Amoebas of Algebraic Curves

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Chapitre 1

Preliminaries

1.1 Definitions

Definition 1.1. *An algebraic curve of a given degree d in $\mathbb{R}P^k$ (respectively $\mathbb{C}P^k$) is the zero set of a homogeneous polynomial p, where p is of three variables and* $\deg(p) = d$.

Definition 1.2. *The logarithmic map, denoted by $\mu : (\mathbb{C}^*)^k \to \mathbb{R}^k$ is the map :*

$$(x_1, \ldots, x_k) \to (\log|x_1|, \ldots, \log|x_k|).$$

Definition 1.3. *Let A be a complex algebraic curve defined by a polynomial p. The logarithmic Gauss map, denoted by $\gamma : A \to \mathbb{C}P^{m-1}$ is the map :*

$$(z_1, \ldots, z_m) \to [z_1 \partial_{z_1} p(z_1, \ldots, z_m) : \ldots : z_m \partial_{z_m} p(z_1, \ldots, z_m)].$$

Definition 1.4. *We call f a Laurent polynomial if $f(x_1, \ldots, x_k) = \sum\limits_{\omega \in \mathbb{Z}^k} a_\omega x^\omega$, where for almost every ω, a_ω is 0.*

Definition 1.5. *The amoeba of a Laurent polynomial f is the subset $\mu(A) \subset \mathbb{R}^k$, where A is the hypersurface defined in $(\mathbb{C}^*)^k$ by f.*

Notation. *It is almost a convention in this report that we denote :*
(1) A is a hypersurface defined in $(\mathbb{C}^*)^k$ by some Laurent polynomial f
(2) $\mathbb{R}A$ is the intersection of some A with $(\mathbb{R}^*)^k$
(3) $\mathbb{R}\bar{A}$ the algebraic curve in $\mathbb{R}P^k$ related to some A, where the Laurent polynomial defining A is changed into a homogeneous polynomial by adding an extra variable.

Definition 1.6. *The Newton polytope N(f) of a Laurent polynomial $f = f(x_1, \ldots, x_k) = \sum\limits_{\omega \in \mathbb{Z}^k} a_\omega x^\omega$ is the convex hull in $\mathbb{R}^k$ of the set $\{\omega : a_\omega \neq 0\}$.*

The amoeba and the Newton polytope for $P(z, w) = 1+z+z^2+z^3+z^2w^3+10zw+12z^2w+10z^2w^2$ are as in the picture.

We introduce the amoebas as tools for the last part, and discuss the basic properties of amoebas, as well as stating the relationships between the amoebas and the Newton polytopes. More precisely, the vertices of Newton polytope are in one-to-one correspondence with the connected components of the complement of the amoebas which contain an affine convex cone with non-empty interior.

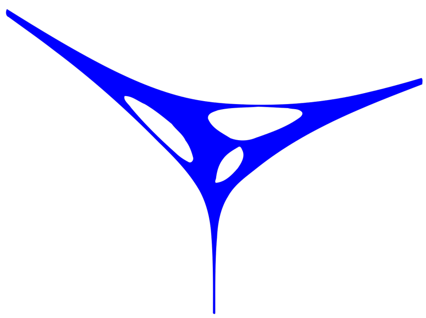

FIGURE 1.1 – Amoeba

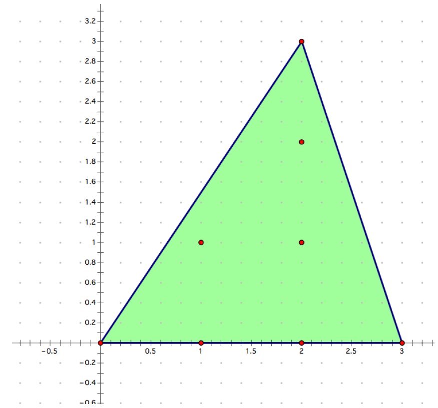

FIGURE 1.2 – Newton polytope

Theorem 1.7. *Vertices of the Newton polytope $\Delta = N(f)$ are in bijection with those connected components of the complement $\mathbb{R}^k - \mu(A)$ which contain an affine convex cone with non-empty interior.*

A deeper relation states that we have a natural injection from the set of the connected components of the complement of the amoeba to the set of interger points belonging to the Newton polytope.

Theorem 1.8. *(Forsberg-Passare-Tsikh) For a polynomial f with m variables, we have a natural injective map :*

$$Ind : \{\textit{the connected component of } \mathbb{R}^m - \mu(A)\} \to (\Delta = N(f)) \bigcap \mathbb{Z}^m$$

Then we step on to use the idea of amoebas to discover a relation between the Newton polytope and the topological type of M-curves, which is the Mikhalkin theorem.

1.2 The Mikhalkin theorem

Definition 1.9. *An algebraic curve $\mathbb{R}\bar{A} \subset \mathbb{R}P^2$, which is the zero set of a polynomial p of degree d, is an M-curve if :*
(1) $\mathbb{R}\bar{A}$ is nonsingular.
(2) the number of connected components of $\mathbb{R}\bar{A}$ is $\frac{(d-1)(d-2)}{2} + 1$.

Definition 1.10. *We say that $\mathbb{R}\bar{A}$ is in maximal position with respect to a collection of n lines $l_1, \ldots, l_n$ if :*
(1) $\mathbb{R}\bar{A}$ is an M-curve.
(2) there are n disjoint arcs $a_1, \ldots, a_n \subset \mathbb{R}\bar{A}$ s.t. :
(a) a_j intersects l_j in d points.
(b) all the arcs belong to the same component of $\mathbb{R}\bar{A}$.
We say that $\mathbb{R}\bar{A}$ is in cyclically maximal position with respect to a collection of n lines $l_1, \ldots, l_n$ if, in addition, the numbered sequence of arcs c_j appears in the cyclic order on the component of $\mathbb{R}\bar{A}$.

The word 'maximal' comes form the following Harnack's Inequality.

