Temps de mélange du processus de Zero-Range

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Contents

1 Introduction

Markov chains (or Markov processes) are very important subjects, which are not only studied in the Theory of Probability but also have a lot of applications in other fields such as computer science, physics, statistics, engineering, etc. First studied by Markov in 1906, after more than 100 years, the theory of Markov processes is still an active and fruitful research domain.

Under some mild conditions, for many chains, there exists an unique stationary distribution, and the law of the chain converges to the stationary distribution. The classical theory of Markov processes studies fixed chains and their rate of convergence to equilibrium. However, in recent decades, a huge amount of work is devoted to studying the case where the size of the state space is large, says tend to infinity. An important family of such chains is interacting particle system (Zero-Range process, Exclusion process, stochastic Ising model, etc), and we are interested in the case where the number of particles tends to infinity. The time it takes for a system to get close to equilibrium with respect to certain target distance is called the mixing times. Studying the comportment of this parameter as the state space grows is a question that draws a lot of attention.

In this report, we recall some basic properties of Markov processes in continuous time and discuss some result on the Zero-Range process.

2 Continuous time Markov processes

Markovian semigroup Let Ω be a finite set which we called the *state space*. A family of transition matrices $(P_t, t \in \mathbb{R}_+)$ on Ω is a *Markovian semigroup* if it satisfies the following conditions.

(i) $P_0 = \mathrm{Id}_\Omega$.

(ii) $\forall s, t \in \mathbb{R}_+, P_t \cdot P_s = P_s \cdot P_t = P_{s+t}$.

We give here a rigorous definition of a Markov process.

Definition 1. *A (càdlàg) Markov process taking values in Ω is given by the following objects:*

- *A measurable space $(F, \mathcal{F})$.*
- *A filtration $(\mathcal{F}_t,\, t \in \mathbb{R}_+)$ on $(F, \mathcal{F})$.*
- *A Markovian semigroup $(P_t, t \in \mathbb{R}_+)$ on Ω, where we write $P_t = (p_t(i,j))_{i,j\in\Omega}$.*
- *A process $X = (X_t, t \in \mathbb{R})$ defined on $(F, \mathcal{F})$ taking values in Ω such that X is adapted to $(\mathcal{F}_t)_{t\geq 0}$.*
- *A collection of probability measures ($\mathbb{P}_\mu, \mu$ is a probability measure on Ω) such that for any μ, under P_μ, X has càdlàg trajectories, and for any $t_0 = 0 < t_1 < \cdots < t_n$ and any $i_0, ..., i_n \in \Omega$,*

$$\mathbb{P}_\mu\left[X_0 = i_0, \ldots, X_{t_n} = i_n\right] = \mu(i_0) \prod_{k=1}^{n} p_{t_k - t_{k-1}}(i_{k-1}, i_k).$$

From now to the end of this section, let $X = (X_t)_{t\geq 0}$ be a Markov process defined on $(F, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0})$, taking values in Ω, with semigroup $(P_t)_{t\geq 0}$. We can view X as a random variable taking values in the set $D(\mathbb{R}_+, \Omega)$ of càdlàg function from $\mathbb{R}_+$ to Ω. The family of time-shift operators,

$$(\theta_t)_{t\geq 0} : D(\mathbb{R}_+, \Omega) \to D(\mathbb{R}_+, \Omega),$$

is defined by $\theta_t(x)(s) = x(t+s)$, $\forall s, t \in \mathbb{R}_+$. Analogously to the discrete time setting, we have the weak and strong Markov properties:

Theorem 2 (Weak Markov property)**.** *For any law μ on Ω, for any bounded measurable function $\Phi : D(\mathbb{R}_+, \Omega) \to \mathbb{R}$,*

$$\mathbb{E}_\mu\left[\Phi(\theta_t(X))|\mathcal{F}_t\right] = \mathbb{E}_{X_t}\left[\Phi(X)\right].$$

Theorem 3 (Strong Markov property)**.** *Let T be a $(\mathcal{F}_t, t \in \mathbb{R}_+)$ stopping time. Then*

