Normal Forms of Vector Fields

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1 Introduction and notations

As a well-known fact, a vector field $X \in \Gamma(TM)$ non-vanishing at a point x_0 can be represented as $\frac{\partial}{\partial s_1}$ for some coordinate chart $(U, \phi = (s_1, s_2, ..., s_n))$ near that point, which means that X can be sent on a locally constant vector field through some diffeomorphism. However, there is not such a good local representation around singular points. The aim of this presentation is to show a relatively good local result formulated in [2], and to show its concrete applications.

Since the problem is local, we may well consider vector fields defined on a neighbourhood of $x_0 = 0 \in \mathbb{R}^n$ and vanishing at 0. We denote this set by $\mathcal{X}$. Note that $\mathcal{X}$ is preserved by the Lie bracket, so $(\mathcal{X}, [\,])$ is a Lie algebra. We also define $\mathcal{G}$ as the group of local diffeomorphisms of $\mathbb{R}^n$ that fix 0. We shall later study the action of $\mathcal{G}$ on $\mathcal{X}$.

We will frequently use the natural identification $T\mathbb{R}^n \approx \mathbb{R}^{n+n}$ without mentioning it. Under this identification, we can also see $X \in \mathcal{X}$ as a function $X : U \ni 0 \to \mathbb{R}^n$. For any function $\phi : U \ni 0 \to \mathbb{R}^n$, we denote $[\phi]^k$ the up to k-th Taylor development of ϕ at 0 and $[\phi]_l$ the l-th homogenous term of its Taylor development at 0. If $\mathcal{Y}$ is a subalgebra of $\mathcal{X}$, we denote $\mathcal{Y}^k := \{[Y]^k, Y \in \mathcal{Y}\}$ and $\mathcal{Y}_k := \{[Y]_k, Y \in \mathcal{Y}\}$. In the following text, we always consider subalgebras $\mathcal{Y}$ that satisfy an additional condition:

$$\forall k \in \mathbb{N},\, \mathcal{Y}_k \subseteq \mathcal{Y},$$

and we call them graded subalgebras as a reminder.

Finally, for $X \in \mathcal{X}$, as $A = [X]^1 = [X]_1$ is the linear part of X at 0, we see it as a matrix of $\mathcal{M}(n, \mathbb{R})$.

2 Statement of the main Theorem

Recall the Jordan-Dunford decomposition of matrix: for any $A \in \mathcal{M}(n, \mathbb{R})$, $\exists ! A_s \in \mathcal{M}(n, \mathbb{R}), A_n \in \mathcal{M}(n, \mathbb{R})$ such that $A = A_s + A_n$ where A_s is diagonalisable, A_n is nilpotent, and $[A_s, A_n] = 0$.

Theorem 1. *Let $\mathcal{Y}$ be a graded subalgebra of $\mathcal{X}$, $X \in \mathcal{Y}$ a vector field with $A = [X]_1$. We consider a matrix B which satisfies one of the two conditions:*

- *$B = A_s$.*
- *$B \in \mathcal{Y}_1$ and B is adjoint to A under some inner product over $\mathbb{R}^n$.*

Then there exists a sequence $Y_i \in \mathcal{Y}_i$, $i \geqslant 2$, such that for any $k \geqslant 2$, $[(\varphi^1_{Y_2+...+Y_k})_* X - A]^k$ commutes with B, where $\varphi^t_{Y_2+...+Y_k}$ is the flow of $Y_2 + ... + Y_k$ at time t.

Remark 1. *We shall see that the field $(\phi^1_{Y_2+...+Y_k})_* X$ is a member of $\mathcal{Y}$.*

The vector field $(\varphi^1_{Y_2+...+Y_k})_* X$ is called the *normal form* of X. We will see that its linear part is still A, but the next terms in its Taylor development have an additional property given by the commutation with B. In the best cases, all the monomial terms of order 2 to k are zero, which means that X, once pushed forward by the diffeomorphism $\varphi^1_{Y_2+...+Y_k}$, is very close to its linear part. Then the dynamics of the flow of X can be approached by the one of A. In the worst cases, e.g. if $B = 0$, the theorem gives no information at all.

3 Proof of the main theorem

We grade the ring $\mathbb{R}[x_1, x_2, ..., x_n]$ by degree, *i.e.*

$$\mathbb{R}[x_1, x_2, ..., x_n] = \bigoplus_{k \geqslant 0} E_k,$$

where E_k is the space of homogenous polynomials of degree k. We define $E^k := E_1 \oplus ... \oplus E_k$ the vector space of polynomials P of degree lower than k such that $P(0) = 0$.

The main idea in this proof is to associate both vector fields $X \in \mathcal{X}$ and diffeomorphisms $\phi \in \mathcal{G}$ with endomorphisms of E^k, $\widehat{X}$ and $\widetilde{\phi}$, which express their behaviour around 0 at order k. We therefore define two morphisms of vector spaces:

$$\begin{array}{rcccl} \widehat{} & : & \mathcal{X} & \longrightarrow & End(E^k) \\ & & X & \longmapsto & (P \in E^k \mapsto [X \cdot P]^k)\,, \end{array}$$

and

$$\begin{array}{rcccl} \widetilde{} & : & \mathcal{G} & \longrightarrow & End(E^k) \\ & & \phi & \longmapsto & (P \in E^k \mapsto [P \circ \phi]^k)\,. \end{array}$$

3.1 Analysis of the kernels

If $[X]^k = 0, X \in \mathcal{X}$, then

$$[XP]^k = \left[\sum_{i=1}^{n} X^i \partial_i P\right]^k = \sum_{i=1}^{n} \left[X^i \partial_i P\right]^k = \sum_{i=1}^{n} \left[[X^i]^k \partial_i P\right]^k = \left[[X]^k P\right]^k = 0.$$