Theorem 1.11. *(Harnack) Let $C \subset \mathbb{R}P^2$ be a curve defined by equation $f(X,Y,T)=0$, where f is homogeneous of degree d. Suppose C is non-singular, then the number of connected components of C does not exceed $g+1$, with $g = \frac{(d-1)(d-2)}{2}$.*

The main questions we are going to ask ourselves in this presentation are : do M-curves in maximal position really exist for some n and d and if they do what are the possible topological types (called *maximal types*) of $(\mathbb{R}P^2, \mathbb{R}A, l_1 \cup ... \cup l_n)$ when we fix n and d? First let's take a look at what those questions mean for some small value of n. If $n=0$ it means that we don't actually make any assumption on the M-curves we consider : this is a part of Hilbert's 16th problem which is still an open question today so we do not discuss this case . While the case where $n=1$ does not appear to be any easier, we have some ideas imposed nevertheless. If $n=1$ the condition of maximality can be translated to : the set $\mathbb{R}A \setminus l_1$ has at least $\frac{(d-1)(d-2)}{2} + d$ components but by Harnack's inequality it can not have more components than that. In this case M-curves in maximal position were classified for d strictly less than 6 (see the figure below for all the maximal topological types for $d=5$).

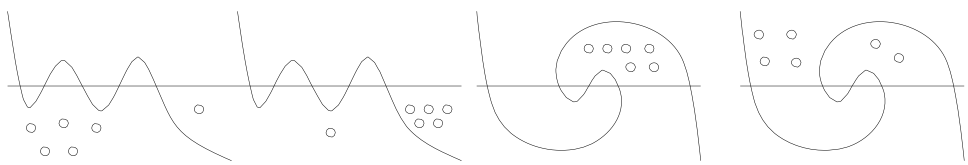

FIGURE 1.3 – Maximal topological types for $n=1$, $d=5$

For $n=2$ the question is still open for $d>4$ but for $d=4$ for instance we can describe all the topological types (see the figure below).

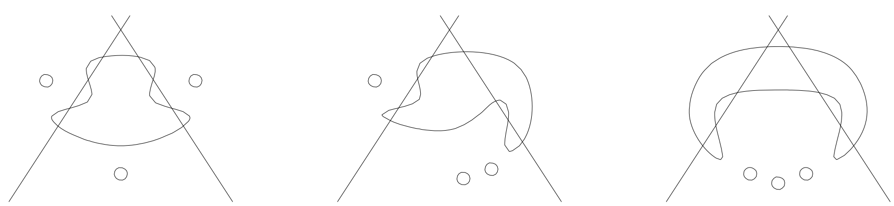

FIGURE 1.4 – Maximal topological types for $n=2$, $d=4$

Now $n=3$ is the case that we are going to deal with in this presentation. The result that we are aiming for is the following :

Theorem 1.12. *If $\mathbb{R}\bar{A}$ is in cyclically maximal position in $\mathbb{R}P^2$ with respect $l_1, \dots, l_n$ then the topological type of $(\mathbb{R}P^2; \mathbb{R}\bar{A}, l_1 \bigcup \dots \bigcup l_n)$ depends only on Δ.*

This theorem will be proved in the last chapter of this presentation. In fact we also have the existence of such maximal types for any d and M-curves having such types can be constructed by a method called "patchworking". This method will be briefly explained in chapter four.

Chapitre 2

Forsberg-Passare-Tsikh Theorem

We first state some basic properties of amoeba :

Property 1. *The amoeba $\mu(A)$ is a closed subset of $\mathbb{R}^2$.*

This is clear since A is closed in $(\mathbb{C}^*)^2$, and μ is proper and continuous, and we have a theorem saying that :

If $f : X \to Y$ is a proper continuous map and Y is a locally-compact Hausdorff space then f is closed.

Property 2. *Each connected component of $\mathbb{R}^k - \mu(A)$ is convex.*

This property is due to two complex analysis results.

Proposition 2.1. *Let $f(x) = \sum_{\omega \in \mathbb{Z}^k} c_\omega x^\omega$ be a (formal) Laurent series in $x_1, \ldots, x_k$ with complex coefficients c_ω (which may be non-zero for all ω).Then the domain of convergence of $f(x)$ in $(\mathbb{C}^*)^k$ has the form $\mu^{-1}(B)$, where $B \subset \mathbb{R}^k$ is a convex subset.*

Proposition 2.2. *If $\varphi(x)$ is a holomorphic function in a domain of the form $\mu^{-1}(B)$, where $B \subset \mathbb{R}^k$ is convex open subset, then there is a unique Laurent series converging to $\varphi(x)$ in this domain.*

proof. Let there be components $C_1, \ldots$. Because we know that C_i is an open set, we choose any $D \subset C_i$ a disk.

Consider $1/f$ on the set $\mu^{-1}(D)$. By proposition 2.2, we know that there is a unique Laurent series $g = \sum_{\omega \in \mathbb{Z}^k} c_\omega x^\omega$ that converges to $1/f$ on $\mu^{-1}(D)$

Moreover, this Laurent series converges on some convex set $\mu^{-1}(B)$ by proposition 2.1. Suppose B is the maximal.

Clearly $B \subset C_i$, but if there is $B \neq C_i$, thus we can find $v \in C_i - B$, s.t $d(v, B) = 0$. Thus we can find $D' \subset C_i \neq \emptyset$.

Because on set $\mu^{-1}(D')$ and $\mu^{-1}(D' \bigcap B)$, there is unique convergent Laurent series g. Thus g converges on $\mu^{-1}(B \bigcup D')$, this contradicts the maximality of B. $\square$

It turns out that there is a strong relation between the Newton polytope and the amoeba. One basic result about the relationship between Newton polytope and the ameoba is the following theorem :

Theorem 2.3. *The vertices of the Newton polytope $\Delta = N(f)$ are in a bijection with those connected components of the complement $\mathbb{R}^k - \mu(A)$ which contain an affine convex cone with non-empty interior.*

proof. The tool we use to detect for an $x \in (\mathbb{C}^*)^k$ if $f(x) \neq 0$ is to try to construct $1/f$.

We want to find an element $l = (l_1, \ldots, l_k) \in \mathbb{Z}^k$ s.t. for all $\omega \in \mathbb{Z}^k \bigcap N(f)$, we have $l.(\omega - \alpha) \ll 0$. This is possible because α is a vertex of $N(f)$.

we build such a cone

$$N_\alpha(\Delta) = \{l | l.(\alpha - \delta) \leq 0, \ \forall \delta \in Q\}$$

We verify this set is a cone with a non-empty interior.