1. *X_T is $\mathcal{F}_T$ measurable.*
2. *The map $\theta_T X : F \to D(\mathbb{R}_+, \Omega)$ is measurable.*
3. *For any probability measure μ on Ω, for any bounded measurable function $\Phi : D(\mathbb{R}_+, \Omega) \to \mathbb{R}$,*

$$\mathbb{E}_\mu\left[\mathbb{1}_{\{T<\infty\}}\Phi(\theta_T X)|\mathcal{F}_{T+}\right] = \mathbb{1}_{\{T<\infty\}}\mathbb{E}_{X_t}\left[\Phi(X)\right].$$

Infinitesimal generator. The classical theory of Markov processes says that there exists a square matrix $\mathcal{L} = (\mathcal{L}_{i,j},\, i, j \in \Omega)$, called the *infinitesimal generator* such that $\mathcal{L}_{i,j} \geq 0$ when $i \neq j$ and

$$\sum_{j\in\Omega} \mathcal{L}_{i,j} = 0,\ \forall i \in \Omega,$$

and $P^t = e^{t\mathcal{L}}, \forall t \in \mathbb{R}_+$. The law of the process X is determined by L. Usually, we often define $\mathcal{L}$ by its action on the observables:

$$\mathcal{L}\varphi(x) = \sum_{y\neq x} \mathcal{L}(x,y)(\varphi(y) - \varphi(x)),\ \forall \varphi : \Omega \to \mathbb{R}.$$

Martingale associated with an observable. For any observable $\varphi : \Omega \to \mathbb{R}$, for any initial distribution, the process $M = (M(t))_{t\geq 0}$ given by

$$M(t) := \varphi(X(t)) - \varphi(X(0)) - \int_0^t \mathcal{L}\varphi(X(u))\mathrm{d}u \tag{1}$$

is a zero-mean martingale, see e.g [4]. Let φ_1, φ_2 be two observables, and let M_1, M_2 be the associated martingales. Then the *predictable covariation* of M_1 and M_2 is given by

$$\langle M_1, M_2\rangle_t = \int_0^t \sum_{y\in\Omega} \mathcal{L}(X(u), y)\left(\varphi_1(y) - \varphi_1(X(u))\right)\left(\varphi_2(y) - \varphi_2(X(u))\right) du. \tag{2}$$

Total variation distance. The *total variation distance* is a metric to measure the difference between measures of probability on Ω, which is defined as

$$\mathrm{d_{TV}}\left(\mu, \nu\right) = \max_{A\subset\Omega} |\mu(A) - \nu(A)| = \frac{1}{2}\sum_{x\in\Omega} |\mu(x) - \nu(x)|.$$

Convergence to equilibrium. We say that L is irreducible if for any $i, j \in \Omega$, there exists a sequence $i = i_0, i_1, \ldots, i_n = j$ such that

$$\prod_{k=1}^{n} L_{i_{k-1},i_k} > 0.$$

If L is irreducible, then the law of the process will converge to the stationnary distribution:

Theorem 4 (Convergence to equilibrium)**.** *Let L be an irreducible infinitesimal generator. Then there exists a unique distribution π such that for any initial distribution μ,*

$$\lim_{t\to\infty} d_{\mathrm{TV}}\left(\mu P^t, \pi\right) = 0.$$

A fundamental question is to understand the speed of the convergence above, which leads to the study of mixing times.

Mixing time. The speed of convergence to equilibrium is characterized by the so-called *mixing times*, defined by

$$\mathrm{t_{MIX}}(x;\epsilon) := \min\{t \geq 0 : \mathrm{d_{TV}}\left(P_x^t, \pi\right) \leq \epsilon\}$$

The *worst-case mixing time* is obtained by maximizing over the initial states $x \in \Omega$.

$$\mathrm{t_{MIX}}(\epsilon) := \max\{\mathrm{t_{MIX}}(x;\epsilon) : x \in \Omega\}.$$

Understand the parameters above is in general a challenging task (see the book [11]).