In the same way, if $[\phi]^k = 0, \phi \in \mathcal{G}$, then

$$\begin{aligned} [P \circ \phi]^k(x) &= \left[P \circ ([\phi]^k(x) + O(|x|^{k+1}))\right]^k = \left[P([\phi]^k(x)) + O(|x|^{k+1})\right]^k \\ &= \left[P \circ [\phi]^k\right]^k(x) = 0, \end{aligned}$$

hence $[P \circ \phi]^k = 0$. Moreover, when $P = \mathrm{id}$, $\widehat{X}(\mathrm{id}) = [X]^k$ and $\widetilde{\phi}(\mathrm{id}) = [\phi]^k$. Thus we have Ker$\widehat{}= \{X \in \mathcal{X}, [X]^k = 0\}$ and Ker $\widetilde{}= \{\phi \in \mathcal{G}, [\phi]^k = 0\}$. We then conclude that the image of $\widehat{}$and $\widetilde{}$are determined by the k-th order development of their preimages.

3.2 Algebraic properties of $\widehat{}$and $\widetilde{}$

Proposition 2. *We have:*

1. $\forall \phi, \psi \in \mathcal{G}$ $(\phi \circ \psi)\widetilde{} = \widetilde{\psi} \circ \widetilde{\phi}$
2. $\forall \phi \in \mathcal{G}, \forall X \in \mathcal{X}$, $\phi_* X \in \mathcal{X}$ *and* $(\phi_* X)\widehat{} = \widetilde{\phi^{-1}} \circ \widehat{X} \circ \widetilde{\phi}$
3. $[X, Y]\widehat{} = \widehat{X} \circ \widehat{Y} - \widehat{Y} \circ \widehat{X}$

Proof. *1.* In fact,

$$\begin{aligned}(\phi\circ\psi)(x) &= [\phi]^k(\psi(x)) + O(|\psi(x)|^{k+1})\\ &= [\phi]^k([\psi]^k(x) + O(|x|^{k+1})) + O(|x|^{k+1})\\ &= [\phi]^k\circ[\psi]^k(x) + O(|x|^{k+1}),\end{aligned}$$

hence for all P in E_k,

$$(\phi\circ\psi)\tilde{}(P) = [P\circ[\phi\circ\psi]^k]^k = [P\circ[\phi]^k\circ[\psi]^k]^k = [[P\circ[\phi]^k]^k\circ[\psi]^k]^k = \widetilde{\psi}(\widetilde{\phi}(P)).$$

2. Take $P\in E_k$, we have $(\phi_* X)P = X(P\circ\phi)\circ\phi^{-1}$, so

$$\begin{aligned}[(\phi_* X)\,P]^k &= \widetilde{\phi^{-1}}\left([X(P\circ\phi)]^k\right) = \widetilde{\phi^{-1}}\left(\sum_{i=1}^n \left[X^i\partial_i(P\circ\phi)\right]^k\right)\\ &= \widetilde{\phi^{-1}}\left(\sum_{i=1}^n \left[X^i L_i\left([P\circ\phi]^{k+1}\right)\right]^k\right),\end{aligned}$$

where $L_i : x_1^{j_1}\dots x_i^{j_i}\dots x_n^{j_n} \longmapsto j_i\, x_1^{j_1}\dots x_i^{j_i-1}\dots x_n^{j_n}$. Since $\forall i, X^i(0)=0$,

$$\begin{aligned}[(\phi_* X)\,P]^k &= \widetilde{\phi^{-1}}\left(\sum_{i=1}^n \left(\left[X^i\partial_i[P\circ\phi]^k\right]^k + \left[X^i\partial_i[P\circ\phi]_{k+1}\right]^k\right)\right)\\ &= \widetilde{\phi^{-1}}\left(\left[X[P\circ\phi]^k\right]^k\right) = \widetilde{\phi^{-1}}\circ\widehat{X}\circ\widetilde{\phi}(P).\end{aligned}$$

3. By the same argument as in *2*, as $X(0)=0$,

$$\begin{aligned}\widehat{XY}(P) &= [X(YP)]^k = [X[YP]^{k+1}]^k = [X[YP]^k]^k + [X[YP]_{k+1}]^k\\ &= [X[YP]^k]^k = \widehat{X}\circ\widehat{Y}(P).\end{aligned}$$

□

3.3 Image of the flow of a vector field

For $X\in\mathcal{X}$, we now consider its flow $\varphi_X : \mathcal{D}\subseteq\mathbb{R}\times\mathbb{R}^n \longrightarrow \mathbb{R}^n$. Since $\varphi_X(t,0)=0$ is defined for all $t\in\mathbb{R}$, we have $\varphi_X^1 = \varphi_X(1,\cdot)$ is defined on a neighbourhood of 0, hence $\varphi_X^1\in\mathcal{G}$.

Proposition 3. *We have* $\widetilde{\varphi_X^1} = \exp(\widehat{X})$.

Proof. Note that $End(E^k)$ is of finite dimension (with any norm on $End(E^k)$), thus the ODE $\alpha'(t) = \alpha(t) \circ \widehat{X}$ with initial condition $\alpha(0) = \mathbb{1}_{E^k}$ has a unique solution. The solution is exactly $\alpha(t) = \exp(t\widehat{X})$.