Because α is a vertex, so we can always find some hyperplane with normal vector all of interger coefficients that can separate α and other interger points belonging to the Newton polytope, and it wouldn't hurt much if we perturb slightly the hyperplan. The collection of the normal directions of these hyperplane is in $N_\alpha(\Delta)$, which is what we want.

So, now the last step is to show that if $\mu(x) \in l + N_\alpha(\Delta)$. To do this, we can construct $1/f$ using Laurent expansion :

$$f(x) = a_\alpha x^\alpha (1 + \sum_{\omega \neq \alpha} \frac{a_\omega}{a_\alpha} x^{\omega - \alpha})$$

Now

$$\begin{aligned}
&\mu(x) \in l + N_\alpha(\Delta) \\
\Rightarrow &\mu(x).(\omega - \alpha) = l.(\omega - \alpha) + \delta.(\omega - \alpha) \ll 0 \\
\Rightarrow &|x^{\omega - \alpha}| \ll 1 \\
\Rightarrow &g(x) = | \sum_{\omega \neq \alpha} \frac{a_\omega}{a_\alpha} x^{\omega - \alpha}| \ll 1 \\
\Rightarrow &R_\alpha = \frac{1}{f} = \frac{1}{1+g} = 1 - g + g^2 - \ldots
\end{aligned}$$

So up to now we have deduce that $l(\alpha) + N_\alpha(\Delta)$ is contained in some connected component of $\mathbb{R}^k - \mu(A)$. Moreover we want to show that if $\alpha \neq \beta$ then if $l(\alpha) + N_\alpha(\Delta) \subset C_\alpha$, and $l(\beta) + N_\beta(Q) \subset C_\beta$, then $C_\alpha \neq C_\beta$.

Otherwise, if $l(\beta) + N_\beta(Q) \subset C_\alpha$, then we can find a normal direction $a = (a_1, \ldots, a_k) \in \mathbb{Z}^k \bigcap N_\beta(Q)$ s.t. $a.\omega < a.\beta$, for all $\omega \in \Delta \bigcap \mathbb{Z}^k$.

Let $x(z) = (z^{a_1} e^{l(\beta)_1}, \ldots, z^{a_k} e^{l(\beta)_k})$. First we note that $x(z) \notin A$ for $|z| \gg 0$, this is because $\mu((z^{a_1} e^{l(\beta)_1}, \ldots, z^{a_k} e^{l(\beta)_k})) = l(\beta) + ta \in l(\beta) + N_\beta(Q) \subset C_\alpha$,

Thus we consider $m(z) = g_\alpha(x(z)) = g_\alpha((z^{a_1} e^{l(\beta)_1}, \ldots, z^{a_k} e^{l(\beta)_k}))$.

$$
\begin{aligned}
g(x(z)) &= |\sum_{\omega\neq\alpha}\frac{a_\omega}{a_\alpha}x^{\omega-\alpha}| \\
&\rangle |\frac{a_\beta}{a_\alpha}x^{\beta-\alpha}| - \sum_{\omega\neq\alpha,\beta}|\frac{a_\omega}{a_\alpha}x^{\omega-\alpha}| \\
&= C_1(e^{(l+ta).(\beta-\alpha)} - C_2\sum_{\omega\neq\alpha,\beta}e^{(l+ta).(\beta-\omega)}) \\
&\to\infty \quad as \quad t\to\infty
\end{aligned}
$$

As R_α converges on $\mu^{-1}(l(\alpha)+N_\alpha(\Delta))$, combined with the proof of 2, we can deduce that R_α converges on $\mu^{-1}(C_\alpha)$. But for the $x(z)\in\mu^{-1}(C_\alpha)$ chosen we see that $1-g+g^2+\ldots$ does not absolutely converge thus a contradiction.

Conversely, if a connected component of $\mathbb{R}^k-\mu(A)$ contains a affine convex cone with non-empty interior, then this convex cone must intersects $l(\alpha)+N_\alpha(\Delta)$ for some vertex α as $N_\alpha(\Delta)$ splits the whole plane, so this component must contains $l(\alpha)+N_\alpha(\Delta)$ thus can be mapped to α.

□

Now we look deeper into the problem and we can find a more subtle structure as the following theorem states.

Definition 2.4. *Let x be a point in the amoeba complement $\mu(A)^c$. The order of x is then defined as the vector $v\in\mathbb{Z}^n$ whose components are*

$$
v_j = \frac{1}{(2\pi i)^n}\int_{\mu^{-1}(x)}\frac{z_j\partial_j f(z)}{f(z)}\frac{dz_1\wedge\ldots\wedge dz_n}{z_1\ldots z_n}, \qquad j=1,\ldots,n
$$

Since the homology class of the cycle $\mu^{-1}(x)$ is the same for all x in the same component C of $\mu(A)^c$, we can call v the order of the component C. When we wish to emphasize the dependence on f and x we will write $v_j(f,x)$ rather than v_j.

Remark. *We note further, that we actually have*

$$
\begin{aligned}
v_j(f,x) &= \frac{1}{2\pi i}\int_{|z_j|=e^{x_j},z_k=e^{x_k},k\neq j}\frac{\partial_j f(z)}{f(z)}dz_j \\
&= \frac{1}{2\pi}\int_0^{2\pi}\frac{e^{i\theta}\partial_j f((e^{x_1},\ldots,e^{x_{j-1}},e^{x_j+i\theta},e^{x_{j+1}},\ldots,e^{x_n}))}{f((e^{x_1},\ldots,e^{x_{j-1}},e^{x_j+i\theta},e^{x_{j+1}},\ldots,e^{x_n}))}d\theta.
\end{aligned}
$$