Dirichlet form, Poincaré constant. We note by $\langle\cdot,\cdot\rangle_\pi$ the scalar product on $l^2(\Omega,\pi)$. The *Dirichlet form* associated to the process is defined by

$$\mathcal{E}(\varphi,\psi) := -\langle\varphi, \mathcal{L}\psi\rangle_\pi .$$

The *Poincaré constant* is defined by

$$\lambda_* := \min\left\{\frac{\mathcal{E}(\varphi,\varphi)}{\mathrm{Var}\left[\varphi\right]}\right\},$$

where the minimum is taken over all non constant observable.

Reversibility. If there exists a distribution π such that for all $x, y \in \Omega$, $\pi(x)\mathcal{L}(x,y) = \pi(y)\mathcal{L}(y,x))$, we says that the generator $\mathcal{L}$ is **reversible**. We can prove that for any $t > 0$, for any $x, y \in \Omega$, $\pi(x)P^t(x,y) = \pi(y)P^t(y,x)$. When $\mathcal{L}$ is reversible, λ_* coincides with the absolute spectral gap:

$$\lambda_* = \lim_{t\to\infty} -\frac{1}{t} \log \max_{x\in\Omega} \mathrm{d}_{\mathrm{TV}}\left(P_x^t, \pi\right).$$

In other words, the convergence to equilibrium is exponentially fast with speed λ_*.

Cutoff phenomenon. We are interested in the case where the state space grows to infinity. More precisely, let $(\Omega_n, L_n)_{n\in\mathbb{N}}$ is a sequence of state spaces and infinitesimal generators. To lighten the notation, we sometimes ignore the subscript n. For many family of chains, it happens that when n tends to infinity, the asymptotic comportment of $\mathrm{t}_{\mathrm{MIX}}(x;\epsilon)$ does not depend on ϵ:

$$\frac{\mathrm{t}_{\mathrm{MIX}}(\epsilon)}{\mathrm{t}_{\mathrm{MIX}}(1-\epsilon)} = 1 + o(1).$$

In other word, starting from the worst initial condition, the total variation distance will stay close to 1 for a long time, and then drops abruptly to zero in a shorter time scale. This phenomenon is known as *cutoff*, discovered by Diaconis and Aldous when study card shuffling in [1], [2].

3 The Zero-Range process

The Zero-Range process, introduced by Spitzer in [10], is a model of interacting particle system in continuous time. It describes the evolution of $m \geq 1$ particles jumping across a set V, whose elements are called sites. More precisely, the state space is

$$\Omega = \{x \in \mathbb{Z}_+^V : \sum_{v\in V} x(v) = m\}.$$

The dynamic is characterized by the following functions:

1. A function $r : V\times\mathbb{N}^* \to \mathbb{R}_{>0}$ called *the potential function*, where $r(v,k)$ is the rate at which site v expels a particle if it is occupied by k particles. For convenience, we let $r(v,0) = 0$, $\forall x \in V$ (no jump from empty sites).
2. An irreducible transition matrix P on V, which defines the geometry on V.

The infinitesimal generator L acts on an observable $f : \Omega \to \mathbb{R}$ as follow:

$$\mathcal{L}f(x) = \sum_{(u,v)\in V\times V} r(u,x(u))P(u,v)(f(x-\delta_u+\delta_v) - f(x)),$$

where $(\delta_v)_{v\in V}$ is the canonical basis of $\mathbb{R}^V$. Let μ be the unique invariant distribution of P. It is easy to see that $\mathcal{L}$ is also irreducible and its invariant distribution is given by

$$\pi(x) \propto \prod_{v\in V}\prod_{k=1}^{x(v)} \frac{\mu(v)}{r(v,k)}.$$

The model defined above is very general. We actually only discuss the case where $r(v,k)$ does not depend on v, which means the speed of a particle only depends on the number of its co-occuppants. The most studied cases in literature is when V is the vertex set of a regular graph (the complete graph, the torus, etc), and P is the transition matrix of the simple random walk on the graph. The system exhibits different behaviors under different potential functions r.

3.1 Mean-field case, $r = 1$

In [9], the author studies the mean-field case for the potential function $r = 1$. He proves a strong bound depending on the density for the spectral gap.