We can also note that $\partial_t(P \circ \varphi_X) = (XP) \circ \varphi_X$. We consider the Taylor developments of $\partial_t(P \circ \varphi_X)$ and $(P \circ \varphi_X)$ at 0 with respect to x for each t:

$$\partial_t(P \circ \varphi_X)(t,x) = \sum_{|\nu| \leqslant k} \frac{1}{\nu!}\, \theta^\nu(t)\, x^\nu + O(|x|^{k+1})$$

and

$$(P \circ \varphi_X)(t,x) = \sum_{|\nu| \leqslant k} \frac{1}{\nu!}\, \beta^\nu(t)\, x^\nu + O(|x|^{k+1}),$$

where $\nu \in \mathbb{N}^n$ are multi-indices, and $\nu! = \prod_{i=1}^n \nu_i!\,$.

Then $(\beta^\nu)'(t) = \theta^\nu(t)$, for $\beta^\nu(t) = \partial_\nu(P \circ \varphi_X)(t,0)$ and

$$\theta^\nu(t) = \partial_\nu(\partial_t(P \circ \varphi_X))(t,0) = \partial_t(\partial_\nu(P \circ \varphi_X))(t,0) = \frac{d}{dt}\beta^\nu(t)$$

Hence we have $\partial_t\widetilde{\varphi_X^t}(P) = \widetilde{\varphi_X^t} \circ \widehat{X}(P)$, and $\varphi_X^0(P) = P$, so $\widetilde{\varphi_X^t} = \exp(t\widehat{X})$. □

3.4 Conjugated operators

We define two more operators. For $V \in End(E^k)$, if V is invertible, we define

$$\begin{array}{rcccl} Ad_V & : & End(E^k) & \longrightarrow & End(E^k) \\ & & M & \longmapsto & V^{-1} \circ M \circ V\,. \end{array}$$

and for any $L \in End(E^k)$,

$$\begin{array}{rcccl} ad_L & : & End(E^k) & \longrightarrow & End(E^k) \\ & & M & \longmapsto & [M,L] = M \circ L - L \circ M\,. \end{array}$$

Note that $Ad_V \in GL(End(E^k)) \subseteq End(End(E^k))$, and that $ad_L \in End(End(E^k))$.

Proposition 4. $\forall L \in End(E^k), Ad_{\exp L} = \exp(ad_L)$.

Proof. Let $\alpha(t) = Ad_{\exp(tL)}$, then

$$\alpha'(t)M = -L \circ (\alpha(t)M) + (\alpha(t)M) \circ L = (ad_L \circ \alpha(t))M.$$

Thus $\alpha(t) = \exp(ad_{tL})$, so $Ad_{\exp L} = \exp(ad_L)$. □

3.5 Proof of the theorem

The commutative statement of the k-th development of vector fields with the field B is now well characterised by the commutativity of their image under $\widehat{}$ by the third assumption of proposition 2. We'll now prove the theorem by induction, with the help of $\widehat{}$ and $\widetilde{}$. Suppose we already have $Y_i \in \mathcal{Y}_i$, $2 \leqslant i \leqslant k-1$, such that $[(\varphi_Z^1)_* X - A]^{k-1}$ commutes with B and $(\phi_Z^1)_* X \in \mathcal{Y}$, where $Z = \sum_{2 \leqslant i \leqslant k-1} Y_i$.

By the second assumption of proposition 2 and propositions 3 and 4, we have

$$\begin{aligned}\big((\varphi_Z^1)_* X\big)\widehat{} &= \widetilde{(\varphi_Z^1)^{-1}} \circ \widehat{X} \circ \widetilde{\varphi_X^1} = \exp(-\widehat{Z}) \circ \widehat{X} \circ \exp(\widehat{Z}) \\ &= Ad_{\exp \widehat{Z}}(\widehat{X}) = \exp(ad_{\widehat{Z}})\widehat{X}\end{aligned}$$

Indeed, $\forall Y \in \mathcal{Y}^k$, $[Z, Y] \in \mathcal{Y}^k$ so $ad_{\widehat{Z}}(\widehat{Y}) = [\widehat{Y}, \widehat{Z}] = [Y, Z]\,\widehat{}$, and from the kernel of $\widehat{}$, we know that $(\mathcal{Y}^k)\widehat{} = \widehat{\mathcal{Y}} \subseteq End(E^k)$. Then $ad_{\widehat{Z}}$ preserves $\widehat{\mathcal{Y}} \subseteq End(E^k)$, and so does $\exp(ad_{\widehat{Z}}) = Ad_{\exp \widehat{Z}}$.

By the induction hypotheses and the preservation argument for dimension $k-1$, we have $((\phi_Z^1)_* X)\widehat{} = Ad_{\exp \widehat{Z}}(\widehat{X}) = \widehat{A} + \widehat{N} + \widehat{R_k}$, where $N \in \mathcal{Y}^{k-1}$ has no linear term and commutes with B, and $R_k \in \mathcal{Y}_k$. For all $Y_k \in \mathcal{Y}_k$, note that Z consists of monomials with degree at least 2, which leads to $\widehat{Z} \circ \widehat{Y_k} = \widehat{Y_k} \circ \widehat{Z} = 0$. Hence we have $\exp(\widehat{Z} + \widehat{Y_k}) = \exp(\widehat{Z}) \circ \exp(\widehat{Y_k})$. We then have equations:

$$\begin{aligned}&\exp(-\widehat{Z} - \widehat{Y_k}) \circ \widehat{X} \circ \exp(\widehat{Z} + \widehat{Y_k}) \\ &\qquad = \exp(-\widehat{Y_k}) \circ (\widehat{A} + \widehat{N} + \widehat{R_k}) \circ \exp(\widehat{Y_k}) \\ &\qquad = (\mathrm{id} - \widehat{Y_k}) \circ (\widehat{A} + \widehat{N} + \widehat{R_k}) \circ (\mathrm{id} + \widehat{Y_k}), \qquad \text{since } \widehat{Y_k} \circ \widehat{Y_k} = 0 \\ &\qquad = \widehat{A} + \widehat{N} + \widehat{R_k} + \widehat{A} \circ \widehat{Y_k} - \widehat{Y_k} \circ \widehat{A}.\end{aligned}$$

