INTRODUCTION TO p-ADIC AND MODULO p LANGLANDS CORRESPONDENCE

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Contents

1. Introduction

This is a brief and limited introduction to p-adic Langlands program which aims towards the author's thesis. We present the basic preliminary and some history of p-adic Langlands program through the most classical and well-known example $\mathrm{GL}_2(\mathbb{Q}_p)$, and then shed some light on what the author is doing. The term p-adic Langlands correspondence is relatively young, whose history is less than 15 years. However, the theory is closely connected to various classical theories 'on the automorphic side' such as p-adic modular forms, smooth representations of p-adic reductive groups with coefficients in both characteristic 0 and p, modular representations of finite group of Lie type, Verma modules, etc; and various p-adic Hodge theories 'on the Galois side' such as ϕ, Γ-modules, (ϕ, N)-modules, Fontaine-Laffaille theory, Breuil modules and Strong divisible modules, Kisin modules etc together with Galois deformation theory. The author will only present the p-adic Langlands correspondence through a simplest language possible and try his best to reach the part where he is working on. As a result, there can be no comprehensive introduction to a single theory listed above (each theory may take one book or at least a series of articles to explain). Instead, the author offers the standard references whenever necessary, and tries his best to tell a fluent story leading to his thesis.

2. Notation

We fix the common notation in this article.

Let K be a p-adic field, namely a finite extension of $\mathbb{Q}_p$. We denote its integer ring, residual field and absolute ramification index by $\mathcal{O}_K$, k and e respectively. Through the construction of Witt vector we have a surjection $\mathrm{Gal}(\overline{K}/K) \twoheadrightarrow \mathrm{Gal}(\overline{k}/k)$ and we denote the kernel of this surjection by I_K, which is the inertial subgroup. We define $K_0 = W(k)[\frac{1}{p}]$ which is the maximal unramified extension of $\mathbb{Q}_p$ inside K. As k is a finite field, we know that $\mathrm{Gal}(\overline{k}/k) \cong \hat{\mathbb{Z}}$ and is topologically generated by the Frobenius element. We define the Weil group W_K to be the unique subgroup of $\mathrm{Gal}(\overline{K}/K)$ fixing into the exact sequence:

$$I_K \hookrightarrow W_K \twoheadrightarrow \mathbb{Z} \tag{1}$$

where $\mathbb{Z}$ is the subgroup of $\mathrm{Gal}(\overline{k}/k)$ generated by the Frobenius element. Here the topology of W_K is induced by the profinite topology of I_K and the discrete topology of $\mathbb{Z}$.

Let F be a number field, namely a finite extension of $\mathbb{Q}$. We use the notation v for any place of F and denote the completion of F at v by F_v. We denote the integer ring and residual field of F_v by $\mathcal{O}_v$ and k_v if v is a nonarchimedean place. We use the notation $\mathbb{A}_F$ and $\mathbb{I}_F$ respectively for the Adele ring and Idele group associated to F. (See [] for detail.) Moreover, we use $\mathbb{A}_F^f$, $\mathbb{I}_F^f$ respectively for the finite Adele and finie Idele.

Let G be a reductive group defined over a p-adic field or a number field. If G is split over its base, we use the notation B and T for a Borel subgroup and a maximal tori such that $T \subset B$.

3. Preliminary on modular forms

In this section we will give some classical background of the whole theory following Chapter 1,2,5 of [19], including the definition of modular curves, modular forms, Hecke operators and so on.

Let $\mathbb{H}$ be the upper half place endowed with the standard holomorphic structure, which is isomorphic to the unit disc as Riemann surface. Then the group $\mathrm{GL}_2(\mathbb{R})^+$ (invertible order 2 real matrix with positive determinant) has a natural action on $\mathbb{H}$ through the fractional transform:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot z = \frac{az+b}{cz+d} \tag{2}$$

for any $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(\mathbb{R})^+$ and $z \in \mathbb{H}$.

Now consider a family of discrete subgroups of the Lie group $\mathrm{GL}_2(\mathbb{R})^+$ which are called congruence subgroups. We define $\Gamma(1) := \mathrm{GL}_2(\mathbb{Z})^+ \subset \mathrm{GL}_2(\mathbb{R})^+$ and $\Gamma \subset \Gamma(1)$ is any subgroup of $\Gamma(1)$ containing $\Gamma(N) := \mathrm{Ker}(\mathrm{GL}_2(\mathbb{Z})^+ \twoheadrightarrow \mathrm{GL}_2(\mathbb{Z}/N\mathbb{Z}))$ for certain integer N.