Theorem 5 (Morris 2006)**.** *Let $\lambda_* = \lambda_*(n)$ be the spectral gap of the mean-field Zero Range process on n vertices. Then there exists a universal constant C such that $\lambda_* \geq \dfrac{C}{(\rho+1)^2}$, where ρ is the density of particles.*

And after that he uses comparison technique to extend the results to the torus model $(\mathbb{Z}/L\mathbb{Z})^d$.

Theorem 6. *Let $\lambda_{*,2}$ be the spectral gap of the Zero-Range process on the lattice $(\mathbb{Z}/L\mathbb{Z})^d$, then there exists a universal constant C such that $\lambda_{*,2} \geq \dfrac{C}{p^2(\rho+1)^2}$.*

3.2 Mean-field case, r increasing bounded

In [5], Hermon and Salez prove cutoff phenomenon for the mean-field case for the case r is increasing but bounded in the regime where the number of sites tend to infinity and the density of particles per site is bounded. They also give the explicit formula for the cutoff time when the density stabilizes. More precisely, upon a rescaling, we can suppose that r increases to 1. In the regime

$$n \to \infty, \quad \frac{m}{n} \to \rho \in [0, \infty), \tag{3}$$

consider the state space

$$\Omega := \left\{ x = (x_1, x_2, ..., x_n) \in \mathbb{Z}_+^n : \sum_{i=1}^{n} x_i = m \right\},$$

and the Markov generator $\mathcal{L}$ acts on an observable $\varphi : \Omega \to \mathbb{R}$ by

$$(\mathcal{L}\varphi)(x) = \frac{1}{n} \sum_{1 \leq i,j \leq n} r(x_i)(\varphi(x - \delta_i + \delta_j) - \varphi(x)).$$

Theorem 7 (Hermon, Salez 2019)**.** *For any $\epsilon \in [0,1)$ fixed,*

$$\frac{t_{\text{MIX}}(\epsilon)}{n} \overset{n\to\infty}{\longrightarrow} \gamma = \gamma(r, \rho),$$

where $\gamma(r,\rho)$ is given explicitly.

By considering the initial configuration where all particles are on a same site, we easily see that the worst-case mixing time is at least of order n. The authors use an argument of *separation of time-scale*. They prove the following theorem.

Theorem 8 (Fast mixing)**.** *There exists a dimension-free constant κ such that*

$$t_{\text{MIX}}(x;\epsilon) \le \kappa \|x\|_\infty + (\log n)^\kappa,$$

for every $x \in \Omega$, and every $\epsilon \in [0,1]$, provided that $n \ge \frac{\kappa}{\epsilon}$.

The theorem above says that from a configuration x such that $\|x\|_\infty = o(n)$, the system reaches equilibrium at a time negligible compare to n. So to estimate the mixing time from any configuration, we just need to estimate the time t such that $\|X(t)\|_\infty = o(n)$. They provide the estimate of that time for all configuration x, and then deduce the constant γ from the initial configuration where all particles are on a same site. We only provide a sketch of the proof of Theorem 8.

Graphical construction. Let Ξ be a Poisson point process of intensity $\frac{1}{n}\mathrm{d}t \otimes \mathrm{d}u \otimes \mathrm{Card} \otimes \mathrm{Card}$ on $[0,\infty)\times[0,1]\times[n]\times[n]$, where Card denotes the counting measure. Define the piece-wise constant process $X = (X(t))_{t\ge 0}$ taking values in Ω as follows: $X(0) = x$, and for each point (t,u,i,j) of Ξ,

$$X(t) := \begin{cases} X(t-) - \delta_i + \delta_j, & \text{if } u \le r(X_i(t-)) \\ X(t-) & \text{otherwise.} \end{cases} \tag{4}$$

Then X is a càdlàg Markov process starting from x with generator $\mathcal{L}$.
We can now couple Zero-Range processes with different number of particle by using the same Poisson point process.