We already proved that $(\phi_{Z+Y_k}^1)_* X$ is a normal form of order k if and only if $\left[\big((\phi_{Z+Y_k}^1)_* X\big)\widehat{} - \widehat{A}, \widehat{B}\right] = 0$. By the computation above, it's equivalent to

$$ad_{\widehat{B}}\left(\widehat{R_k} - ad_{\widehat{A}}(\widehat{Y_k})\right) = 0.$$

The following proposition implies $ad_{\widehat{B}} \circ ad_{\widehat{A}}(\widehat{\mathcal{Y}_k}) = ad_{\widehat{B}}(\widehat{\mathcal{Y}_k})$, hence the equation has a solution $Y_k \in \mathcal{Y}_k$, which finishes the proof.

Proposition 5. $\widehat{\mathcal{Y}_k} = (\widehat{\mathcal{Y}_k} \cap ker(ad_{\widehat{B}})) + ad_{\widehat{A}}(\widehat{\mathcal{Y}_k})$.

Proof. We give a proof only in the case where A is diagonalisable and $B = A$; the general case is quite more difficult. For this case, note that $\widehat{A} = \widehat{B}$ is diagonalisable. For simplicity, we omit the hats in the rest of the proof.

In an eigenbasis of A (of eigenvalues (λ_i)), we take $M = (m_{ij}) \in End(V)$ such that $[A, M] \in ker(ad_B) = ker(ad_A)$. This condition, written in coordinates, gives

$$\forall i, j,\ 0 = [[M, A], A]_{ij} = (\lambda_j - \lambda_i)^2\, m_{ij},$$

which implies $(\lambda_j - \lambda_i)\, m_{ij} = 0 = [M, A]_{ij}$. Hence, we have

$$ker(ad_A) \cap im(ad_A) = 0.$$

Moreover, $dim(ker(ad_A)) + dim(im(ad_A)) = dim(End(V))$. Thus we arrived to the equation. □

Remark 2. *When B is adjoint of A for the inner product $<,>$, the same proof applies : if $[[M, A], B] = 0$,*

$$\begin{aligned} 0 &=< [[M, A], B], M >=< [M, A]B > - < B[M, A], M > \\ &=< [M, A], MA > - < [M, A], AM >= ||[M, A]||^2 \end{aligned}$$

so $[M, A] = 0$ and $ker(ad_A) \cap im(ad_A) = 0$.

4 Link with the Poincaré-Dulac theorem

In this section, we show that the main theorem includes the following one, exposed in [1] :

Theorem 6. *(Poincaré-Dulac) Let $\mathcal{X}' = \{X \in \Gamma(T\mathbb{C}^n)$, holomorphic, such that $X(0) = 0\}$, $\mathcal{G}'$ the group of local diffeomorphisms of $\mathbb{C}^n$ that fix 0, $X \in \mathcal{X}'$, $A = [X]_1$ its linear part, $\lambda_1, \ldots, \lambda_n$ the complex eigenvalues of A, $(e_1, \ldots e_n)$ the eigenbasis of A in $\mathbb{C}^n$, and $k \geqslant 2$. Then*

$$\exists \phi \in \mathcal{G}',\, \exists (\alpha_{m,s}) \in \mathbb{C}^{\mathcal{R}},\, \phi_* X(x) = Ax + \sum_{(m,s)\in\mathcal{R}} \alpha_{m,s}\, x^m\, e_s + O(|x|^{k+1})\,,$$

where $\mathcal{R} = \{(m, s) \in \mathbb{N}^n \times [\![1, n]\!], \lambda^m = \lambda_s$ and $2 \leqslant |m| \leqslant k\}$ and $x^m = \prod_{i=1}^{n} x_i^{m_i}$.

The last expression is the normal form of X. The monomials of the form $x^m\, e_s$, with (m,s) in $\mathcal{R}$, are called resonant terms. The dynamics of the flow strongly depends on $\mathcal{R}$, as we shall see in further examples. Thus this theorem tells which terms can be removed without changing the behaviour of φ_X.

Remark 3. *The main theorem applies in $\mathbb{R}^n$, but this one is only valid for holomorphic vector fields in $\mathbb{C}^n$. Thus, in the proof, we identify $\mathbb{C}^n$ with $\mathbb{R}^{2n}$ through an isomorphism Φ.*

In fact, the main theorem shows the similarity between Poincaré-Dulac and Poincaré-Birkhoff normal form theorems by reformulating both of them in a commmon context.

Proof. We just apply the main theorem in the case $B = A_s$.

We set $\mathcal{X} = \{X \in \Gamma(T\mathbb{R}^{2n}),\ X(0) = 0\}$ and $\mathcal{Y} = \Phi_*(\mathcal{X}')$. As the Lie bracket of holomorphic vector fields is an holomorphic vector field, $\mathcal{Y}$ is a sub-algebra of $\mathcal{X}$. We also have $[\Phi^*X]_1 = \Phi^*[X]_1 = \Phi^*A$, and Φ^*B is the diagonalisable part of Φ^*A, so we can apply the main theorem.

We get $\phi = \Phi^*(\varphi^1_{Y_2+\cdots+Y_k}) \in \mathcal{G}'$ such that $[\Phi_*(\phi_*X - A)]^k$ commutes with $\Phi_*(B)$, which is equivalent to the commutation of $[\phi_*X - A]^k$ and B.