Definition 3.0.1. *A modular form of level Γ and weight k is defined to be a holomorphic function f over $\mathbb{H}$ such that*

$$f(\frac{az+b}{cz+d}) = (cz+d)^{-k} f(z) \tag{3}$$

for each $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$. We use the notation $j(\begin{pmatrix} a & b \\ c & d \end{pmatrix}, z) = cz+d$

By definition we know that there is an integer N such that $\Gamma(N) \subset \Gamma$, and hence $\begin{pmatrix} 1 & N \\ 0 & 1 \end{pmatrix} \in \Gamma$. As a result, modular form f with level Γ must satisfy $f(z+N) = f(z)$ and has a Fourier expansion of the following form:

$$f(z) = \sum_{n \in \mathbb{Z}} a_n e^{\frac{2\pi i z}{N}}. \tag{4}$$

Definition 3.0.2. *f is called holomorphic if $a_n = 0$ for all $n < 0$. f is called a cusp form if $a_n = 0$ for all $n \leq 0$.*

We denote the space of cusp modular forms of level Γ and weight k by $S_k(\Gamma)$. From now on we fix the integer N such that $\Gamma(N) \subset \Gamma$. For each prime p such that $(p, N) = 1$, we have a Hecke operator defined as a bicoset operator as follows:

Definition 3.0.3. *Given two congruence subgroup Γ_1, Γ_2, and $g \in \mathrm{GL}_2(\mathbb{Q})^+$ we have a natural bicoset operator:*

$$[\Gamma_2 g \Gamma_1] : S_k(\Gamma_1) \to S_k(\Gamma_2), f \mapsto \sum_{\gamma g \Gamma_1 \in \Gamma_2 g \Gamma_1} \gamma \cdot f \tag{5}$$

In particular, we define the Hecke operators for prime p satisfying $(p, N) = 1$ as follow:

$$T_p := [\Gamma t_1 \Gamma] : S_k(\Gamma) \to S_k(\Gamma) \tag{6}$$

where $t_1 = \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}$; and for each integer d with $(d, N) = 1$ we have:

$$< d >:= [\Gamma t_2 \Gamma] : S_k(\Gamma) \to S_k(\Gamma) \tag{7}$$

where $t_2 = \begin{pmatrix} d & 0 \\ 0 & d \end{pmatrix}$

The quotient $Y(\Gamma) := \mathbb{H}/\Gamma$ exist as a Riemann surface as long as Γ has a free action on $\mathbb{H}$, which is true whenever Γ is small enough. In fact, there are only three fix points in $\mathbb{H}$ whose stablizer in $\mathrm{GL}_2(\mathbb{Z})^+$ is nontrivial. There three points are i, ω and ω^2 where $\omega^3 = 1$. As a result, Γ has a free action on $\mathbb{H}$ whenever it does not contain the stablizer of the stablizers (which are finite groups of order 2, 3 and 3 respectively) of these three points. An interesting but less important fact is that quotient of Riemann surface by some discrete group with only stablizers with finite order is still a Riemann surface, and therefore finally we do not need to assume Γ to be small enough.

A careful study of the action of Γ near the boundary $\mathbb{R}$ of $\mathbb{H}$ shows us that in fact $Y(\Gamma)$ can be compactified naturally by adding finite number of cusps, and we denote the compactification by $\overline{Y}(\Gamma)$. (See Chapter 2 of [19] for detail.)

Remark 3.0.1. *We choose the most classical and down to earth definition of modular forms and their related conceptions. However, in some sense it is definitely not the right method to present them. The conceptual method is to realize non compact modular curves as solution to moduli problems of elliptic curves over $\mathbb{Q}$ with level structures, (compact) modular curves as the solution to moduli problems of generalized ellptic curves over $\mathbb{Q}$, modular forms as global section of some natural invertible sheaves on modular curves, and Hecke operators through geometric Hecke correspondence which comes naturally from natural operations on level structures of elliptic curves. We refer to Chapter 3 of* [16] *for quick survey of this language. Through the moduli realization of moduli curves, one can see a natural $\mathbb{Q}$ structure on modular curves (or finite union of them). From now on we use $X(\Gamma)$ to denote the compact moduli curve over $\mathbb{Q}$ coming from the moduli problem. The geometric connected component of $X(\Gamma)$ is isomorphic to $\overline{Y}(\Gamma)$. In particular, if $\Gamma = \Gamma(N)$, we use the notation $X(N)$ for $X(\Gamma(N))$.*

4. Short preliminary on p-adic Hodge theory

Roughly speaking, p-adic Hodge theory aims to compare the p-adic etale cohomology and the de Rham or Crystalline cohomology of projective schemes over some p-adic field K. The main classical theorem is the p-adic comparison theorem.

Theorem 4.0.1 ([24])**.** *Given a projective scheme X over K, then we have the following isomorphism*

$$H^*_{dR}(X) \otimes_K B_{dR} \cong H^*_{et}(X \otimes_K \overline{K}) \otimes_{\overline{K}} B_{dR}. \tag{8}$$

where B_{dR} is the de Rham period ring ([26]*) with a filtration ,a action of a Frobenius and a action of $\mathrm{Gal}(\overline{K}/K)$, $H^*_{dR}(X)$ has a filtration on it, and $H^*_{et}(X \otimes_K \overline{K})$ has a natural action of $\mathrm{Gal}(\overline{K}/K)$. The isomorphism is compatible with all these actions and the filtration on both sides.*

The above theorem is the de Rham version of p-adic comparison theorem, there are also other versions like Crystalline version or semi-stable version, which we do not mention here for simpleness.