Mean-field jump rate. At all time, the rates of arriving into each site are equal and given by

$$\zeta(t) := \frac{1}{n}\sum_{j=1}^{n} r(X_j(t)). \tag{5}$$

Path coupling via tagged particle. For $k \in \mathbb{Z}_+$, define

$$\Delta(k) := r(k+1) - r(k) \ge 0. \tag{6}$$

Let Θ be a Poisson point process of intensity $\frac{1}{n}\mathrm{d}t \otimes \mathrm{d}u \otimes \mathrm{Card}$ on $\mathbb{R}_+ \times \mathbb{R}_+ \times [n]$, independent of the Poisson processes used in the graphical construction of X. For a site $i \in [n]$, define an $[n]$- valued process $I = (I(t))_{t\ge 0}$ by setting $I(0) = i$ and for each (t,u,k) in Θ,

$$I(t) := \begin{cases} k & \text{if } u \le \Delta(X_I(t-)) \\ I(t-) & \text{otherwise,} \end{cases} \tag{7}$$

where $X_I(t) := X_{I(t)}(t)$. This definition means that conditionally on X, the process $I(t)$ will jump with the time-varying rate $\Delta(X_I(t))$, and the destination is uniformly chosen among all sites. A simple but important observation is that $(X(t) + \delta_{I(t)})_{t\ge 0}$ has the same distribution as a Zero-Range process starting at $x + \delta_i$. We call I a tagged particle. For j another site, similarly we can construct a second tagged particle J starting from $J(0) = j$ using the same process Θ. Thus we have a coupling $(X(t) + \delta_{I(t)}, X(t) + \delta_{J(t)})_{t\ge 0}$ of two Zero-Range processes starting at $x + \delta_i$ and

$x+\delta_j$ respectively. We note $\mathbb{P}_{x,i.j}$ for the law of the process (X,I,J) starting from (x,i,j) and $\mathbb{E}_{x,i,j}$ for the expectation taken w.r.t $\mathbb{P}_{x,i,j}$. Let τ be the coalescence time of I and J:

$$\tau := \inf\{t \geq 0 : I(t) = J(t)\}. \tag{8}$$

By the classical relation between $\mathrm{d_{TV}}(\cdot,\cdot)$ and coupling, we have:

$$\mathrm{d_{TV}}\left(P^t_{x+\delta_i}, P^t_{x+\delta_j}\right) \leq \mathbb{P}_{x,i,j}\left[\tau > t\right]. \tag{9}$$

They estimate the mixing time via the coalescence time of the path coupling above.

Sketch of the proof. The coalescence time depends on the jump rates of the tagged particles, which can be very small (think of the case $r \equiv 1$), so we should think of a smart scenario of coalescence. By a dimension-free constant, we refer to a real number that only depends on r, ρ. First important observation is that there exists dimension free constant $\epsilon > 0$ such that for any $t \geq 1$,

$$\mathbb{P}\left[\zeta(t) > 1-\epsilon\right] \leq e^{-\epsilon n}.$$

$1-\epsilon$ is smaller than $r(k)$ for k large enough. So considering the observable $\varphi(x) = e^{\theta x_i}$, we can prove the drift follow.

$$\mathbb{E}_x\left[e^{\theta X_i(t)}\right] \leq \kappa(1 + e^{\theta(x_i - \delta t)}), \tag{10}$$

for some dimension-free constant κ, θ, δ. This gives us a very good concentration:

$$\mathbb{P}_x\left[X_i(t) \geq a\right] \leq 4e^{-\theta a}, \text{ when } t \geq \|x\|_\infty / \delta. \tag{11}$$

It implies that after a time $\Theta(\|x\|_\infty)$, the system reaches a set of height $\mathcal{O}(\log n)$ w.h.p, by an argument of union bound over all the site. Then they prove that there exists a dimension-free constant κ such that for the path coupling above,

$$\mathbb{P}\left[\tau \geq \kappa(\|x\|_\infty \vee (\log n)^\kappa)\right] \leq \frac{\kappa}{n^2}. \tag{12}$$

This provides an upper bound for the total variation distance of the laws of two processes starting from two adjacent configurations. By triangle inequality, this can be extended to two arbitrary configurations $\mathrm{d_{TV}}\left(P^t_x, P^t_y\right) \leq \frac{m\kappa}{n^2}$ for $t = \kappa(\|x\|_\infty \vee \|y\|_\infty \vee (\log n)^\kappa)$. Note that the set $\{y : \|y\|_\infty > (\log n)^\kappa\}$ is of small probability under π, thanks to (10). Then we can prove that

$$\mathrm{d_{TV}}\left(P^t_x, \pi\right) = o(1),$$

for $t = \kappa(\|x\|_\infty + (\log n)^\kappa$, which implies theorem 8.