We write the Taylor development of ϕ_*X:

$$\exists(\alpha_{m,s}) \in \mathbb{C}^{\mathbb{N}^n\times[\![1,n]\!]},\ \phi_*X(x) = Ax + \sum_{\substack{(m,s)\\ 2\leqslant|m|\leqslant k}} \alpha_{m,s}\, x^m\, e_s + O(|x|^{k+1})\,.$$

As $B(x) = \sum_{s=1}^n \lambda_s\, x_s\, e_s$, we compute :

$$\begin{aligned}
0 &= \left[[\phi_*X - A]^k, B\right](x)\\
&= \sum_{\substack{(m,s)\\ 2\leqslant|m|\leqslant k}} \alpha_{m,s}\, B(x)^m\, e_s - B\left(\sum_{\substack{(m,s)\\ 2\leqslant|m|\leqslant k}} \alpha_{m,s}\, x^m\, e_s\right) + O(|x|^{k+1})\\
&= \sum_{\substack{(m,s)\\ 2\leqslant|m|\leqslant k}} \alpha_{m,s}\, \lambda^m\, x^m\, e_s - \sum_{\substack{(m,s)\\ 2\leqslant|m|\leqslant k}} \alpha_{m,s}\, x^m\, \lambda_s\, e_s + O(|x|^{k+1})\\
&= \sum_{\substack{(m,s)\\ 2\leqslant|m|\leqslant k}} \alpha_{m,s}\, (\lambda^m - \lambda_s)\, x^m\, e_s + O(|x|^{k+1}).
\end{aligned}$$

The family $(x^m\, e_s)_{1\leqslant |m|\leqslant k, 1\leqslant s\leqslant n}$ is a base of E^k, so all the terms of the sum are zero. If $(m,s)\in\mathcal{R}$, as $\lambda^m - \lambda_s \neq 0$, we conclude that $\alpha_{m,s}=0$, which gives us the expected result. □

5 A direct proof of Poincaré-Dulac theorem

As the main theorem is quite abstract and more complex than Poincaré-Dulac theorem, we give an elementary proof of the latest in the case of a diagonalisable linear part $A=B$. This proof shows how to choose the morphism ϕ. We once again use an induction on k. The case $k=1$ is trivial. Let $k\geqslant 2$ such that the theorem holds for $k-1$. We know that

$$\exists \phi \in \mathcal{G}',\ \exists(\alpha_{m,s})\in\mathbb{C}^{\mathcal{R}},\ \phi_* X(x) = Ax + \sum_{(m,s)\in\mathcal{R}} \alpha_{m,s}\, x^m\, e_s + v(x) + O(|x|^{k+1})\,,$$

with v a homogeneous polynomial of degree k containing no resonant term. We want to find another a homogeneous polynomial h of degree k such that the change of variables $y = x - h(x)$ leads to

$$\phi_* X(y) = Ay + \sum_{(m,s)\in\mathcal{R}} \alpha_{m,s}\, y^m\, e_s + O(|y|^{k+1})\,.$$

We compute

$$\begin{aligned}
\phi_* X(y) &= \phi_* X(x-h(x)) = \phi_* X(x) - d_x(\phi_* X)(h(x)) + O(|x|^{2k})\\
&= Ax + \sum_{(m,s)\in\mathcal{R}} \alpha_{m,s}\, x^m\, e_s + v(x) - \partial_x h\, Ax + O(|x|^{k+1})\\
&= Ay + Ah(x) + \sum_{(m,s)\in\mathcal{R}} \alpha_{m,s}\, y^m\, e_s + v(x) - \partial_x h\, Ax + O(|x|^{k+1}).
\end{aligned}$$

Thus we get the good result if and only if $v(x) = \partial_x h\, Ax - Ah(x) =: L_A h(x)$. The operator L_A is linear, so we can decompose

$$v(x) = \sum_{\substack{(m,s)\notin\mathcal{R}\\ |m|=k}} \alpha_{m,s}\, x^m\, e_s$$

and look for a solution of each equation $L_A h(x) = x^m\, e_s$. But

$$\begin{aligned} L_A(x^m\, e_s) &= \sum_{i=1}^{n} \partial_i x^m\, e_s A x_i - A\, x^m\, e_s \\ &= \sum_{i=1}^{n} m_i\, \frac{x^m}{x_i}\, e_s\, x_i - \lambda_s\, x^m\, e_s = (\lambda^m - \lambda_s)\, x^m\, e_s. \end{aligned}$$

The $\lambda^m - \lambda_s$ are different from zero for $(m,s) \notin \mathcal{R}$, so we just have to take

$$h(x) = \sum_{\substack{(m,s)\notin\mathcal{R} \\ |m|=k}} \frac{\alpha_{m,s}}{\lambda^m - \lambda_s}\, x^m\, e_s.$$

Finally, the change of variables $\psi(x) = y = x - h(x)$ is a local diffeomorphism, and $\psi_* \phi_* X$ is the normal form of X at order k, which concludes the proof.

Remark 4. *Note that the last computations are the same in the two proofs : indeed, $L_A(h)$ is equal to the bracket of h with B in the case $B = A_s$.*

6 Examples

In this section, we just study a few examples to understand the link between the previous theorems and the dynamics of the flow.