We are not going into the p-adic geometry involved anyway. We just need these theory as a setting to motivate p-adic Langlands correspondence.
Parallel to the comparison theorem above, it is natural to expect a procedure to transform p-adic Galois representations of $\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$ into some 'linear' object corresponding to the de Rham cohomology or Crystalline cohomology in different settings.

Fontaine introduced this procedure without the geometric setting using various period rings B_{dR}, B_{cris}, B_{st}...(See [26] for definition and basic properties. Here the notation abbreviate the fixed p-adic field K in the definition of these period rings. Of course, one can define the relative version of these period rings in general). We summarize briefly the structures on these period rings as follows:
B_{dR} has a filtration and an action by $\mathrm{Gal}(\overline{K}/K)$. B_{cris} and B_{st} are subrings of B_{dR} stable by $\mathrm{Gal}(\overline{K}/K)$ and therefore inherit the action of $\mathrm{Gal}(\overline{K}/K)$ and a filtration from B_{dR}. Moreover, both of them also has an action by Frobenius operator. B_{st} is endowed with the action of a monodromy operator as a derivation.

Definition 4.0.4 (cf section 2.1 of [28])**.** *Let B be a unital commutative ring with an action of $\mathrm{Gal}(\overline{K}/K)$ such that $g(ab) = g(a)g(b)$ for any $a, b \in B$. Assume moreover that $L := B^{\mathrm{Gal}(\overline{K}/K)}$ is a field. A p-adic representation (ρ, V) of $\mathrm{Gal}(\overline{K}/K)$ is called B-admissible if*

$$\dim_{\overline{\mathbb{Q}}_p} V = \dim_L (B \otimes V)^{\mathrm{Gal}(\overline{K}/K)}. \tag{9}$$

In particular ρ is called de Rham, crystalline or semi-stable, if it is B admissible where B equals B_{dR}, B_{cris} or B_{st} respectively. We denote the right side of the equation by $D_{dR}(V)$, $D_{cris}(V)$ and $D_{st}(V)$ respectively. As $B_{dR}^{\mathrm{Gal}(\overline{K}/K)} = K$, $B_{cris}^{\mathrm{Gal}(\overline{K}/K)} = B_{st}^{\mathrm{Gal}(\overline{K}/K)} = K_0$, we note that $D_{dR}(V)$ is a K vector space while $D_{cris}(V)$ and $D_{st}(V)$ are K_0 vector spaces.

Definition 4.0.5 (cf section 6.4.1 [28])**.** *A (ϕ, N) module D is a finite dimensional K_0 vector space endowed with two operators ϕ and N such that ϕ is semilinear* *with respect to the Frobenius on* K_0*,* N *is* K_0*-linear and we have* $N\phi = p\phi N$*. A filtered* (ϕ, N) *module is is a* (ϕ, N) *module* D *endowed with a decreasing, separated, exhausted filtration* $\mathrm{Fil}^{\cdot}$ *on* $D \otimes_{K_0} K$*. We refer the definition of weakly admissibility to section 6.4.3 of* [28]*.*

Now we can state the following equivalence of category

Theorem 4.0.2 (cf section 6.5 of [28])**.** *The functor* $(\cdot \otimes B_{st})^{\mathrm{Gal}(\overline{K}/K)}$ *induce the following equivalence of category*
{ semi-stable p*-adic representation of* $\mathrm{Gal}(\overline{K}/K)$ *}* $\rightleftharpoons$ *{ admissible filtered* (ϕ, N) *modules over* K_0 *}*
Here K_0 *is the maximal unramified extension of* $\mathbb{Q}_p$ *inside* K*.* ϕ *is the Frobenius operator and* N *is the monodromy operator. The dimension of the representation on the left side corresponds to the dimension of the* K_0 *vector space on the right side.*
Moreover, a quasi inverse of this functor is given by $(\cdot \otimes B_{st})^{\phi=1, N=0}$*.*

Remark 4.0.2. *Here we use the term admissible on the right side of the correspondence and therefore it is not difficult to prove the theorem above. However, the proof of the equivalence of admissible and weakly admissible is a hard theorem proven by various methods (cf* [15]*). One can find one of the proofs in section 6.5 of* [28]*. The author admits that* [28] *is not the original reference, while it is one of the readable material for beginners.*

Remark 4.0.3. *Of course the theorem above can be stated in a more general situation when the left side is potentially semi-stable representations. In the case we need to introduce filtered* (ϕ, N) *modules with descent data. Also it is not necessary for us to present the formulation here for simpleness, we mention that in the author's thesis we are actually using filtered* (ϕ, N) *modules with tamely ramified descent data.*

Now we give some basic examples

Example 4.0.1. *Suppose D is a weakly admissible filtered (ϕ, N) module.*
If $\dim_{K_0} D = 1$ *with a basis e. Then N is necessarily zero and ϕ acts by a scalar on e. The filtration has only one gap, which is the Hodge-Tate weight of the corresponding p-adic Galois character.*
If $\dim_{K_0} D = 2$*, then the situation is a bit more subtle. In the generic situation, we can find a basis e, f of D such that ϕ acts on e and f by distinct scalar, $N = 0$ and the filtration has two gaps with the filtrations between the two gaps being the* 1 *dimensional subspace generated by e.*