Lemma 2.5. *For any vector $s\in\mathbb{Z}^n-\{0\}$, the direction order $\langle s,v(f,x)\rangle$ is equal to the number of zeros (minus the order of the pole at the origin) of the one-variable Laurent polynomial*

$$
\varphi : w\to f(c_1w^{s_1},\ldots,c_nw^{s_n})
$$

inside the unit circle $|w|=1$. Here $c\in(\mathbb{C}^*)^n$ is any vector with $\mu(c)=x$

proof. By argument principle, the value we are going to calculate is

$$
\frac{1}{2\pi i}\int_{|w|=1}\partial\log f(c_1w^{s_1},\ldots,c_nw^{s_n}).
$$

But we have $[\varphi] = [s_1\gamma_1 + \ldots + s_n\gamma_n \text{ in } H_n(\mu^{-1}(x))]$, where $\gamma_j : \theta \to (e^{x_1}, \ldots, e^{x_{j-1}}, e^{x_j + i\theta}, e^{x_{j+1}}, \ldots, e^{x_n})$. We therefore have

$$\begin{aligned}
\int_{|w|=1} \partial \log f(cw^s) &= \sum_{j=1}^{n} s_j \int_{\gamma_j} \partial \log f(z) \\
&= \sum_{j=1}^{n} s_j \int_{|z_j|=e^{x_j}} \frac{\partial_j f(z)}{f(z)} dz_j \\
&= 2\pi i \sum_{j=1}^{n} s_j v_j.
\end{aligned}$$

This proves the lemma.

□

Proposition 2.6. *The order v of any component of $\mu(A)^c$ is contained in the Newton polytope Δ.*

proof. It suffice to show that $\langle s, v\rangle \leq \max_{\alpha\in\Delta}\langle s, \alpha\rangle$ for any vector $s \in \mathbb{Z}^n - \{0\}$. From 2.5 we know that $\langle s, v\rangle$ is equal to the number of zeros of $w \to f(cw^s)$ inside the unit circle. Since the top degree of this one-variable Laurent polynomial is equal to $\max_{\alpha\in\Delta}\langle s, \alpha\rangle$, the proposition follows. □

Proposition 2.7. *Two different components C and C' of $\mu(A)^c$ cannot have equal orders v and v'.*

proof. Take two points x and x' in $\mathbb{Q}^n - \mu(A)$ and let $s \in \mathbb{Z}^n - \{0\}$ be the direction from x to x', so that $x' = x + rs$ for some $r > 0$. We shall show that $\langle s, v'\rangle > \langle s, v\rangle$. Indeed, by 2.5 these two numbers coincide with the number of zeros inside $|w| = 1$ of the one-variable polynomials $w \to f(c'w^s)$ and $w \to f(cw^s)$ respectively, where $\mu(c') = x'$ and $\mu(c) = x$. Now, since $c'_j/c_j = e^{rs_j}$, and hence $c'w^s = c(e^r w)^s$, we may also interpret $\langle s, v'\rangle$ as the number of zeros of $f(cw^s)$ inside the larger circle $|w| = e^r$. But if there would be no zero of this polynomial in the ring $1 < |w| < e^r$, then the line segment $[x, x']$ would not intersect the amoeba $\mu(A)$. □

Theorem 2.8. *(Forsberg-Passare-Tsikh Theorem)For a polynomial f with m variables, we have an natural injective map :*

$$Ind : \{\textit{the connected component of } \mathbb{R}^m - \mu(A)\} \to (\Delta = N(f)) \bigcap \mathbb{Z}^m$$

proof. We just set Ind to take the order of the component, and the theorem follows from all the lemmas and propositions above. □

Chapitre 3

Harnack's Inequality

One thing to notice is that the word 'maximal' comes from the following theorem of Harnack which describe an extreme case of the topology of the algebraic curves in the real projective plane.

Definition 3.1. *For any connected component of non-singular algebraic curve of dimension 1 in $\mathbb{R}P^2$ we can divide them into two kinds : (1) bounded disks, later we refer to them as ovals. (2) one-sided circle.*

Lemma 3.2. *Any two one-sided circle must intersects.*

proof. Let these two circles be γ_1 and γ_2. Any one-sided circle corresponds to the element $[1] \in \mathbb{Z}/2\mathbb{Z} \simeq H_1(\mathbb{R}P^2, \mathbb{Z}/2\mathbb{Z})$, and any these two circles intersect. This is because if we consider $\pi : S^2 \to \mathbb{R}P^2$ the canonical covering map, then $\pi^{-1}(\gamma_1)$ divide the sphere into two parts, however, if $\pi^{-1}(\gamma_2)$ lies in only one part, and as γ_1 and γ_2 are two different curves, thus there is one point $v \in \gamma_2 - \gamma_1$, thus $-v$ lies in different part, thus $\pi^{-1}(\gamma_1)$ intersects with $\pi^{-1}(\gamma_2)$, thus γ_1 intersects with γ_2. $\square$

Lemma 3.3. *Let $C \subset \mathbb{R}P^2$ be a curve defined by equation $f(X, Y, T) = 0$, where f is homogeneous polynomial of degree d. Suppose C is non-singular, then C is a union of several separated components of dimension 1, C has at most one component which is a one-sided circle.*

proof. Suppose the contrary, then we have γ_1 and γ_2 both are one-sided circle, but they intersect, so f is singular on $\gamma_1 \bigcap \gamma_2$, thus a contradiction. $\square$

Theorem 3.4. *(Bezout)X and Y are hypersurface in $\mathbb{R}P^2$ defined by two polynomials, whose polynomial greatest common divisor is a constant. Then the total number of intersection points of X and Y counted with their multiplicities, is smaller than or equal to the product of the degrees of X and Y.*

Theorem 3.5. *(Harnack) Let $C \subset \mathbb{R}P^2$ be a curve defined by equation $f(X, Y, T) = 0$, where f is homogeneous polynomial of degree d. Suppose C is non-singular in $\mathbb{R}P^2$, then the number of connected components of C p does not exceed $g+1$, with $g = \frac{(d-1)(d-2)}{2}$.*

proof. If $f = f_1 f_2$ is reducible, let $\deg f_1 = d_1$, $\deg f_2 = d_2$ then denote C_1, C_2 the curve defined by f_1 and f_2, thus we have p_1, p_2 the number of connected components of C_1 and C_2, thus if we have shown for all irreducible case, we have

$$p = p_1 + p_2 \leq \frac{(d_1 - 1)(d_1 - 2)}{2} + 1 + \frac{(d_2 - 1)(d_2 - 2)}{2} + 1 \leq \frac{(d_1 + d_2 - 1)(d_1 + d_2 - 2)}{2} + 1 = g + 1.$$

If $d \leq 2$, the claim is trivail, so we suppose $d > 2$.