6.1 A system with only one resonance

Let $p, q \geqslant 2$ integers, $c \in \mathbb{R}\backslash\{0\}$, and $X : (x,y) \mapsto (x, py + cx^q)$ a holomorphic vector field of $\mathbb{C}^2$. Its linear part is $A = \begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix}$. The dynamical systems associated to A and X are respectively

$$\begin{cases} \dfrac{d}{dt} x = x \\ \dfrac{d}{dt} y = py \end{cases} \quad \text{and} \quad \begin{cases} \dfrac{d}{dt} x = x \\ \dfrac{d}{dt} y = py + cx^q \end{cases}$$

These systems can be solved explicitly. The first one has solutions of the form $x(t) = x_0\, e^t$, $y(t) = y_0\, e^{p\,t}$, which gives a trajectory $y(x) = y_0 \left(\frac{x}{x_0}\right)^p$.

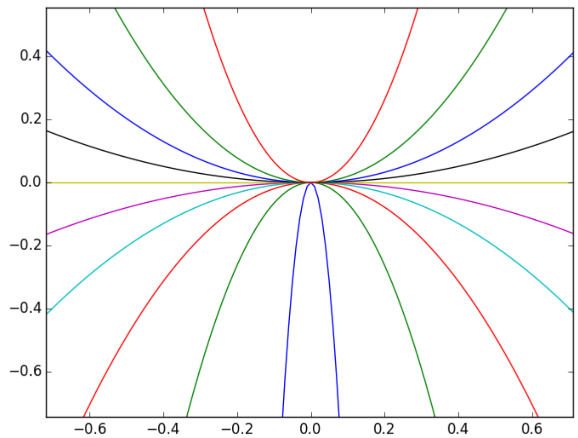

Figure 1: The trajectories of φ_A in the case $p = 2$, with $c = 1$. They all converge at 0, and three trajectories (including the one staying at 0) can be joined to form the C^∞ colored curbs.

For the second one, the solution depends on the values of p and q.

- If $p = q$, we have $x(t) = x_0\, e^t$, and $y(t) = (y_0 + c\, x_0^p\, t)\, e^{p\,t}$, so $y(x) = y_0 \left(\frac{x}{x_0}\right)^p + c\, x^p \log(\frac{x}{x_0})$.

- If $p \neq q$, we have $x(t) = x_0\, e^t$, and $y(t) = \left(y_0 - \frac{c\, x_0^q}{q-p}\right) e^{p\,t} + \frac{c\, x_0^q}{q-p}\, e^{q\,t}$, so $y(x) = \left(y_0 - \frac{c\, x_0^q}{q-p}\right) \left(\frac{x}{x_0}\right)^p + \frac{c}{q-p}\, x^q$.

Here, as $\lambda_1 = 1$ and $\lambda_2 = p$, the only resonance is $\mathcal{R} = \{(m, s)\} = \{((p, 0), 2)\}$. The resonant term $x^p\, e_y$ is non-zero only in the case $p = q$. Thus Poincaré-Dulac theorem states that X is locally isomorphic to A at order k for all $k \geqslant 2$ whenever $p \neq q$. Indeed, the holomorphic application

$$\begin{array}{rccc} \phi : & \mathbb{C}^2 & \longrightarrow & \mathbb{C}^2 \\ & (x, y) & \longmapsto & \left(x, y + \frac{c}{q-p}\, x^q\right) \end{array}$$

is a local diffeomorphism, and $\phi_* A = X$.

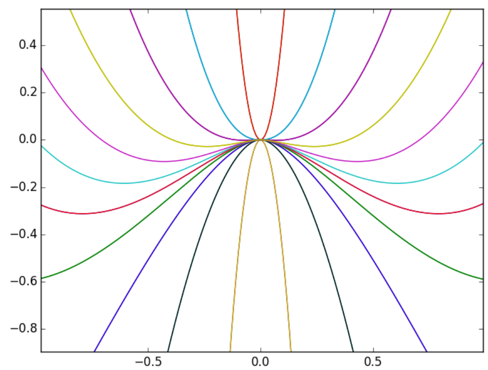

Figure 2: The trajectories of φ_X for $p = q = 2$. They are not C^2 at 0.

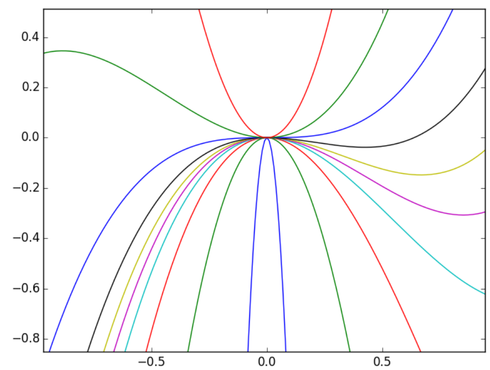

Figure 3: The trajectories of φ_X for $p = 2$ and $q = 3$. They form C^∞ curbs.

At contrary, in the case $p = q$, we can find a C^{q-1}-diffeomorphism

$$\begin{array}{cccc} \phi & : & \mathbb{C}^2 & \longrightarrow & \mathbb{C}^2 \\ & & (x,y) & \longmapsto & (x, y + c\,x^p\,\log x) \end{array}$$

but there is no C^q-diffeomorphism sending φ_A on φ_X. Indeed, we can join 3 trajectories of φ_A (the ones passing at (x_0, y_0), $(0,0)$ and $(-x_0, (-1)^p\, y_0)$ respectively) to make the C^∞-curb $\left\{\left(x, y_0\left(\frac{x}{x_0}\right)^p\right)\right\}$, but that's not possible with φ_X because the function $y(x)$ written above is not C^p at 0. We then use the proposition below :

Proposition 7. *With the same notations as before, let $\phi \in \mathcal{G}_\mathcal{Y}$ and $X, Y \in \mathcal{Y}$ such that $\phi_* X = Y$. If the trajectories of φ_X can be joined to form curbs C^k at 0, then the trajectories of φ_Y can be joined to form curbs C^k at 0 too.*