5. Eichler-Shimura theorem

We present the classical Eichler-Shimura theorem in this section with some remarks.
With the notation and definition given in a former section, the Eichler-Shimura theorem states the following:

Theorem 5.0.3. *Given a modular cusp new form f of level N and weight* 2*, we can associate to it a* 2*-dimensional irreducible ℓ adic representation $\rho(f)$ of* $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$. *Assume that $T_p f = a_p f$ for $p \nmid N$, then we have $a_p = \mathrm{trace}(\mathrm{Frob}_p)$ for each $p \nmid \ell N$.*

Remark 5.0.4. *This theorem can be found in Chapter 8 and 9 of* [19] *starting with the classical name Eichler-Shimura relation. One can also check section 3.2 of* [16] *for a quick survey.*

The generalization of Eichler-Shimura theorem to general weight $k \geq 2$ is a theorem by Deligne in [17]. On the other hand, the case for weight 1 is treated by Deligne-Serre in [18] using congruence between modular forms.

6. Converse to Eichler Shimura theorem, modularity

Eichler-Shimura theorem states that we can associate to a holomorphic modular cusp new form a 2-dimensional irreducible ℓ-adic representation of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ so that certain data of this modular cusp new form can be seen from the Galois representation. A natural question arises that is there a converse of the Eichler-Shimura theorem, namely which family of 2-dimensional irreducible ℓ-adic representations of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ is coming from a holomorphic modular form through the Eichler-Shimura theorem and its generalization to general weights?

Before we try to present the precise formulation of these classical conjecture, let us list all the properties shared by the Galois representations coming from modular cusp new forms through Eicher-Shimura theorem and its generalizations as follows:

(i) these 2-dimensional Galois representations are odd in the sense that $\det\rho(c) = -1$ where c is the complex conjugation in $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$

(ii) the restriction of these Galois representations to $\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$ for $p \neq \ell$ is unramified for all p with finite exceptions. We use the term almost everywhere for everywhere with only finite exceptions.

(iii) the restriction of these Galois representations to $\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$ for $p = \ell$ is de Rham.

Definition 6.0.6 (modularity)**.** *Given a 2-dimensional ℓ adic representation ρ of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$. If there exist a modular new form f such that $a_p = \mathrm{trace}(\mathrm{Frob}_p)$ for almost every prime p, then we say that ρ is modular. We say that a modulo ℓ representation $\overline{\rho}$ is modular if it has a modular lift ρ.*

Now we state the following case of modularity conjecture

Conjecture 6.0.1 (modularity)**.** *If ρ is absolutely irreducible 2-dimensional ℓ adic representation of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ and satisfies the three conditions above, then ρ is modular.*

The statement of the conjecture above is not quite standard compared to the classical reference [16]. However, the only difference is the concrete conditions used at $p = \ell$. We mentioned the condition de Rham so that the statement is in the philosophy of Fontaine-Mazur conjecture.

Conjecture 6.0.2 (Serre conjecture [43])**.** *If $\overline{\rho}$ is an absolutely irreducible odd 2-dimensional $\overline{\mathbb{F}}_p$ representation of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$, then $\overline{\rho}$ is modular. Moreover, assume that $\overline{\rho} \cong \overline{\rho(f)}$ for certain modular new form f, the minimal possible weight $k(\overline{\rho})$ and level $N(\overline{\rho})$ can be determined explicitly by $\overline{\rho}$.(See section 7.1 of [16]) In fact, $k(\overline{\rho})$ depends only on $\overline{\rho}\,|_{I_{\mathbb{Q}_p}}$.*

Remark 6.0.5. *These conjectures are usually directly related to Shimura-Taniyama-Weil conjecture (now a theorem by [7]) which is the modularity for elliptic curves*

over $\mathbb{Q}$. In fact, one simply consider the 2-dimensional ℓ adic representation of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ *associated to a given elliptic curve over $\mathbb{Q}$ to pass from Shimura-Taniyama-Weil conjecture to the form above. The conjecture above is related to the Fontaine-Mazur conjecture* [27] *which asserts that Galois representations satisfying conditions similar to (ii) and (iii) are coming from subquotient of ℓ adic representation of certain algebraic variety defined over a number field. The Serre conjecture is proven in* [37] *and* [38]*. And the first conjecture is the combination of Serre conjecture and automorphic lifting theorems. Automorphic lifting theorems are a series of theorems stating results of the form:*
If a $\overline{\mathbb{F}}_p$ Galois representation is modular, then any of its lifting to p-adic Galois representation with certain de Rham conditions at the places over ℓ is modular (or automorphic).
There are tons of articles around the topic of automorphic lifting theorems, and they remain one of the most important tools to attack p-adic Langlands correspondence. We mention the article [2] *here and ask the reader to check the list of references inside them as there are too many. The author apologize for this as he is not familiar with these theorems in the sense that he never use them seriously himself.*

7. p-adic Langlands correspondence

Among the discussion in the former sections, we observe that there is actually a correspondence between holomorphic modular cusp new forms and certain family of 2-dimensional irreducible ℓ-adic representation of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ such that the level of the modular form corresponds to the Artin conductor of the Galois representation. We wish to present these results in a representation theoretic language, so that the whole strategy could be generalized more naturally.