If C has components $C_1, \ldots, C_{g+2}$, we find a contradiction.

First, we consider the space of plane curves of degree $d-2$, it is a space of dimension $\frac{(d-2)(d+1)}{2}$, so if we fix $q = \frac{(d-2)(d+1)}{2}$ points in $\mathbb{R}P^2$, there is a curve of degree $d-2$ through those points.

But as $d > 2$, we have $q > g+1$, let us take $g+1$ points $P_i \in C_i (1 \leq i \leq g+1)$, and q-g-1 other distinct points on C_{g+2}. By 3.3 , we can suppose C_i are all ovals for $(1 \leq i \leq g+1)$ thus there is C', a curve of degree d-2 passing through these points. We notice that, for each component C_i, as C_i is an oval, which means the intersection number of C_i and C' is at least two(multiplicity counted). Thus we have the total intersection number of at least $2g+2+q-g-1 = q+g+1$ points(counted with multipicity).

But as C is irreducible so the intersection points of C and C' are isolated of at most of number $d(d-2)$. So we have $d(d-2) \geq q+g+1 = (d-1)^2$ which is a contradiction.

□

The inequality is actually achievable, for $d = 1$, any homogeneous polynomial of degree 1 is of the form $ax + by + ct$, thus defines a line, thus satisfies the claim. For $d = 2$, we can consider $x^2 + y^2 = z^2$, where its component is an oval, for greater d, the method to construct an M-curve is described in the later section.

Chapitre 4

Introduction to tropical geometry

Besides giving a proof of the theorem we can also give some more insights on why heuristically this result should be true through what is called tropical geometry.

4.1 Introduction to tropical geometry

The tropical semi-field denoted by $\mathbb{T}$ is the set of real numbers endowed with what is called the tropical addition (denoted by $+_\infty$) and the tropical multiplication (denoted by $\times_\infty$) which are respectively defined by :

$$x +_\infty y = max(x, y) \qquad\qquad x \times_\infty y = x + y \tag{4.1}$$

to which we add $-\infty$ so that our new addition possesses a neutral element. Tropical geometry studies algebraic curves on the tropical semi-field but since it is not a field (the reader may have noticed that the new addition still doesn't have any inverses) we need a new definition of algebraic curves. First notice that tropical polynomial functions in two variables are now defined by expressions such as :

$$\max_{i,j}(a_i + ix + jy) \tag{4.2}$$

Such functions are piecewise linear and globally convex. It means that their graph is an unbounded polygon with vertices and edges. We define the set of roots of a tropical polynomial function to be the projection of the edges of the polygon on the plane. This is the set of points (x, y) such that the maximum defining the polynomial is obtained at least twice at (x, y). Here are some examples of tropical curves in the plane :

Now the interesting thing about tropical curves is that the relation between a tropical curve and its Newton polygon is much more explicit than in the classical case. Let P be a tropical 2-dimensional polynomial, A be its tropical curve and Δ its Newton polygon. By definition of a tropical curve in the plane the vertices of A correspond to the set of couples (x, y) where the maximum defining P is obtained at least for three monomials. For each vertex δ of A denote by Δ_δ the convex hull of the integer points (i, j) for which the monomial $a_{ij} + ix + jy$ is maximal in δ. Each Δ_δ is a subset of Δ and by the convexity of the graph of P we can see that the family of Δ_δs forms in fact subdivisions of Δ. The figure below shows some tropical curves and their Newton polygon subdivisions :

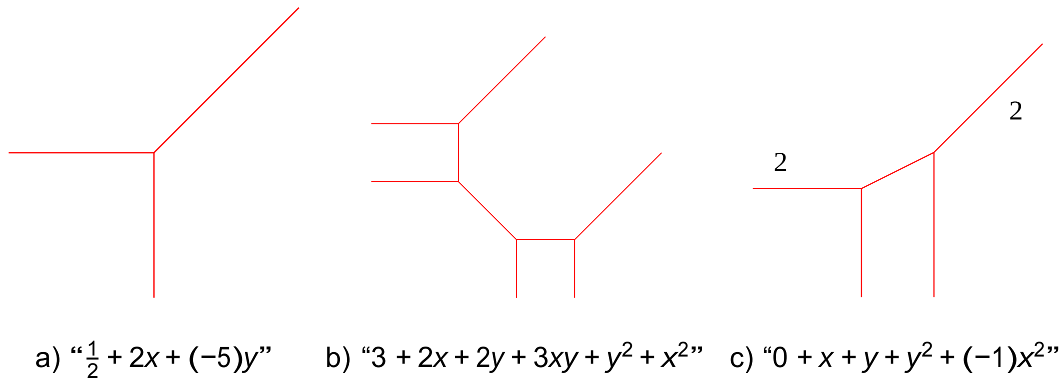

FIGURE 4.1 – Some tropical curves

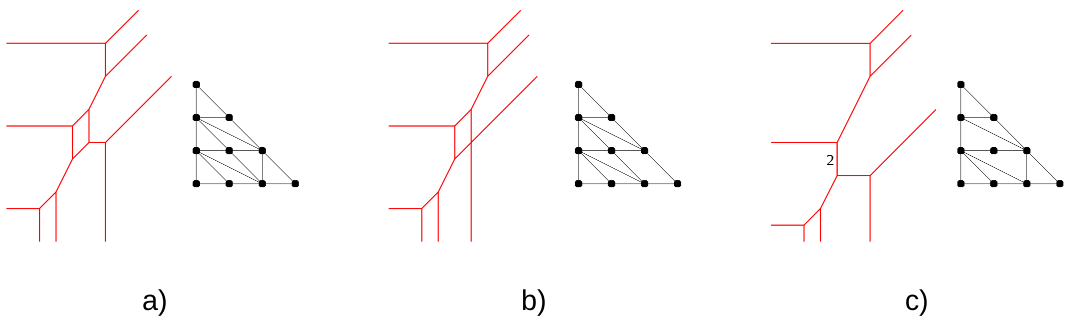

FIGURE 4.2 – Some tropical curves and their subdivisions

The edges of this subdivision form a graph in $\mathbb{R}^2$ for which every face corresponds to a vertex of A, every vertex corresponds to a face of A and edges correspond to (orthogonal) edges. In terms of graph theory our graph is the dual graph of A. But the faces of A are exactly the components of $\mathbb{R}^2 \setminus A$ and the vertices of the subdivision graph are integer points of Δ : we have just proved the Forsberg-Passare-Tsikh theorem for tropical curves. What remains to highlight is the link between tropical and classical geometry.