3.3 Mean-field case, r increases to infinity at sublinear speed

In the same setting as subsection 3.2, except we consider the case where potential function r increases to infinity at sublinear speed, i.e. , r satisfies the following condition.

$$r(k+1) \geq r(k),\ \forall k \in \mathbb{Z}_+, \tag{13}$$

$$\lim_{k\to\infty} r(k) = \infty, \tag{14}$$

$$\sup_{k\in\mathbb{Z}_+} \frac{r(k)}{k} < \infty. \tag{15}$$

Let $R : \{1, 2, \dots\} \to \mathbb{R}$ be the function defined by:

$$\forall k \geq 1;\ R(k) = \sum_{i=1}^{k} \frac{1}{r(i)}.$$

In the regime 3, we have the following theorem.

Theorem 9 (T., 2021+)**.** *For $\epsilon \in (0,1)$ fixed, for any initial state x,*

$$t_{\text{MIX}}(x;\epsilon) \leq (1+o(1))R(\|x\|_\infty) + \mathcal{O}(\log n). \tag{16}$$

In addition, if the initial state $x = x^{(n)}$ satisfies $\left\|x^{(n)}\right\|_\infty \xrightarrow{n\to\infty} \infty$, then

$$t_{\text{MIX}}(x;\epsilon) \geq (1-o(1))R(\|x\|_\infty). \tag{17}$$

Maximizing over all initial states x, we obtain

Corollary 10 (Cutoff)**.** *Suppose additionally that $R(m) \gg \log n$. Then for $\epsilon \in (0,1)$ fixed,*

$$\frac{t_{\text{MIX}}(\epsilon)}{R(m)} = 1 + o(1). \tag{18}$$

In other words, the system exhibits cutoff at time $R(m)$.

The conditions imposed on r implies that the mean-field jump rate is always upper bounded by a dimension free constant. As the potential function r is not bounded, we expect a stronger drift than the case previous case. The proof of the lower bound in Theorem 9 is obtained by studying the marginal law of a site at equilibrium, and to look at the observable that counts the number of particles at the initially highest site. The proof of the upper bound in Theorem 9 also relies on an argument of *separation of time scaling*. We prove that after a time $t = (1+o(1))R(\|x\|_\infty)+\mathcal{O}(\log n)$, $\|X(t)\|_\infty = \mathcal{O}(\log n)$. From such configuration, the system reaches equilibrium in $\mathcal{O}(\log n)$. The strategy is similar to that in the previous case, except for now we need a stronger version of (12): There exists a dimension free constant κ such that for $\|x\|_\infty = \mathcal{O}(\log n)$,

$$\mathbb{P}_x[\tau > \kappa \|x\|_\infty] \leq \frac{\kappa}{n^2}. \tag{19}$$

We can also prove that the Poincaré constant is bounded away from zero and infinity.

Theorem 11 (Poincaré constant)**.** $\lambda_* = \Theta(1)$.

Mean-field case, r is linear. When $r(k) = k, \forall k \in \mathbb{N}$, we see that all particles jump independently. Then the system reaches equilibrium once every particle has jumped at least one time. We can see easily that the model exhibits cutoff at time $\log m$, which is due to the concentration of sum of independent exponential variables. This implies that the upper bound in (16) is sharp, at least up to the order. It happens that if we start from a very flat configuration, for example, $m = n; x_i = 1, \forall i \in [n]$, the system also needs a time $\Theta(\log n)$ to reach equilibrium.