Remark 5. *As the trajectories are solutions to $\frac{d}{dt}x(t) = Y(x(t))$, the trajectories are C^∞ at any regular point.*

Proof. Let $c :]-\varepsilon, \varepsilon[\mapsto \mathbb{R}^n$ a curb following trajectories of φ_X, i.e. such that $\forall \theta \in]-\varepsilon, \varepsilon[$, $\frac{d}{d\theta}c(\theta)$ is colinear to $X(c(\theta))$. We assume that $c(0) = 0$ and c is C^k at 0. Then $\phi \circ c$ follows trajectories of Y because

$$\forall \theta \in]-\varepsilon, \varepsilon[,\ \frac{d}{d\theta}\phi \circ c(\theta) = d\phi_{c(\theta)}\,\frac{d}{d\theta}c(\theta)$$

is colinear to $Y(c(\theta)) = \phi_* X(c(\theta)) = d\phi_{c(\theta)}\,X(c(\theta))$. □

Remark 6. *The proposition remains true in $\mathbb{C}^n$ with holomorphic vector fields, using the same argument as in Poincaré-Dulac theorem.*

6.2 Harmonic oscillator

We now study a harmonic oscillator with a non-linear term, of the form

$$\begin{cases} \dfrac{d}{dt}x = y \\ \dfrac{d}{dt}y = -x + c\,x^a\,y^b \end{cases}$$

We know that the solutions of the linear system are of the form $x(t) = c_1\,e^{i\,t} + c_2\,e^{-it}$, $y(t) = c_1\,e^{i\,t} - c_2\,e^{-it}$, hence $|x|^2 + |y|^2 = 2(|c_1|^2 + |c_2|^2)$ is constant, so the trajectories are circles.

Here, $A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, so $\lambda_1 = i$ and $\lambda_2 = -i$. Hence

$$\mathcal{R} = \bigcup_{k\geqslant 0} \{((k+1,k),1)\,,((k,k+1),2)\}\ .$$

Consequently, the system above is resonant if and only if $b = a+1$. When multiplying the second equation by y, we get

$$y\,\dot{y} = -x\,\dot{x} + c\,x^a\,y^{b+1}\,,$$

so

$$\frac{1}{2}\frac{d}{dt}\left(|x|^2 + |y|^2\right) = x\,\dot{x} + y\,\dot{y} = c\,x^a\,y^{b+1}.$$

If a is even and $b = a+1$, $x^a\,\dot{x}^{b+1} = (x\,\dot{x})^a\,\dot{x}^2 \geqslant 0$ and this term is different from zero (except at $x = 0$ or $y = 0$), so the radius $r = \sqrt{|x|^2 + |y|^2}$ depends monotonously on time at a finite order, increasing if $\Re(c) > 0$ and decreasing if $\Re(c) < 0$, and the trajectories are spirals.

On the other hand, when $b \neq a+1$ the Poincaré-Dulac theorem implies that the trajectories remain close to cycles at any order k, as they are C^k diffeomorph to the circles of the linear system at order k.

To make it more clear, define $D(x,y) = |x(2\pi) - x(0)|^2 + |y(2\pi) - y(0)|^2$ to measure the evolution of the coordinates after a time 2π. In the linear case, φ_A is 2π-periodic, so $D(x,y) = 0$. With a non-linear term such that $b \neq a+1$, using Poincaré-Dulac theorem, $\forall k$, $D(x, y) = o(r^k)$. Finally, when $b = a+1$, as the variations of r^2 are of order $a + b + 1 = 2a + 2$, $D(x, y) = \Theta(r^{a+1})$.

We now understand better the choice of the word *resonance*: a monomial term is resonant when he corresponds to a perturbation of the linear system that can affect the properties of the dynamics around the origin.

6.3 Coupled oscillators

We now study a system composed of two harmonic oscillators coupled through their position :

$$\begin{cases} \dot{x}_1 = \omega_1\, y_1 + c\, x_2 \\ \dot{y}_1 = \omega_1\, x_1 \\ \dot{x}_2 = \omega_2\, y_2 + c\, x_1 \\ \dot{y}_2 = \omega_2\, x_2 \end{cases}$$

with $\omega_1 > \omega_2 > 0$. Of course, each oscillator has its own resonances, as in the previous example. What we want to know is whether they are non-trivial resonances caused by the interaction between the oscillators. Let's compute the eigenvalues of A:

$$\begin{aligned} \det(A - uI) &= \det \begin{pmatrix} -u & \omega_1 & c & 0 \\ -\omega_1 & -u & 0 & 0 \\ c & 0 & -u & \omega_2 \\ 0 & 0 & -\omega_2 & -u \end{pmatrix} \\ &= (u^2 + \omega_1^2)(u^2 + \omega_2^2) - c^2\, u^2 \\ &= u^4 + (\omega_1^2 + \omega_2^2 - c^2)\, u^2 + \omega_1^2\, \omega_2^2. \end{aligned}$$

For c sufficiently small, the eigenvalues are

$$\lambda_{1,2} = i \left(\frac{\omega_1^2 + \omega_2^2 - c^2 \pm \sqrt{(\omega_1^2 + \omega_2^2 - c^2)^2 - 4\, \omega_1^2 \omega_2^2}}{2} \right)^{\frac{1}{2}}$$

and their complex conjugates. The non-trivial resonances are the ones of the form $a\, \lambda_1 + b\, \lambda_2 = 0$, with $(a, b) \in \mathbb{N}^2 \setminus \{(0, 0)\}$. The function $f(c) = \frac{\lambda_1}{\lambda_2}(c)$ is continuous, monotonic, defined on $[0, \sqrt{\omega_1^2 + \omega_2^2}]$, and reaches $\frac{\omega_1^2}{\omega_2^2}$ at 0 and 1 at $\sqrt{\omega_1^2 + \omega_2^2}$, so f reaches rational values on an infinite countable set of values of c.