Definition 7.0.7. *A topological group is called profinite if it is isomorphic to a projective limit of a sequence of finite discrete groups. A topological group is called locally profinite if it has a profinite open subgroup. In particular any compact subgroup of a locally profinite group is profinite.*
A representation of a locally profinite topological group is called smooth if the stablizer of any vector contains a open compact subgroup. It is admissible if the fixed

subspace of any open compact subgroup is finite dimensional. In particular, representations of finite dimension is always admissible.

Example 7.0.2. *By definition, the absolute Galois group of any field is a profinite group. On the other hand, the Weil group W_K for a p-adic field K is locally profinite as the inertial group I_K is profinite.*
On the other hand, the set of K points of a p-adic reducitive group defined over K is a locally profinite group, while the set of $\mathcal{O}_K$ points form a profinite group.

Example 7.0.3. *Now we present some basic examples of smooth admissible representations of both p-adic reductive groups and Weil groups.*
A character $\chi : K^\times \to E^\times$ is smooth if and only if the image of $\mathcal{O}_K^\times$ is finite. Here E is any coefficient field, namely $\mathbb{C}$, $\overline{\mathbb{F}}_p$,etc.
On the other hand, a character $\chi : W_K \to E^\times$ is smooth if and only if the image of I_K is finite. In fact, the local class field theory says that there is a bijective correspondence between these two set of characters (The reciprocity law in the local class field theory is usually stated in other manners, which are equivalent to this one after a direct translation).
Given a smooth $\mathbb{C}$ character $\chi := \chi_1 \otimes \chi_2 \otimes ... \otimes \chi_n$ of $K^{\times,n} := K^\times \times ... \times K^\times$ is n copy of $K^\times$ which is the set of K points of the maximal tori of $\mathrm{GL}_n/_{\mathbb{Z}}$. We consider following space
$\mathrm{Ind}_{B(K)}^{\mathrm{GL}_n(K)}\chi :=$

$$\{f : \mathrm{GL}_n(K) \to \mathbb{C} \text{ locally constant }, s.t. f(gb) = \chi(b)f(g)\forall b \in B(K)\}. \tag{10}$$

This representation is smooth as each element inside is a locally constant function, which is fixed by some open compact subgroup of $\mathrm{GL}_n(K)$ which is small enough. This representation is admissible due to the Iwasawa decomposition of p-adic reductive groups. The irreducibility of this representation is analyzed carefully in []. This construction is a special case of the so called smooth parabolic induction.

Definition 7.0.8. *We say that an irreducible smooth representation of $\mathrm{GL}_n(K)$ is unramified if it has a nontrivial vector fixed by $\mathrm{GL}_n(\mathcal{O}_K)$. A character of $K^\times$ is unramified if and only if the action of $\mathcal{O}_K$ is trivial. Such a character corresponds* *to a smooth character of* W_K *with trivial action of* I_K *by local class field theory, hence explains the term 'unramified'.*

Remark 7.0.6. *By the classical Satake isomorphism (cf* [31] *for survey), unramified irreducible smooth representations of* $\mathrm{GL}_n(K)$ *are all subquotients of* $\mathrm{Ind}_{B(K)}^{\mathrm{GL}_n(K)}\chi_1 \otimes ... \otimes \chi_n$ *for some unramified characters* $\chi_1, \chi_2, ..., \chi_n$*.*

Definition 7.0.9. *Given a p-adic reductive group* G *define over* K*. The corresponding Hecke algebra* $\mathcal{H}(G(K))$ *is defined to be the space of all locally constant functions on* $G(K)$ *with compact support where the multiplication is defined by convolution (See section 1.2 of* [36] *for the case* GL_2*, in fact the definition is true in general). We refer the definition of smoothness and admissibility of a representation (or module) of Hecke algebra to page 16 of section 1.2 of* [36]*.*

Remark 7.0.7. *It is shown in section 1.2 of* [36] *(even section 1.2 of* [36] *treats the case* GL_2*, the same proof holds in general) that there is a equivalence of category between smooth admissible representations of* $G(K)$ *and the category of smooth admissible representations of* $\mathcal{H}(G(K))$*.*

Remark 7.0.8. *For the infinite places, we have to use the language of* $(\mathfrak{g}, \mathrm{K})$ *modules (See section 1.5 and 1.6 of*[36] *for the case* GL_2 *and* [5] *in general). One can also define Hecke algebras for reductive groups over archimedean fields, and then show that there is an equivalence of category between representations of Hecke algebra and* $(\mathfrak{g}, \mathrm{K})$ *modules (See* [36] *for the case* GL_2*).*

