INTRODUCTION À UN DOMAINE DE RECHERCHE

Generic Actions of Finitely Generated Groups in $\mathrm{Diff}^{\infty}_{\mathrm{vol}}(\mathrm{M})$

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Contents

My research focuses on *Differential dynamical systems*. Before getting into more details about my research field, I start by introducing some basic concepts.

1 Basic Concepts of Dynamical Systems

Definition 1.1. *A measure-preserving dynamical system* $(X, \mathcal{B}, \mu, T)$ *is a probability space with a measure-preserving transformation* T *on it, i.e., for each* $A \in \mathcal{B},\ \mu(T^{-1}A) = \mu(A)$.

Definition 1.2. *The measure-preserving transformation* T *as defined above is called ergodic if for every* $T-$*invariant set* $A \in \mathcal{B}$*, either* $\mu(A) = 0$ *or* $\mu(A) = 1$.

Example 1. *An irrational rotation on* $\mathbb{R}/\mathbb{Z}$*, is preserving Lebesgue measure. Moreover, it is an ergodic transformation.*

Theorem 1.1. (Birkhoff-Khinchin) *Let* f *be measurable,* $\mathbb{E}(|f|) < \infty$*, and* T *be a measure-preserving map. Then with probability* 1*:*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} f(T^k x) = \mathbb{E}(f|\mathcal{C}),$$

where $\mathbb{E}(f|\mathcal{C})$ *is the conditional expectation given the* σ*-algebra* $\mathcal{C}$ *of invariant sets of* T.

Corollary 1.1. (Pointwise Ergodic Theorem): *In particular, if* T *is also ergodic, the* $\mathcal{C}$ *is the trivial* σ*-algebra, and thus with probability* 1*:*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} f(T^k x) = \mathbb{E}(f).$$

Example 2. *Suppose* f *is a measurable* $1-$*periodic function on* $\mathbb{R}$*,* θ *is an irrational number, Then for almost every* $x \in \mathbb{R}$ *:*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} f(x + k\theta) = \int_0^1 f(x)dx$$

Definition 1.3. *A topological dynamical system is defined as a complete metric space* X*, which together with a continuous transformation* T *of that space.*

Given a topological dynamical system (X, T)*, if for every two open subsets* U, V*, there exists an integer* n *for which* $T^n(U) \cap V \neq \emptyset$*, we call* (X, T) *topologically transitive. If every point has a dense orbit, then we say that* (X, T) *is minimal.*

In fact, T is transitive if and only if there exists a point in X which has a dense orbit. Another good remark here is that T is minimal if and only if every T-invariant closed set is either empty or X itself.

The central problem in dynamical systems is to consider whether a dynamcial system is ergodic or a topological dynamical system is transitive or minimal. In particular, we consider these questions at the same time for conservative diffeomorphism (area preserving diffeomorphism).

2 Differential Dynamical Systems

Definition 2.1. *We say that a diffeomorphism $f(C^r, 0 \leq r \leq \infty)$ on a smooth manifold M is conservative if it preserves a fixed volume form ω on M. The space of C^r conservative diffeomorphism on M is noted as $\mathit{Diff}^r_{vol}(\mathrm{M})$.*

Example 3. *All the area preserving diffeomorphisms on surfaces are conservative. A symplectic diffeomorphism f of a symplectic manifold (M, ω) (here we denote the symplectic form ω) are conservative since they preserve the volume form ω^n if M has dimension $2n$.*

Corollary 2.1. *If a volume preserving diffeomorphism is ergodic, then almost every point has a dense orbit.*

Corollary 2.1 can be proved by using Birkhoff ergodic theorem at almost every point and indicate functions of arbitrary nonempty open set.

It would be very difficult for us to study an arbitrary area preserving diffeomorphism, hence we try to study generic behaviors in $\mathit{Diff}^\infty_{vol}(\mathrm{M})$.

Definition 2.2. *We say a set in $\mathit{Diff}^r_{vol}(\mathrm{M})$,$0 \leq r \leq \infty$ is residual if it is a countable intersection of open dense sets. We call a property is generic in $\mathit{Diff}^r_{vol}(\mathrm{M})$if it holds on a residual set.*

Here is an example:

Theorem 2.1. (Oxtoby-Ulam)[1] *For any generic homeomorphism $f \in \mathit{Diff}^0_{vol}(M)$, $\mathit{Diff}^0_{vol}(M)$ denotes the group of homeomorphisms which preserve volume of M, the invariant measure v is ergodic.*

The theorem Oxtoby-Ulam is established mainly for the reason that the C^0 topology is so weak that we can use a homeomorphism sufficiently chaos to approach an arbitrary homeomorphism. The KAM theory tells us when $r \geq 4$, the same conclusion is false for $\mathit{Diff}^r_{vol}(\mathrm{M})$, and that the generic transitivity is not true when $r \geq 4$.