Question 1. *Can we prove that* (16) *is sharp for every configuration x, which means from any configuration x, $t_{\text{MIX}}(x;\epsilon) = \Theta(\log n)$?*

3.4 Comparision technique

In [6], Hermon and Salez prove a comparison technique that allows us to extend effortlessly the result on the spectral gap in the mean-field model to other geometries. This comparison technique is more general and stronger than the one used by Morris in [9]. More precisely, let P and Q be two irreducible transition matrices on V which admit the same stationary distribution μ. Let $ZRP(P, r, m)$ and $ZRP(Q, r, m)$ denote the Zero-Range processes with m particles, potential function r, and transition matrices P and Q respectively. Let $\mathcal{E}_P$ and $\lambda(P)$ (resp. $\mathcal{E}_Q$ and $\lambda(Q), \mathcal{E}_{ZRP(P,r,m)}$ and $\lambda(P, r, m)$, $\mathcal{E}_{ZRP(Q,r,m)}$ and $\lambda(Q, r, m)$) be the Dirichlet form and Poincaré constant associated to P (resp. $Q, ZRP(P, r, m), ZRP(Q, r, m)$) respectively. We have the following theorem.

Theorem 12 (Hermon, Salez 2019)**.** *We have*

$$\inf\left\{\frac{\mathcal{E}_{ZRP(P,r,m)}(f,f)}{\mathcal{E}_{ZRP(Q,r,m)}(f,f)}\right\} = \inf\left\{\frac{\mathcal{E}_P(\varphi,\varphi)}{\mathcal{E}_Q(\varphi,\varphi)}\right\}, \tag{20}$$

where the infimum of the left-hand side (resp. right-hand side) is taken over all non constant function on Ω (resp. on V).

The proof involves a combinatorial identity that relates the Dirichlet form of the Zero-Range process with that of the underlying random walk. The statement is true for any potential function r. If we take $Q = \Pi$ to be the matrix such that all of its rows are equal to μ, then the right-hand side is just the Poincaré constant $\lambda(P)$. Then we obtain

Corollary 13.

$$\mathcal{E}_{ZRP(P,r,m)}(f,f) \geq \lambda(P)\mathcal{E}_{ZRP(\Pi,r,m)}(f,f)$$

In particular, if P is doubly stochastic, then μ is actually the uniform mesure on V. In particular, $ZRP(\Pi, r, m)$ is exactly the mean-field model. Dividing both side by the variance of f under stationary distribution and optimizing, we deduce that

$$\lambda(P, r, m) \geq \lambda(P)\lambda(\Pi), \tag{21}$$

The authors prove another result for P doubly stochastic.

Lemma 14. *Assume that P is doubly stochastic, then*

$$\lambda(P, r, m) \leq \left(1 - \frac{1}{n}\right)\lambda(P)\frac{\mathbb{E}\left[r(x_1)\right]}{Var\left[x_1\right]},$$

where the expectation and the variance is taken with respect to the stationary measure.

Thanks to these comparison, we can obtain results on Poincaré constant of many models by studying the mean-field model.

3.5 When r is not increasing

For completeness, we mention an article discussing the case where the function r is not increasing. This is from the paper [3] (for more recent result, see [7], [8]). The setting in this case is a little different from other cases above. Here we fix a vertex set V, and we fix a transition matrix P, which we assume irreducible and reversible w.r.t the a probability measure μ, on V. The function r is defined by

$$r(0) = 0, r(1) = 1, \text{ and } r(k) = \left(\frac{k}{k-1}\right)^{\alpha}, \ k \geq 2,$$

for some constant $\alpha > 1$. Consider the case where the number of particles m tend to infinity. This system exhibits different kind of behavior to the above cases, notably *condensation*. More precisely, let

$$\mu_* = \max\{\mu(v), v \in V\}, \ V_* = \{v \in V : \mu(v) = \mu_*\}, \ \kappa_* = |V_*|.$$

At equilibrium, with high probability, all but a finite number of particles are located on a same site in V_*. We call the highest site the condensate. At time scale $N^{1+\alpha}$, the evolution of the condensate converges to a certain random walk on V_*.

Question 2. *Study the mixing time of this Zero-Range process ?*

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