4.2 How tropical geometry is related to classical geometry

A summarised answer to that question is that the tropical semi-field is obtained as a limit of classical semi-fields. To see this recall that we have the well-known semi-field $(\mathbb{R}_+, +, \times)$ and the family of functions $(\log_t)_{t>1})$:$\mathbb{R}_+ \to \mathbb{T}$ which defines a family of semi-field structures on $\mathbb{T}$ by the formulas :

$$\forall t > 1 \qquad x +_t y = log_t(t^x + t^y) \qquad x \times_t y = log_t(t^x t^y) = x + y. \tag{4.3}$$

But since in $\mathbb{R}_+$ we have $\max(x, y) \leq x + y \leq 2\max(x, y)$ by applying $\log_t$ we get :

$$\forall t > 1, \max(x, y) \leq x +_t y \leq \max(x, y) + \log_t 2. \tag{4.4}$$

Thus $t \to +\infty$ shows that the tropical semi-field can be obtained as a limit of semi-fields arising naturally from the classical semi-field. Now applying that to geometry we can obtain tropical objects as limit deformations of classical objects simply by taking the logarithmic map of classical curves(see chapter one for the definition) -which gives us an amoeba- and then rescaling the base of the logarithm to ∞. For instance the figure below show how we can deform a classical line in the plane to a tropical line.

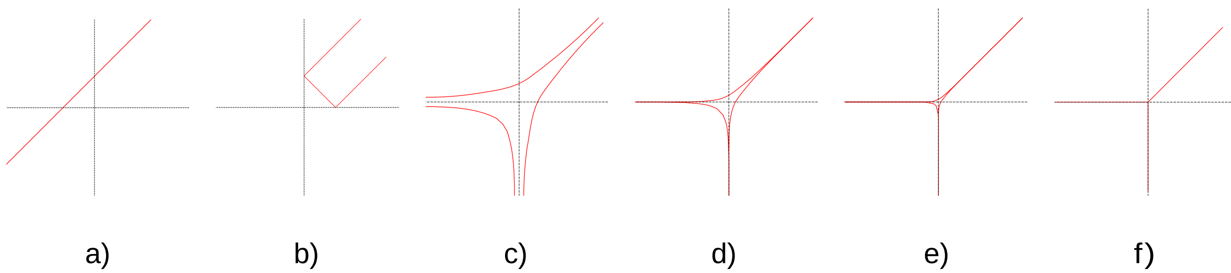

FIGURE 4.3 – From a classical line to a tropical line. a)b)c)Apply the logarithmic map d)e)f)Rescale $t \to \infty$

This process is called dequantization. However one can remark that doing this will only enable us to obtain a certain class of tropical curves : those with only one vertex. To obtain all the diversity of tropical curves the solution is to modify the polynomial as we rescale the amoeba. More precisely take a family P_t of polynomials indexed by $t > 1$, denote by $\mathcal{A}_t$ the image of the algebraic curve of P_t by the mapping $(x, y) \to (\log_t(|x|)\log_t(|y|))$ and look at the limit as t tends to infinity. The following theorem states that all tropical curves can be obtained using this technique :

Theorem 4.1. *Let $P_\infty(x, y) = \sum_{i,j} a_{i,j} x^i y^j$ be a tropical polynomial. For each coefficient $a_{i,j}$ different from $-\infty$ choose a finite set $I_{i,j} \subset \mathbb{R}$ such that $a_{ij,} = \max\ I_{i,j}$ and choose $\alpha_{i,j}(t) = \sum_{r \in I_{ij}} \beta_{i,j,r} t^r$ with $\beta_{i,j,a_{i,j}} \neq 0$. Define $P_t(x, y) = \sum_{i,j} \alpha_{i,j}(t) x^i y^j$. Then $\mathcal{A}_t(P_t)$ tends to the tropical curve defined by P_∞ when t tends to $+\infty$.*

4.3 Viro's patchworking

In this section we describe a method which was developed by Oleg Viro to prove the existence of certain algebraic curves having some wanted topological type via purely combinatorial means and tropical geometry (in fact the formulation we are going to use here is not Viro's since he developed this method before tropical geometry had taken shape). The method consists in starting from a tropical curve and modify it to obtain at the end a subset in the plane which is neither a tropical curve nor a classical curve but which has the same topological type as a certain classical curve. Let us describe now this method (one could see it as the reverse action of dequantization).

Define $\mathbb{R}^2_{(-1)^{j_1},(-1)^{j_2}} = \{(x_1, x_2) | (-1)^{j_1} x_1 > 0, (-1)^{j_2} x_2 > 0\}$. Let C be a tropical curve. We suppose that C verifies some technical properties : for every edge of C we have that $\max(\gcd(|i -$

$k|, |j-l|))$ where the maximum is taken on all the couples of pairs $((i,j),(k,l))$ such that the two faces which are on both sides of the edge are respectively given by the monomials (i,j) and (k,l) is an odd integer. Furthermore C only has vertices with 3 adjacent edges. Now the steps are the following : first- Shift the curve so that every vertices of C belongs to a single quadrant of $(\mathbb{R}\setminus 0)^2$ and erase what doesn't fit in this quadrant then second- copy the result on all the 3 other quadrants by means of reflexions, then third - for every edge of the initial curve erase two out of the four copies of this curve following two rules : if e' is erased then the other copy erased is in the quadrant $\mathbb{R}^2_{(-1)^{i_1},(-1)^{i_2}}$ where the couple (i_1,i_2) defines a vector tangent to the initial edge. For each vertex of C and for each quadrant either one or three out of the three adjacent edges are erased. See the figures below for some examples :

