Introduction au domaine de recherche.

Two Probability Distributions on Random Sets and their Possible Universality.

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Contents

This document features two probability distributions on random sets and two areas of mathematics where they both feature as limit distributions in similar ways.

1. In the first part, these two probability distributions are described.
2. In the second part, one of the two areas of mathematics is introduced.
3. In the third part, the other area is introduced.

Part I. The Airy ensembles

1 Point processes.

Let Σ be the smallest σ-algebra on the set $\mathcal{P}(\mathbb{R})$ of subsets of $\mathbb{R}$ such that for all $K \subset \mathbb{R}$ compact, $f_K : \begin{array}{ccc} \mathcal{P}(\mathbb{R}) & \to & [\![0, +\infty]\!] \\ F & \mapsto & \text{card } (F \cap K) \end{array}$ is measurable.

A point process on $\mathbb{R}$ is a probability measure μ on the σ-algebra Σ such that $\mu \{F \subset \mathbb{R} : \forall K \subset \mathbb{R} \text{ compact}, \text{card } (F \cap K) < +\infty\} = 1$

Theorem 1.1. *A point process μ of canonical random variable F is fully determined by the family of numbers indexed by finite sets of compacts:*

$$r_{K_1,\dots,K_n} = \mathbb{E} \left(\text{card } \{(x_1, \dots, x_n) \in F^n \cap K_1 \times \dots \times K_n : \forall i \neq j \in [\![1, n]\!], x_i \neq x_j\}\right).$$

Let λ be the Lebesgue measure on $\mathbb{R}$.

Let $n \in \mathbb{N}$. If $r_{K_1,\dots,K_n} = \int_{(x_1,\dots,x_n) \in K_1 \times \dots \times K_n} f_n(x_1, \dots, x_n) \lambda(d\,x_1) \dots \lambda(d\,x_n)$, then f_n is called its n-point correlation function. The sequence of n-point correlation functions, if it exists, define the point process.

A determinantal point process on $\mathbb{R}$ of kernel $K : \mathbb{R}^2 \mapsto \mathbb{R}$ is a point process such that its n-point correlation function $\rho^{(n)}$ is

$$\rho^{(n)}(x_1, \dots x_n) = \det \left[K(x_i, x_j)\right]_{(i,j) \in [\![1,n]\!]^2}.$$

See [16] for a survey about determinantal point processes.

A pfaffian point process on $\mathbb{R}$ of kernel $K : \mathbb{R}^2 \mapsto \mathbb{M}(2, 2)$, where $\mathbb{M}(m, n)$ is the set of matrices of size (m, n), is a point process such that its n-point correlation function $\rho^{(n)}$ is

$$\rho^{(n)}(x_1, \dots x_n) = \sqrt{\det \left[K(x_i, x_j)\right]_{(i,j) \in [\![1,n]\!]^2}}.$$

($\left[K(x_i, x_j)\right]_{(i,j) \in [\![1,n]\!]^2} \in \mathbb{M}(2, 2)$ and is a block matrix whose blocks are the $(K(x_i, x_j))_{(i,j) \in [\![1,n]\!]^2}$)

2 The Airy function

Formula 2.1. The Airy function is the function :

$$\mathrm{Ai}\,(x) = \frac{1}{\pi} \lim_{a \to +\infty} \int_0^a \cos(\frac{t^3}{3} + xt)d\,t$$

Some classical properties of this function are listed in the section 10.4 of [18]. It is, up to a multiplicative constant, the unique solution of the equation :

$$\frac{d^2 f}{dx^2}(x) - xf(x) = 0$$

which tends to zero at $+\infty$.

Formula 2.2. The Airy kernel is the function

$$\mathrm{A}\,(x,y) = \begin{cases} \dfrac{\mathrm{Ai}\,(x)\,\mathrm{Ai}\,'(y) - \mathrm{Ai}\,'(x)\,\mathrm{Ai}\,(y)}{x-y} & \text{if } x \neq y \\ \left(\mathrm{Ai}\,'(x)\right)^2 - \mathrm{Ai}\,''(x)\,\mathrm{Ai}\,(x) & \text{if } x = y \end{cases}$$