Definition 7.0.10 (cf [42] I.2.17 for definition of automorphic forms)**.** *We define the automorphic representations of* $G(\mathbb{A}_F)$ *to be subquotients of the function space over* $G(\mathbb{A}_F)$ *containing all the functions satisfying:*
(i) each function is fixed by some open compact subgroup of $G(\mathbb{A}_F^f)$
(ii) the action of the center $Z(\mathbb{A}_F^f)$ *on each function generate a finite dimensional space*
(iii) each function is invariant under the left action of $G(F)$
(iv) each function is smooth (in the ordinary case) and slowly increasing at infinity places
The left action by $G(\mathbb{A}_F^f)$ *on this function space is induced by the right multiplication*

of $G(\mathbb{A}_F^f)$ on the space. At the infinite place, we have the $\mathfrak{g}, \mathrm{K}$ modules structure instead of representation of $G(\mathbb{R})$ or $G(\mathbb{C})$.

Remark 7.0.9. *With no doubt the subject of automorphic representation for general reductive groups over global fields takes a series of books to introduce. The classical reference for a list of survey is [4]. Here our only goal is to translate Eichler-Shimura theorem into a representation theoretic language.*

Theorem 7.0.4 (Restricted tensor product decomposition cf [36] section 2.9 for the case GL_2)**.** *Given an irreducible automrophic representation Π, then there are a family of smooth irreducible representations Π_v of $\mathcal{H}(G(F_v))$ such that*

$$\Pi = \otimes' \Pi_v \tag{11}$$

where $\otimes'$ denote the restrict tensor product.

Example 7.0.4. *We give some simplest examples of automorphic representations and their restricted tensor product decomposition.*
Given a character $\eta := \eta_1 \otimes ... \otimes \eta_n$ of $(F^\times \backslash \mathbb{I}_F^f)^n$, we consider the following $\mathrm{Ind}_{B(\mathbb{A}_F^f)}^{\mathrm{GL}_n(\mathbb{A}_F^f)} \eta :=$

$$\{f : \mathrm{GL}_n(K) \to \mathbb{C} \text{ locally constant }, s.t. f(gb) = \chi(b) f(g) \forall b \in B(K)\}. \tag{12}$$

A character χ_i of $\mathbb{I}_F^f$ is the same thing as $\otimes' \chi_{i,v}$ where $\chi_{i,v}$ is a character of $F_v^\times$ for each finite place v. We use the notation $\eta_v := \chi_{1,v} \otimes ... \chi_{n,v}$, and then we claim that

$$\mathrm{Ind}_{B(\mathbb{A}_F^f)}^{\mathrm{GL}_n(\mathbb{A}_F^f)} \eta = \otimes'_{v \text{ finite}} \mathrm{Ind}_{B(F_v)}^{\mathrm{GL}_n(F_V)} \eta_v \tag{13}$$

Note that we restrict ourselves to the finite places as we do not want to bother about the $(\mathfrak{g}, \mathrm{K})$ modules appearing at infinite places. Of course we can state this example in the language of modules of Hecke algebra so that infinite places come into the picture as well. However, we want to emphasis principal series representations here.

Then we explain how can we translate certain modular form into an automorphic representation of $\mathrm{GL}_2(\mathbb{A}_{\mathbb{Q}})$ following section 3.A of [30].
A modular form is a holomorphic function on $\mathbb{H}$ by definition. Note that we have the isomorphism $\mathrm{SL}_2(\mathbb{R})/\mathrm{SO}_2(\mathbb{R}) \cong \mathbb{H}$, and therefore functions on $\mathbb{H}$ can be viewed as functions over $SL_2(\mathbb{R})$ which are invariant under the action of $\mathrm{SO}_2(\mathbb{R})$ from the right. Fix an open compact subgroup U of $\mathrm{GL}_2(\mathbb{A}_{\mathbb{Q}}^f)$ such that its intersection with $\mathrm{GL}_2(\mathbb{Q})$ is $\Gamma(N)$. We observe that the operators $< d >$ offer us a character of $(\mathbb{Z}/N\mathbb{Z})^\times$, and therefore a grossencharacter of $\mathbb{Q}^\times\backslash\mathbb{I}_{\mathbb{Q}^\times}^f$ and therefore a character of $\mathrm{GL}_2(\mathbb{A}_{\mathbb{Q}}^f)$ by factoring through the determinant. We have the decomposition (by strong approximation principle) $g = \gamma g_\infty u$ where $\gamma \in \mathrm{GL}_2(\mathbb{Q})$, $g_\infty \in \mathrm{SL}_2(\mathbb{R})$ and $u \in U$. Then we define:

$$\phi_f(g) := f(g_\infty(i))j(g_\infty, i)^{-k}\psi(u) \tag{14}$$

In particular, the condition that ϕ_f is fixed by certain open compact subgroup of $\mathrm{GL}_2(\mathbb{A}_{\mathbb{Q}}^f)$ is ensured by the fact that modular form f is invariant under certain $\Gamma(N)$ up to a factor. Note that a congruence subgroup is simply the intersection of certain open compact subgroup of $\mathrm{GL}_2(\mathbb{A}_{\mathbb{Q}}^f)$ and $\mathrm{GL}_2(\mathbb{Q})$. As we have achieve a function on $\mathrm{GL}_2(\mathbb{A}_{\mathbb{Q}})$, we can consider the representation generate by this function and then arrive at an automorphic representation. We denote this automorphic representation by $\pi(f)$ where f is the original modular form.