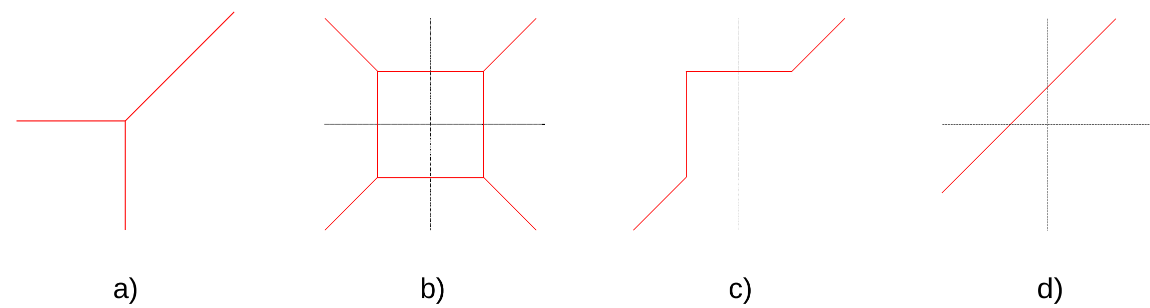

FIGURE 4.4 – Viro's patchworking applied to a tropical line

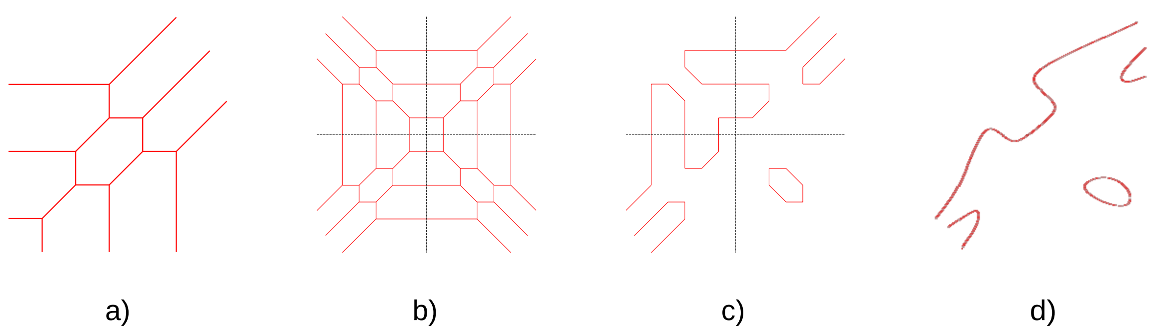

FIGURE 4.5 – Viro's patchworking applied to a degree three tropical curve

We have then the following theorem :

Theorem 4.2 (O. Viro)**.** *For any curve C satisfying the technical properties for the patchworking method the patchworking result has the same topological type as some real algebraic curve in the plane with the same degree.*

We will not prove this theorem here but it can be found in Viro's paper [2].

Chapitre 5

Proof of Mikhalkin's theorem

In this part we prove the results that were stated in the first part concerning the topological type of real algebraic M-curves in cyclical maximal position in the real projective plane :

Theorem 5.1 (G. Mikhalkin)**.** *If $\mathbb{R}\bar{A}$ is in cyclically maximal position in $\mathbb{R}P^2$ with respect to $l_1, \ldots, l_n$ then the topological type of $(\mathbb{R}T; \mathbb{R}\bar{A}, l_1 \bigcup \ldots \bigcup l_n)$ depends only on Δ.*

We are going to present the proof given by Mikhalkin in [3]. We only deal with the case $n = 3$ and we restrict ourselves to M-curves whose Newton polygon is the convex hull of the points $(0,0), (0,d), (d,0)$ (a triangle).

For the rest of the chapter let A be a complex non singular algebraic curve in the complex projective plane given by a polynomial with real coefficients. Suppose that $\mathbb{R}A$ is an M-curve in cyclical maximal position with regard to the x-axis, the y-axis and the infinite line. Suppose also that the Newton polygon of A is the triangle described above. The proof will consist in demonstrating the two following propositions :

Proposition 5.2. *$\mu_{|\mathbb{R}A}$ is an embedding.*

Proposition 5.3. ind *(described in chapter three) is a bijection that maps unbounded components of $\mathbb{R}^2 \setminus \mu(A)$ to exterior integer points of Δ and bounded components to interior integer points of Δ. Furthermore each half-plane is bounded by an arc of $\mu(\mathbb{R}A)$ and each disk is bounded by an oval of $\mu(\mathbb{R}A)$.*

For the first one we will use the logarithmic Gauss map which was introduced in the first chapter. The main result we need to know about the logarithmic Gauss map is the following :

Lemma 5.4. *The degree of the logarithmic Gauss map $\gamma : \overline{A} \to \mathbb{C}P^1$ is* $2\mathrm{Area}(\Delta)$.

The proof of this result is omitted but can be found in [1]. We can already notice that since γ is a holomorphic map between smooth complex varieties the lemma gives us the exact cardinal of each fiber of γ (where each preimage is counted with its multiplicity).

Now denote by F the set of critical points of $\mu_{|A}$. We have the following lemma :

Lemma 5.5. $F = \mathbb{R}A$

proof. First we show that $F{=}\gamma^{-1}(\mathbb{R}P^1)$. Indeed by the definition of μ a point P is critical for $\mu_{|A}$ if and only if the tangent space to A at P contains $\partial\theta_1$ or $\partial\theta_2$ which is equivalent to the tangent space at $\log(P)$ (where log is one of the branches of the holomorphic logarithm) having a purely imaginary vector which is equivalent to the fact that the complex direction normal to this tangent space is real. Thus locally we have $F{=}\left(\mathrm{G}\circ\log\right)^{-1}(\mathbb{R}P^1)$ and we have locally $\gamma{=}\ \mathrm{G}\circ\log$ which gives the result. Now since by definition of γ and by the fact that we consider real coefficiented polynomials we have $\gamma(\mathbb{R}A)\subset\mathbb{R}P^1$ which gives the inclusion $\mathbb{R}A\subset F$.