3 The 1-Airy ensemble

Formula 3.1. Let $\mu^{(1)}$ be the pfaffian point process on $\mathbb{R}$ whose kernel is equal to :

$$\begin{array}{rcl} K_{1,1}(x,y) & = & \mathrm{A}\,(x,y) + \frac{1}{2}\,\mathrm{Ai}\,(x) \int_{-\infty}^{y} \mathrm{Ai}\,(t)d\,t \\ K_{2,2}(x,y) & = & \mathrm{A}\,(x,y) + \frac{1}{2}\,\mathrm{Ai}\,(x) \int_{-\infty}^{y} \mathrm{Ai}\,(t)d\,t \\ K_{1,2}(x,y) & = & -\frac{1}{2}\,\mathrm{Ai}\,(x)\,\mathrm{Ai}\,(y) - \frac{\partial}{\partial y}\,\mathrm{A}\,(x,y) \\ K_{2,1}(x,y) & = & \int_0^{+\infty} \int_{x+u}^{+\infty} \mathrm{Ai}\,(v)d\,v\,\mathrm{Ai}\,(x+u)d\,u - \frac{1}{2}\,\mathrm{sign}(x-y) + \frac{1}{2} \int_y^x \mathrm{Ai}\,(u)d\,u \\ & + & \frac{1}{2} \int_x^{+\infty} \mathrm{Ai}\,(u)d\,u \int_{-\infty}^{y} \mathrm{Ai}\,(v)d\,v \end{array} .$$

(from [17])

Theorem 3.2. *With probability* 1*, the random set corresponding to* $\mu^{(1)}$ *has a finite maximum in* $\mathbb{R}$*.*

This result can be proven by using Theorems 4a and 4e of [16] and the properties of the Airy function.

Thanks to that theorem, we can put the points of the random set in decreasing order: $\lambda_1 > \lambda_2 > \ldots$.

The 1-Airy ensemble is then defined as the set of these λ_n for $n \in \mathbb{N}$.

4 The 2-Airy ensemble

Let $\mu^{(2)}$ be the determinantal point process on $\mathbb{R}$ whose kernel is equal to the Airy kernel.

Theorem 4.1. *With probability* 1*, the random set corresponding to* $\mu^{(2)}$ *has a finite maximum in* $\mathbb{R}$*.*

This result can be proven by using Theorems 4a and 4e of [16] and the properties of the Airy function.

Thanks to that theorem, we can put the points of the random set in decreasing order: $\lambda_1 > \lambda_2 > \dots$.

The 2-Airy ensemble is then defined as the set of these λ_n for $n \in \mathbb{N}$.

Part II. Young diagrams and β-Plancherel measure

Combinatorial objects called Young diagrams and Young tableaux were introduced by Alfred Young (1873-1940) in 1900, and were subsequently studied by Frobenius in [5] and [6]. Their use for studying increasing and decreasing subsequences of permutations is more recent. An algorithm found in 1961 in [14] results in a simple relationship between the output (a Young Tableau) and the longest increasing subsequence of the input (a sequence of distinct entries). The algorithm is usually called the Robinson-Schensted because of [12], but the algorithm outlined in Robinson's article is different from the one used in Schensted's article. This algorithm was then completed in 1970 in [9], who expanded the output making the algorithm injective. This algorithm naturally induces a probability measure on Young diagrams, which has been called "Plancherel measure" in [20] in 1977 from Michel Plancherel (1885-1967). This probability measure can be expressed simply in terms of a well-known variable called the dimension of a Young tableau, which has uses in representation theory (see for example [7]).

For a survey on Young diagrams and Young tableaux, see [13].

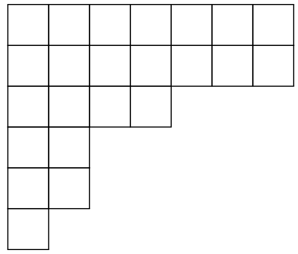
5 Definitions

A k-increasing sequence is the union of k increasing sequences. It can also be defined as a sequence with no decreasing subsequence of length $k+1$.

A Young diagram is a finite non-increasing sequence of natural numbers ($[\![1, +\infty[\![$).

For example, $(7, 7, 4, 2, 2, 1)$. Its size is the sum of the elements of the finite sequence. The size of the example is $7 + 7 + 4 + 2 + 2 + 1 = 23$.