Suppose $f \in \mathit{Diff}^4_{vol}(\mathrm{M})$, with an elliptic fixed point x. We can prove that, there exists a neighborhood U of x and a C^4diffeomorphism h from U into a neighborhood V of the origin in $\mathbb{R}^2$ such that,

$$(hfh^{-1})(r, \theta) = (r, \theta + \alpha_0 + \alpha_1 r) + o(\|r\|^3 + \|\theta\|^3)$$

where (r, θ) the polar coordinates of a point in $h(U \cap f^{-1}(U))$.

Definition 2.3. *Under the above assumptions , if* $\alpha_0 \neq 0, \pm\frac{\pi}{2}, \pi, \pm\frac{2\pi}{3}, \alpha_1 \neq 0$, *we say that the elliptic fixed point* x *of diffeomorphism* f *is non-degenerate.*

Theorem 2.2. (Kolmogorov-Arnold-Moser)[2][1] *Let* f *be a volume-preserving diffeomorphism of class* $C^r, r \geq 4$ *of a surface* M. *If* x *is a non-degenerate elliptic fixed point, then for every* $\epsilon > 0$ *there exist an arbitrarily small neighborhood* U *of* x *and a set* $U_0 \subset U$ *with the following properties:*

- U_0 *is a union of f-invariant simple closed curves of class* C^{r-1} *containing* x_0 *in their interior.*
- *The restriction of* f *to each of these curves is topologically equivalent to an irrational rotation.*
- *Denoting by* λ *the measure associated with the volume form of* M, *we have*

$$\lambda(U - U_0) \leq \epsilon\lambda(U).$$

Notice that if g is a diffeomorphism close to f enough in $Diff^r_{vol}$(M), then there exists a unique nondegenerate ellipitic fixed point x_g of g close to x. As a corollary of KAM theorem, for every such g, there exists a $g-$invariant disc D_g which has positive measure and nonempty interior. In fact, we can choose D_g such that

$$\partial(D_g) = \gamma_g$$

where γ_g is one of the $g-$invariant circle.

Corollary 2.2. *Let* M *be a compact surface without boundary,* ω *a volume form on* M *and* $\mathrm{Diff}^r_{vol}(\mathrm{M})$ *the space of* C^r *diffeomorphisms of* M *that preserve* ω, *endowed with the* C^r *topology. Then the set of* $f \in \mathrm{Diff}^r_{vol}(\mathrm{M})$ *which are ergodic (with respect to the probability measure* λ *associated to* ω*) is not dense in* $\mathrm{Diff}^r_{vol}(\mathrm{M})$ *for* $r \geq 4$.

For $1 \leq r \leq 3$ (The KAM theory is true for all $r > 3$), the problem of generic ergodicity is still open, we get some partial results under certain assumptions of hyperbolicity.

Definition 2.4. *Let* M *be a manifold and* $f : M \to M$ *a diffeomorphism,* f *is called uniformly hyperbolic or an* ***Anosov diffeomorphism*** *if for each* $x \in M$ *there exists a splitting of the tangent space* $T_xM = E^s(x) \oplus E^u(x)$ *and there are constants* $C > 0$, $\lambda \in (0, 1)$ *such that for all* $n \in \mathbb{N}$ *we have:*

$$\|Df^n(v)\| \leq C\lambda^n \|v\|, \forall v \in E^s(x)$$

$$\left\|Df^{-n}(v)\right\| \leq C\lambda^{-n} \|v\|, \forall v \in E^u(x)$$

[1]The strongest version of Theorem KAM is due to Herman

It is not hard to prove that Anosov diffeomorphism forms an open set in $\mathrm{Diff^r(M)}$, for all $r \geq 1$. Moreover, there exists (un)stable manifolds tangent to (un)stable distribution at all points in M.