We've just seen that $F{=}\gamma^{-1}(\mathbb{R}P^1)$ and $\mathbb{R}A\subset F$ so what remains to show is that all the preimages of a real direction by γ are indeed in $\mathbb{R}A$ and furthermore by 5.4 we have an upper bound of the cardinal of the preimage so it is sufficient to find enough preimages in $\mathbb{R}A$ (enough being exactly $2\mathrm{Area}(\Delta)$).

Denote by A the area of Δ, a the number of lattice points in the interior of Δ and b the number of lattice points on the boundary of Δ. Pick's formula gives us the relation : $2A = 2a + b - 2$. Since we consider only M-curves we know that the number of components of $\mathbb{R}\overline{A}$ is exactly $a+1$ and by the maximality assumption a is the number of ovals in $\mathbb{R}A$. The remaining component of $\mathbb{R}\overline{A}$ is a collection of h arcs which go from axis to axis. By the cyclical maximality we know that the number of arcs which go from one axis to the same axis is exactly $\sum_j(d_j-1) = b-n$ where d_j is the number of lattice points on the side j of the Newton polygon. We have n remaining arcs which respectively go from axe l_i to axe l_{i+1}. Now let x be any element in $\mathbb{R}P^1$. Each one of the a ovals add at least 2 elements in $\gamma^{-1}(x)\cap\mathbb{R}A$, each one of the first category of arcs add at least one element and the last n arcs give together at least $n-2$ preimages (see the figure below). Summing up the three terms we obtain exactly $2a-b-2$ preimages which is what we needed. $\square$

So in the end we have $\gamma^{-1}(\mathbb{R}P^1) = F = \mathbb{R}A$ and $\mathbb{R}A$ is non singular so γ does not have any real critical point and thus $\mu(\mathbb{R}A)$ does not have any inflection point. From this we can deduce :

Lemma 5.6. $\partial\mu(A) = \mu(\mathbb{R}A)$

We are now ready to prove 5.2 :

proof. By its definition $\mu_{|\mathbb{R}A}$ is an immersion. Suppose it is not injective : there exists a point in the plane where two branches of $\mu(\mathbb{R}A)$ collide but since $\mu(\mathbb{R}A)$ does not have any inflexion point and $\mu(\mathbb{R}A) = \partial\mu(A)$ one of the branches must intersect $\mathbb{R}^2\setminus\mu(\mathbb{R}A)$ which is a contradiction. $\square$

Now by 5.2 and 5.6 to each component of $\mathbb{R}^2\setminus\mu(A)$ we can associate a component of $\mathbb{R}A$ (given by the inverse image of the boundary of the component by μ) and this association is bijective. Furthermore we know that the number of components of $\mathbb{R}A$ is precisely the cardinal of $\Delta\cap\mathbb{Z}^2$. Thus ind is an injective map between finite sets of same cardinal : it is a bijection. Besides we can verify that it sends unbounded components to boundary lattice points and bounded components to interior lattice points. We have just proved 5.3. We now detain a lot of information on the number and the location of the components of $\mathbb{R}A$ but what is still left to do is to determine how those components are distributed among the quadrants of $(\mathbb{R}\setminus\{0\})^2$. Denote $\Omega_{j_1,j_2} = \mathrm{ind}^{-1}((j_1,j_2))$ and denote $\mathbb{R}^2_{(-1)^{j_1},(-1)^{j_2}} = \{(x_1,x_2)|(-1)^{j_1}x_1>0,(-1)^{j_2}x_2>0\}$. Fix a point $y\in\mathbb{R}A$; we have $\mu(y)\in\partial\Omega_{j_1,j_2}$ for some (j_1,j_2). By a simple change of coordinates we can assume that $y\in\mathbb{R}^2_{(-1)^{j_1},(-1)^{j_2}}$. It turns out that after this choice of signs the same is true for all $\mathbb{R}A$:

Lemma 5.7. *If $x\in\mathbb{R}A$ and $\mu(x)\in\partial\Omega_{i_1,i_2}$ then $x\in\mathbb{R}^2_{(-1)^{i_1},(-1)^{i_2}}$*

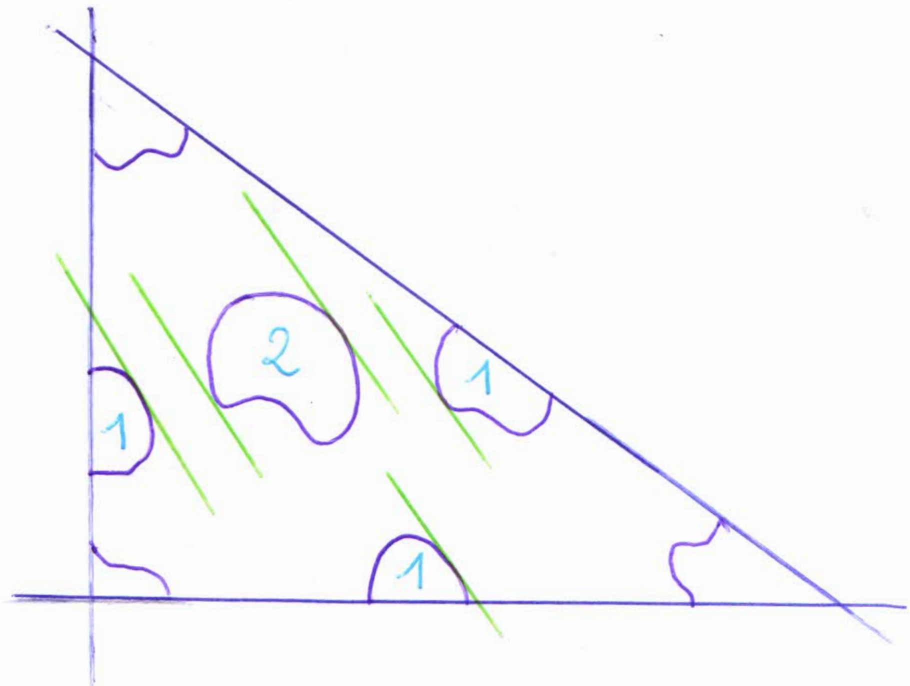

FIGURE 5.1 – The first category gives two preimages each and the second category one each

Thus we can describe the arrangement of every component of $\mathbb{R}A$ through the data given by Δ : this ends the proof of the main result.

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