It is usually represented as an empty table whose row lengths are the elements of the finite sequence. The example would be drawn as:



A standard Young tableau is a Young diagram D filled with all the elements of $[\![1, \text{size } D]\!]$ where all the rows and columns are increasing. For example,

1	2	7	11	13	18	22
3	4	15	16	17	21	23
5	6	19	20			
8	9					
10	14					
12						

The dimension of a Young diagram is the number of standard tableaux of the same shape. The dimension of the example of Young diagram is 1249248.

The β-Plancherel measure of size n is the probability measure $M^\beta_{(n)}$ such that for all Young diagrams D, $M^\beta_{(n)}(D) = \frac{(\dim D)^2}{K^\beta_n}$, where K_β is a normalization constant which ensures that $M^\beta_{(n)}$ is a probability measure.

For $\beta = 1$, 2, the Plancherel measure has some nice properties:

Theorem 5.1.

- *The length of the longest increasing subsequence of a uniform random involution on $[\![1, n]\!]$ has the same distribution as the length of the first row of a random Young diagram generated under the* 1*-Plancherel measure.*

- *The lengths of the longest i-increasing subsequences and j-decreasing subsequences for $i, j = 1, \ldots, k$ of a uniform random involution on $[\![1, n]\!]$ have the same joint distribution as the sum of the lengths of respectively the i first rows and the j first columns of a random Young diagram generated under the* 1*-Plancherel measure.*

Theorem 5.2.

- *The length of the longest increasing subsequence of a uniform random permutation on $[\![1, n]\!]$ has the same distribution as the length of the first row of a random Young diagram generated under the* 2*-Plancherel measure.*

- *The lengths of the longest i-increasing subsequences and j-decreasing subsequences for $i, j = 1, \ldots, k$ of a uniform random permutation on $[\![1, n]\!]$*

have the same joint distribution as the sum of the lengths of respectively the i first rows and the j first columns of a random Young diagram generated under the 2-Plancherel measure.

The latter theorem is a direct consequence of the theorem proved in [8] and the construction in [9]. The former theorem also uses a known property of Knuth's construction, see for example [2].

6 Asymptotic results

Theorem 6.1. *Let $k \in \mathbb{N}$. For $n \in \mathbb{N}$, let u_1^n, u_2^n be the lengths of the 2 first rows of a random Young diagram generated under the 1-Plancherel measure.*

Let v_1^n, v_2^n be such that $\forall i \in [\![1, 2]\!], u_i^n = 2\sqrt{n} + v_i^n n^{\frac{1}{6}}$.

Then (v_1^n, v_2^n) converges in distribution to the 2 greatest numbers of the 1-Airy ensemble.

This result was proven in [2]. It implies the following:

Corollary 6.2. *Let $k \in \mathbb{N}$. For $n \in \mathbb{N}$, let $(l_i^n)_{i\in[\![1,k]\!]}$ be respectively the lengths of the longest i-increasing subsequences of a uniform random involution on $[\![1, n]\!]$.*

Then $(\left(\frac{l_i^n}{\sqrt{n}} - 2\right) n^{1/3})_{i\in[\![1,k]\!]}$ converges in distribution to $(\sum_{j=1}^{i} \lambda_j)_{i\in[\![1,k]\!]}$ where $(\lambda_j)_{i\in[\![1,k]\!]}$ are distributed according to the 1-Airy ensemble.

Theorem 6.3. *Let $k \in \mathbb{N}$. For $n \in \mathbb{N}$, let $u_1^n, \dots, u_k^n$ be the lengths of the k first rows of a random Young diagram generated under the 2-Plancherel measure.*

Let $v_1^n, \dots, v_k^n$ be such that $\forall i \in [\![1, k]\!], u_i^n = 2\sqrt{n} + v_i^n n^{\frac{1}{6}}$.

Then $(v_1^n, \dots, v_k^n)$ converges in distribution to the k greatest points of the 2-Airy ensemble.

This result was first proven in [11], and then, in a more detailed way, in [3]. It implies the following:

Corollary 6.4. *Let $k \in \mathbb{N}$. For $n \in \mathbb{N}$, let $(l_i^n)_{i\in[\![1,k]\!]}$ be respectively the lengths of the longest i-increasing subsequences of a uniform random permutation on $[\![1, n]\!]$.*

Then $(\left(\frac{l_i^n}{\sqrt{n}} - 2\right) n^{1/3})_{i\in[\![1,k]\!]}$ converges in distribution to $(\sum_{j=1}^{i} \lambda_j)_{i\in[\![1,k]\!]}$ where $(\lambda_j)_{i\in[\![1,k]\!]}$ are distributed according to the 2-Airy ensemble.