Lemma 7.0.1. *If f is a modular cusp new form, then $\pi(f)$ is irreducible and cuspidal.*

We refer the proof of this lemma to section 5.A and 5.B of [30].

Theorem 7.0.5 (Eichler-Shimura theorem via classical local-global compatibility)**.** *Let f be a a modular cusp new form of level N, and $\pi(f)$ is the cuspidal automorphic representation of $\mathrm{GL}_2(\mathbb{A}_{\mathbb{Q}})$ associated to f , then we can associate to $\pi(f)$ a 2-dimensional ℓ adic representation $\rho(f)$ of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ for any fixed ℓ. We have a smooth irreducible representation π_p coming from restricted tensor product decomposition of $\pi(f)$ for each finite place p. If $(p, \ell N) = 1$, then π_p is unramified and π_p corresponds to $\rho(f) \mid_{\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)}$ via the Satake correspondence (cf [31]).*

Through the local-global compatibility version of Eichler-Shimura theorem, we can see clearly that information of unramified local component of a automorphic representation corresponds perfectly to information of restriction of ℓ adic Galois representation to unramified places. Of course one wish to get a local-global compatibility result at each place. For places p such that $p \mid N$ and $p \neq \ell$, the local-global compatibility is true in the sense that we need to link supercuspidal representations of $\mathrm{GL}_2(\mathbb{Q}_p)$ to irreducible 2-dimensional ℓ adic representations of $\mathrm{GL}_2(\mathbb{Q}_p)$ by identifying their L functions. This is the central content of classical local Langlands correspondence, and is proven for general GL_n in [32] together with local-global compatibility in a special global setting. (The case GL_2 can even be down through explicit classification of supercuspidal representations in [11]).

When we come to the place $p = \ell$, the situation is much more mysterious. The reason is that the representation $\rho(f) \mid_{\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)}$ for $p = \ell$ is much more complicated than the case $p \neq \ell$ in the sense that p-adic Hodge theory come into play. We can show that $\rho(f) \mid_{\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)}$ is de Rham (by the p-adic comparison theorem [24] and the fact that these Galois representations are realized in the ℓ adic cohomologies of modular curves), and therefore by Fontaine's theory we can associate a filtered (ϕ, N) module (probably with descent data) to it. Here we do not distinguish between de Rham and potentially semi-stable, whose equivalence is a hard theorem proved my many different methods (See [15], [3], [39],[25] etc). In fact one knows that curves have potentially semi-stable reduction (See [1] for example) so we do not even need this equivalence here.

Once we have a filtered ϕ, N module, we can combine the action of ϕ and N together and therefore get a Weil-Deligne representation, which is here a smooth representation of the Weil group $W_{\mathbb{Q}_p}$ together with a monodromy operator N such that the action of $\mathrm{Frob}_p = \phi$ is semi-simple and $N\phi = p\phi N$. (See [44] for definition through Weil-Deligne groups). As a result, comparing to the case $p \neq \ell$, where ℓ adic Galois representations are equivalent to Weil-Deligne representations by Grothendieck monodromy theorem (cf [35]), we have the new informatinon: p-adic Hodge filtration.

For example, in the generic situation for 2-dimensional weakly admissible filtered (ϕ, N), the only information other than Frobenius eigenvalues is the set of Hodge-Tate weights. In other word, we need to find the information of Hodge-Tate weights inside certain representation of $\mathrm{GL}_2(\mathbb{Q}_p)$. The method here to recover the information of Hodge-Tate weight is to enlarge $\pi_p(f)$ to certain locally algebraic representation. (Note that we are considering p-adic representations of p-adic groups, so the terms algebraic and analytic make sense, and in fact they are crucial.) As we have enlarged $\pi_p(f)$, we also need a new global setting where a new compatibility is available such that this enlarged locally algebraic representation is inside the global object. This is the place where complete cohomology come into our sight.

Definition 7.0.11 ([21] section 5.1)**.** *Fix a level N such that $p \nmid N$. Then we have a projective system of modular curves $X(Np^n)$ for each integer n. Then the complete cohomology with tame level N and constant coefficient is defined to be:*

$$\hat{H}^*(N, \mathbb{Z}_p) := (\lim_{\leftarrow_m} \lim_{\rightarrow_n} H^*(X(Np^n), \mathbb{Z}/p^m\mathbb{Z})) \otimes E. \tag{15}$$

The definition of complete cohomology can be easily generalized to a general projective system of space (See [13] *for example). Here the cohomology is either singular cohomology or etale cohomology (they are isomorphic here).*

Now we need to define Banach representations, which usually appear as the closure of $\pi_p(f)$ inside the complete cohomology. By the way, the complete cohomology above has natural action of $\mathrm{GL}_2(\mathbb{Q}_p)$ which makes it into an admissible Banach representation.

