparison theorem, the volume growth is at most Euclidean), with an additional assumption on the holonomy, the authors gave a description of the geometry at infinity.

There are also some examples in higher dimensions. The method of Cherkis and Kapustin produces higher dimensional spaces. And a series of works of Ronan J. Conlon and Hans-Joachim Hein studies the asmptotically conical Calabi-Yau manifolds (see [CH13] [CH15]).

Guided by the 4 dimensional case, one could ask a lot of questions. For example, do there exists any (non-flat) ALF metric? Let us say, a complete 8 dimensional Ricci-flat metric with a 7 dimensional tangent cone at infinity and a decaying Riemannian curvature. Note that one cannot take the product of 4 dimensional ALE and ALF space as example, since the Riemannian curvature of the product space does not decay with the radius.

On the other hand, since the 4 dimensional case is so special, one expects to see some new phenomenons when looking at the case of higher dimensions.

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