Part III. Random matrices

Random matrices were first studied by Hsu and Wishart in the thirties, for example in [23]. Interest for this field grew because of the work of Wigner in the fifties who studied their eigenvalues in connection with nuclear physics, for example in [21] and [22]. In the nineties, explicit limit results were found, for example in [4] and [19].

For a survey of the field, see [10] or [1].

7 Gaussian Orthogonal Ensemble

Let $(\xi_{i,j})_{(i,j)\in\mathbb{N}^2}$ be real independent Gaussian random variables of mean 0 and variance 1.

The distribution of a random matrix M of size $n \times n$ such that

$$M_{i,j} = \begin{cases} \sqrt{2}\xi_{i,i} & \text{if } i = j \\ \xi_{i,j} & \text{if } i < j \\ \xi_{j,i} & \text{if } i > j \end{cases}$$

is called the Gaussian Orthogonal Ensemble. It is called "Orthogonal" because it is invariant under conjugation by an orthogonal matrix. This is, up to a multiplication by a constant, the only distribution on real symmetric matrices which satisfies these two properties:

1. invariance by conjugation by an orthogonal matrix.

2. independence of the entries on or below the diagonal.

The joint probability density of its eigenvalues is $\frac{1}{Z_n^{(1)}} \prod_{k=1}^{n} e^{-\frac{1}{4}\lambda_k^2} \prod_{i<j} |\lambda_j - \lambda_i|$, with $Z_n^{(1)}$ a normalization constant. It was first found in [21].

8 Gaussian Unitary Ensemble

Let $(\xi_{i,j})_{(i,j)\in\mathbb{N}^2}$ be real independent Gaussian random variables of mean 0 and variance 1.

The distribution of random matrix M of size $n \times n$ such that

$$M_{i,j} = \begin{cases} \xi_{i,i} & \text{if } i = j \\ \dfrac{\xi_{i,j} - i\xi_{j,i}}{\sqrt{2}} & \text{if } i < j \\ \dfrac{\xi_{j,i} + i\xi_{i,j}}{\sqrt{2}} & \text{if } i > j \end{cases}$$

is called the Gaussian Unitary Ensemble. It is called "Unitary" because it is invariant under conjugation by an unitary matrix. This is, up to a multiplication by a constant, the only distribution on Hermitian matrices which satisfies these two properties:

1. invariance by conjugation by an unitary matrix.

2. independence of the entries on or below the diagonal.

The joint probability density of its eigenvalues is $\frac{1}{Z_n^{(2)}} \prod_{k=1}^{n} e^{-\frac{1}{2}\lambda_k^2} \prod_{i<j} |\lambda_j - \lambda_i|^2$, with $Z_n^{(2)}$ a normalization constant. It was first found in [22].

9 Asymptotic results

Theorem 9.1. *Let $k \in \mathbb{N}$. For $n \in \mathbb{N}$, let $\lambda_1^n, \ldots, \lambda_k^n$ be the k greatest eigenvalues of a random matrix from the Gaussian Orthogonal Ensemble.*

Let $v_1^n, \ldots, v_k^n$ be such that $\forall i \in [\![1, k]\!], \lambda_i^n = 2\sqrt{n} + v_i^n n^{\frac{1}{6}}$.

Then $(v_1^n, \ldots, v_k^n)$ converges in distribution to the k greatest points of the 1-Airy ensemble.

This theorem was first proven, in another form, in [19].

Theorem 9.2. *Let $k \in \mathbb{N}$. For $n \in \mathbb{N}$, let $\lambda_1^n, \ldots, \lambda_k^n$ be the k greatest eigenvalues of a random matrix from the Gaussian Unitary Ensemble.*

Let $v_1^n, \ldots, v_k^n$ be such that $\forall i \in [\![1, k]\!], \lambda_i^n = 2\sqrt{n} + v_i^n n^{\frac{1}{6}}$.

Then $(v_1^n, \ldots, v_k^n)$ converges in distribution to the k greatest points of the 2-Airy ensemble.

This theorem was first proven, in another form, in [4].

In [15], very similar results featuring the 1-Airy and 2-Airy ensembles are shown for a larger class of random matrix ensembles.

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