As a conclusion, non-trivial resonant terms exist only for specific values of c, ω_1 and ω_2. We can see the eigenvalues as modified frequences, because $\lambda_1 \longrightarrow_{c\to 0} \omega_1$, and $\lambda_2 \longrightarrow_{c\to 0} \omega_2$. The resonances appear whenever the ratio of these frequences is rationnal.

7 Generalisation to a family of vector fields

It's very natural to consider the regularity of the normal forms of a family of vector fields. That's to say if we have a family of vector fields X_λ singular at 0 parameterised by $\lambda \in \Lambda$, and depends on λ with some regularity, then we want to know if there's some family (Y_λ^k) of normal forms of the X_λ with the same regularity on λ. We're going to formalise the problem and give it an affirmative answer.

Definition 1. *Assume Λ is a Banach space, we say the family of vector fields X_λ depends on λ C^r regularly, if the map*

$$\begin{array}{rcl} \Lambda \times U & \longrightarrow & \mathbb{R}^n \\ (\lambda, x) & \longmapsto & X_\lambda(x) \end{array}$$

is C^r smooth as a map from an open subset of the product Banach space $\Lambda \times \mathbb{R}^n$ to $\mathbb{R}^n$. Here $U \ni 0$ is an open set of $\mathbb{R}^n$ and we regard $X_\lambda \in \Gamma(TU)$ as a map $X_\lambda : U \to \mathbb{R}^n$ by the natural identification $T\mathbb{R}^n \approx \mathbb{R}^{n+n}$ mentioned in the beginning.

The answer to the problem is stated here.

Theorem 8. *Let Λ be a Banach space, X_λ a C^r family of vector fields, $A \in \mathcal{Y}_1$ the linear part of X_0 at 0, and we choose B as in the main theorem. Then, there exists a C^r family of vector fields $Y_\lambda^k \in \mathcal{Y}^k$, such that Y_0^k has no linear term, and there exists an open set $V \subseteq \Lambda$, such that $\forall \lambda \in V, [[(\phi^1_{Y_\lambda^k})_* X_\lambda - A]^k, B] = 0$.*

Proof. Similarly to the proof of the main theorem, we prove it by induction on k. Suppose that we have built C^r vector fields Y_λ^{k-1}, $k \geqslant 2$. We then, by the same computation, get the equation:

$$ad_{\widehat{B}}(\widehat{R_\lambda}) - ad_{\widehat{B}} \circ ad_{\widehat{A_\lambda}}(\widehat{Y_\lambda}) = 0,$$

where R_λ and A_λ are the homogenous term of degree k and the linear term of $(\phi^1_{Y_\lambda^{k-1}})_* X_\lambda$ respectively.
The induction hypotheses implies that R_λ and A_λ are C^r families. By proposition 5, we have $ad_{\widehat{B}} \circ ad_{\widehat{A}} : \widehat{\mathcal{Y}_k} \longrightarrow ad_{\widehat{B}}(\widehat{\mathcal{Y}_k})$ is surjective. We may choose a $K \subseteq \widehat{\mathcal{Y}_k}$ as the complementary of its kernal, thus

$$ad_{\widehat{B}} \circ ad_{\widehat{A}} : K \longrightarrow ad_{\widehat{B}}(\widehat{\mathcal{Y}_k})$$

is an isomorphism. By the regularity of A_λ, $\exists V \subseteq \Lambda$ open set, such that $\forall \lambda \in V, ad_{\widehat{B}} \circ ad_{\widehat{A_\lambda}} : K \longrightarrow ad_{\widehat{B}}(\widehat{\mathcal{Y}_k})$ is an isomorphism. We denote

$$\theta_\lambda = (ad_{\widehat{B}} \circ ad_{\widehat{A_\lambda}})^{-1} : ad_{\widehat{B}}(\widehat{\mathcal{Y}_k}) \longrightarrow K,$$

then θ_λ is C^r on λ. Take $Y_\lambda = -\theta_\lambda \circ ad_{\widehat{B}}(R_\lambda)$, which is a C^r family of vector fields and solves the equation.
We finally need to set the initial condition, i.e. to show the existence of a C^r family of vector fields $L_\lambda \in \mathcal{Y}_1$, such that $L_0 = 0$ and

$$\forall \lambda \in V,\ [\exp(-\widehat{L_\lambda})\widehat{A_\lambda}\exp(\widehat{L_\lambda}) - \widehat{A}, \widehat{B}] = 0.$$

Suppose $K \subseteq \widehat{\mathcal{Y}_1}$ a complementary of $ker(ad_{\widehat{B}} \circ ad_{\widehat{A}})$. Consider the map

$$\begin{array}{rccc} F & : & K \times \Lambda & \longrightarrow & ad_{\widehat{B}}(\widehat{\mathcal{Y}_1}) \\ & & (L, \lambda) & \longmapsto & ad_{\widehat{B}}(\exp(-\widehat{L})\widehat{A_\lambda}\exp(\widehat{L}) - \widehat{A}) \,. \end{array}$$

We have $\partial_L F(0,0) = ad_{\widehat{B}} \circ ad_{\widehat{A}}$, which is an isomorphism from $K \approx T_0 K$ to $ad_B(\widehat{\mathcal{Y}_1}) \approx T_0(ad_B(\widehat{\mathcal{Y}_1}))$. Then by the implicit function theorem, we have a C^r family of linear maps $\widehat{L_\lambda}$ which lifts back to a C^r family of vector fields L_λ.

□

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