Example 4. *Let $M = \mathbb{R}^2/\mathbb{Z}^2$ be a torus , f is a linear toral transformation on M,*

$$f(x) = A \cdot x, A = \binom{2 \quad 1}{1 \quad 1}$$

Then f is Anosov diffeomorphism, since the two eigendirections of A form (un)stable distributions of f. And there exists a neighborhood $U \subset \mathrm{Diff^r(M)}$ of f such that $\forall g \in U$, g is Anosov.

By Hopf argument [9][8], we know when $r > 1$, all volume preserving Anosov diffeomorphisms are ergodic. But it is still unknown for $r = 1$.

We finish our disccussion about the measure-theoretical behavior of generic diffeomorphisms in $Diff^r_{vol}$. Now we will consider the topological dynamical behavior. An early but remarkable result is:

Theorem 2.3. *(Pugh[13] [14], closing lemma) For any generic diffeomorphism $f \in Diff^1_{vol}$, the periodic points of f are dense in M.*

And an even better result is:

Theorem 2.4. (Generic transitivity [10]) *Any generic diffeomorphism $f \in \mathrm{Diff}^1_v$ is transitive: there is a G_δ and dense subset $\mathcal{G} \subset M$ such that the forward orbits of any $x \in \mathcal{G}$ is dense in M.*

A remark here is that, the conclusions above are open for arbitrary $r > 1$, it might even be extremely difficult. As what we mentioned before, generic transitivity is false for $r > 3$.

3 Group Actions and Homogeneous Dynamics

In this section, we will generalize our conception of dynamical systems.

Definition 3.1. *Let A be a set of diffeomorphisms (not necessarily volume-preserving) on a manifold M, and Γ the semigroup (or group) generated by A. The action by Γ is transitive if there exists a point which has dense orbit, is minimal if all points have dense orbits, and is ergodic with respect to a volume v if Γ preserves v and any G-invariant set has full measure or 0 measure.*

Here are some examples: When M in the previous definition is a Lie group G, or a homogeneous space G/H, A is defined as $\{L_{g_i} : g_i \in G\}$, where L_g means the action by left multiplication of g. The volume v is always chosen as the homogeneous measure on G/H. Then the problem whether $\Gamma-$action is transitive corresponds to the behavior of $\overline{\Gamma \cdot H}$ in G. Obviously, when G is not abelian, if A contains only one element f, then the $\Gamma-$action is a special case in section 2. However f-action maynot be transitive for general G/H, this is one reason why we consider $\Gamma-$actions.

Definition 3.2. *If there exists a probability measure μ on A (when A is finite, we can give each element the same weight $\frac{1}{|A|}$), we call a measure ν on M is μ-stationary if for all continuous function ϕ, $\int \phi(x)d\nu(x) = \int \phi(g \cdot x)d\nu(x)d\mu(g)$.*

Remark 3.1. *There always exists stationary measure, the question is when there is only one or very few stationary measures? When Γ preserves a volume v and there exists an unique stationary measure, then it must be v and it implies easily that the Γ action is ergodic and minimal. That explains why we care about uniqueness of stationary measure.*

Here is an example of stationary measure given by Furstenberg [3] and Margulis[4].

Example 5. *Suppose $A \subset SL(2,\mathbb{R})$, with a probability measure μ on A, acts on projective space $\mathbb{P}^1$, such that the support of μ is Zariski dense in $SL(2,\mathbb{R})$. Then there exists an unique stationary measure ν and ν is non-atomness. Moreover, if*

$$\int \log \|g\| \, d\mu(g) < \infty,$$

then the Lyapunov exponent *is positive, i.e.:*

$$\lim_{n\to\infty} \frac{1}{n} \int \log \|b_n b_{n-1} ... b_1\| > 0,$$

where $b_n, ..., b_1$ is the random product of the i.i.d elements under the law μ.

Recently Y. Benoist and F. Quint proved a hard theorem about stationary measure:

Theorem 3.2. [12] *Let G be a real simple Lie group, and Λ a lattice of G. Suppose μ is a probability measure with compact support on G, and* $\mathrm{Supp}(\mu)$ *generates a Zariski dense subgroup of G. Then every μ-stationary probability without atoms on G/Λ is G-invariant.*

In other words, Benoist-Quint theorem provides a classification for non-atomness $\mu-$stationary measures.

4 My Domain of Research

My research interest lies mainly in generic actions of finitely generated groups $Diff_{vol}^{\infty}(\mathrm{M})$, studying properties like transitivity, minimality, ergodicity and uniqueness of stationary measure.

When the finitely generated group $G \subset Diff_{vol}^{\infty}(\mathrm{M})$ is a cyclic group, we know from KAM theory that the G-action cannot be generically transitive and ergodic. Thus we consider the case when number of generator of G is bigger than 1, for which we only get the result about transitivity now.

Theorem 4.1. [5] *Suppose that M is a smooth surface. Then there exists a residual set $\mathcal{R} \subset \mathrm{Diff}_{vol}^{r}(M) \times \mathrm{Diff}_{vol}^{r}(M)(r \in \mathbb{N} \cup \{\infty\})$ with the product C^r topology such that if $(f, g) \in \mathcal{R}$, then the iterated function system* $\mathrm{IFS}(f, g)$ *is transitive.*

Here IFS means iterated function systems, which means in fact the semigroup generated by f, g. The main techniques used in the theorem are KAM theory, discussions about hyperbolic fixed points, and some usual techniques in topology of surface such as prime -ends.

Unfortunately, it still remains very hard to prove that all points have dense orbits simply by improving the theorem, and this paper does not have any measure theoretical argument.

Now we consider the measure theoretical properties fof $G-$actions. On one hand, Benoist-Quint shows us how to prove the uniqueness of of non-atomness stationary measure under the condition of f, g acting on homogeneous space. Given an positive lower bound for the Lyapunov exponents of the random product of elements in G, it is possible to extend Benoist-Quint theorem partially to the case of diffeomorphisms.

On the other hand, with small Lyapunov exponents, the following two theorems help to illustrate some interesting phenomena. Although it is limited to analytical cases, we can generalize it to C^k cases by similar arguments.

Definition 4.1. *Let $R : \Gamma \to \mathrm{Diff}(V)$ be a homomorphism of a (discrete) group Γ, and V a compact manifold. We say that R is recurrent if for all $x \in V$, there exists a sequence $(\gamma_i)_{i \geq 1}$ of distinct elements in Γ such that $R(\gamma_i)(x)$ converges to x.*

Theorem 4.2. (Ghys)[6] *Let Γ be a group containing a free non abelian subgroup and S a generating part of Γ. Assume that M is a real analytic compact manifold and $\mathrm{Diff}^{w}(M)$ the group of its analytical diffeomorphisms, with its natural topology. Then there exists a neighbourhood $\mathcal{U}$ of identity in $\mathrm{Diff}^{w}(M)$ such that every homeomorphism $R : \Gamma \to \mathrm{Diff}^{w}(M)$ which maps S in $\mathcal{U}$ is recurrent.*

Theorem 4.3. (Rebelo)[7] *There is an open neighbourhood $\mathcal{U}$ of identity in $\mathrm{Diff}^{w}(S^1)$ with the following property: if G is a non solvable subgroup of $\mathrm{Diff}^{w}(S^1)$ and G admits a finite set of generators contained in $\mathcal{U}$, then there exists a (local and nowhere zero) vector field $\mathcal{X}$ defined in a neighbourhood of any point $p \in S^1$, provided that p is not* *a periodic point for G. Yet this vector field $\mathcal{X}$ defines a local flow which is contained in the C^∞-closure of the restriction of elements in G to this neighbourhood.*

The previous two theorems tell us that it is useful to obtain some elements near identity from small Lyapunov exponents condition, but how to obtain it is a quite interesting problem. One possible method is to use **Pesin theory** (nonuniformly hyperbolic) since it provides some useful distortion estimations. Due to limited space, we only present in this paper the important *Oseledec Theorem* and the definition of nonuniformly hyperbolic systems, see the Appendix.

For the rest of my research, I hope to gain new understanding and results about generic actions of finitely generated groups $Diff^\infty_{vol}$(M) by using tools like stationary measure, Lyapunov exponents (Pesin theorey), KAM theory, etc.

Appendix A Lyapunov Exponents and Nonuniformly Hyperbolic Dynamical Systems

Instead of prescribing bounds for the expansion and contraction of vectors, the theory of nonuniformly hyperbolic dynamical systems measures the infinitesimal asymptotic exponential relative behavior of points by the Lyapunov exponent.

Definition A.1. *For a diffeomorphism $f : M \to M$ the forward Lyapunov exponent of a vector v at a point x is defined by*

$$\mathcal{X}^+(x, v) := \overline{\lim_{n\to\infty}} \frac{1}{n} \log \|D_x f^n(v)\| .$$

At any x this takes only finitely many values $\mathcal{X}^+_1(x) < ... < \mathcal{X}^+_{p^+(x)}(x)$ that determine vector subspaces $V^+_i(x) := \{v \in T_xM | \mathcal{X}^+(x, v) \leq \mathcal{X}^+_i(x)\}$of T_xM which are nested:

$$\{0\} = V^+_0(x) \subsetneq ... \subsetneq V_{p^+(x)}(x) = T_xM.$$

Definition A.2. *The multiplicity of $\mathcal{X}^+_i(x)$ is defined as $k^+_i(x) := \dim V^+_i(x) - \dim V^+_{i-1}(x)$; both of functions $\mathcal{X}^+_i$ and k^+_i are f-invariant. The sum of the positive Lyapunov exponents is*

$$\sum(x) := \sum_{\mathcal{X}^+_i(x)>0} k^+_i(x)\mathcal{X}^+_i(x)$$

Taking the corresponding limits as $n \to -\infty$ yields the backwards Lyapunov exponents $\mathcal{X}^-$, for which the corresponding results hold.

Theorem A.1. (The Oseledec Multiplicative Ergodic Theorem) *For a smooth diffeomorphism of a compact manifold M and any invariant Borel probability measure* *almost every point x is Lyapunov-Perron regular, i.e.,*

$$\begin{aligned}
p^+(x) &= p^-(x) := p(x), \\
T_xM &= \oplus_{i=1}^{p(x)} E_i(x), \text{where } E_i(x) := V_i^+(x) \cap V_i^-(x), \\
\lim_{n\to\pm\infty} \frac{1}{n} \log \|D_x f^n(v)\| &= \mathcal{X}_i^+(x) = -\mathcal{X}_i^- =: \mathcal{X}_i(x) \text{ uniformly in } \{v \in E_i(x) | \|v\| = 1\}, \\
\lim_{n\to\pm\infty} \frac{1}{n} \log |\det D_x f^n| &= \sum_{i=1}^{p(x)} \mathcal{X}_i(x) \dim E_i(x).
\end{aligned}$$

Definition A.3. *A diffeomorphism f is said to have nonzero exponents on an invariant set Λ if for each $x \in \Lambda$ there is an $s = s(x)$ such that*

$$\mathcal{X}_1(x) < ... < \mathcal{X}_s(x) < 0 < \mathcal{X}_{s+1}(x) < ... < \mathcal{X}_{p(x)}(x). \tag{1}$$

An f-invariant Borel probability measure is said to be *hyperbolic* if equation (1) holds for almost every $x \in M$.

Definition A.4. *A Borel set $\Lambda \subset M$ is called nonuniformly partially hyperbolic (in the broad sense) if there exist*

- *a Df-invariant decomposition $T_xM = E^1(x) \oplus E^2(x)$ for $x \in \Lambda$ (i.e., $DfE^i(x) = E^i(f(x))$ for $i = s, u$*
- *f-invariant Borel functions $\lambda_1, \lambda_2 : \Lambda \to \mathbb{R}^+$ (i.e., $\lambda_i \circ f = \lambda_i$for $i = 1, 2$) such that either $\lambda_1 < \min\{1, \lambda_2\}$ or $\lambda_2 > \max\{1, \lambda_1\}$ with the following properties:*

  For every $\epsilon_0 > 0$ there are an f-invariant Borel function $\epsilon : \Lambda \to (0, \epsilon_0)$ and positive Borel functions C, K on Λ such that for every $x \in \Lambda$ we have

  $$\angle(E^2(x), E^1(x)) \geq K(x),$$

  $$\|Df_x^n v\| \leq C(x)(\lambda_1(x))^n \|v\|, v \in E^1(x), n \in \mathbb{N},$$
  $$\left\|Df_x^{-n} v\right\| \leq C(x)(\lambda_2(x))^{-n} \|v\|, v \in E^2(x), n \in \mathbb{N},$$

  and for every $n \in \mathbb{Z}$ and $x \in \Lambda$ we have

  $$C(f^n(x) \leq C(x)e^{\epsilon(x)|n|}, K(f^n(x)) \geq K(x)e^{\epsilon(x)|n|}.$$

We say that Λ is nonuniformly (completely) hyperbolic if $\lambda_1 < 1 < \lambda_2$.

We see easily that for $\dim M = 2$, if for almost every $x \in M$, Lyapunov exponent for f is nonzero at point x, then there exists a full measure Λ which is nonunformly hyperbolic.

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