Definition 7.0.12. *Let G be a p-adic reductive group defining over K. Then a Banach representation (Π, V) of $G(K)$ is a p-adic Banach space V over a sufficiently large finite extension E of K, together with a continuous action of $G(K)$ on V with the p-adic topology. By dualizing a $\mathcal{O}_E$ lattice in the Banach representation, it can be viewed as a module over the Iwasawa algebra $\mathcal{O}_E[[G(K)]]$, which is a noetherian ring by* [40]*. Such a Iwasawa module is called coadmissible if it is finite generated, and the original lattice is called admissible if its dual is coadmissible. We say that Π is admissible if it has an admissible lattice. ((*[21] *section 1.2, definition 3.1.1 and Appendix C))*

Remark 7.0.10. *In particular, if the lattice is admissible, and its modulo p reduction is smooth admissible. On the other hand, Banach representations arising from pieces of complete cohomologies are admissible due to section 4.2 of* [20].

Example 7.0.5. *Any continous character of $K^\times$ with value in E is an admissible Banach representation as it is* 1 *dimensional. We also have parabolic induction in the setting of Banach representation. Given a continuous character $\chi : (K^\times)^n \to \mathcal{O}_E^\times$, we define:*
$\mathrm{Ind}_{B(K)}^{\mathbb{GL}_n(K)}\chi :=$

$$\{f : \mathrm{GL}_n(K) \to E \textit{ continuous }, f(gb) = \chi(b)f(g), \forall b \in B(K)\} \tag{16}$$

This is an admissible Banach representation as on the dual level of Iwasawa modules, the parabolic induction is just tensor $\mathcal{O}_E[[\mathrm{GL}_n(K)]]$ over the $\mathcal{O}_E[[B(K)]]$ module $\mathcal{O}_E(\chi)$.

Now we can state the local-global compatibility theorem by Emerton in a limited version.

Theorem 7.0.6 (Local-global compatibility for p-adic Langlands correspondence for $\mathrm{GL}_2(\mathbb{Q}_p)$, section 1.1 of [21])**.** *If f is a modular cusp new form and $\rho(f)$ is the* 2*-dimensional irreducible ℓ adic representation of* $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$*. Then we have:*

$$\mathrm{Hom}_{E[\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})]}(\rho(f), \hat{H}(N, \mathbb{Z}_p)) \cong B(f) \bigotimes (\otimes'_{p\neq\ell})\pi_p(f) \tag{17}$$

Here E is a finite extension of $\mathbb{Q}_\ell$ and $B(f)$ is a Banach representation with coefficent E containing $\pi_\ell(f)$.

Example 7.0.6. *If $\rho(f)\mid_{\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)}$ is of the form:*

$$\begin{pmatrix} \chi_1 & * \\ 0 & \chi_2 \end{pmatrix} \tag{18}$$

where $\chi_1 \neq \chi_2\varepsilon, \chi_2\varepsilon^{-1}$ and the $*$ means the unique nonsplit extension of these two characters. (The existence and uniqueness is due to calculation of Galois cohomology.) Then the representation $B(f)$ is a indecomposable Banach representation of length 2 *with socle* $\mathrm{Ind}_{B(\mathbb{Q}_p)}^{\mathrm{GL}_2(\mathbb{Q}_p)}\chi_1 \otimes \chi_2\varepsilon$ *and with cosocle* $\mathrm{Ind}_{B(\mathbb{Q}_p)}^{\mathrm{GL}_2(\mathbb{Q}_p)}\chi_2 \otimes \chi_1\varepsilon$.

8. Serre weight and modulo p Langlands correspondence

As long as we have a p-adic Langlands correspondence for $\mathrm{GL}_2(\mathbb{Q}_p)$ ([14] for the proof) between certain Banach representations of $\mathrm{GL}_2(\mathbb{Q}_p)$ and 2-dimensional p-adic continuous representation of $\mathrm{Gal}(\overline{\mathbb{Q}_p}/\mathbb{Q}_p)$, we take lattices on both sides and then take modulo p reduction of both of them. Therefore there should be a modulo p version of Langlands correspondence between certain smooth admissible $\overline{\mathbb{F}}_p$-representation of $\mathrm{GL}_2(\mathbb{Q}_p)$ and 2-dimensional $\overline{\mathbb{F}}_p$-representation of $\mathrm{Gal}(\overline{\mathbb{Q}}_p\mathbb{Q}_p)$. Here we need to be a little cautious in the following sense: passing from p-adic correspondence to modulo p correspondence is by taking a lattic and modulo p; however, torsion cohomology might contain more information than p-adic cohomology, and hence modulo p correspondence is not necessarily contained in or weaker than p-adic correspondence.

This modulo p correspondence actually exists for $\mathrm{GL}_2(\mathbb{Q}_p)$, as partially shown by explicit calculation in [6] and [10], then follows from [14] for the general case. Comparing to p-adic correspondence, modulo p correspondence has certain technical advantages. In fact, one can not attack Banach representations directly. One attacks locally analytic representations (and their corresponding objects on the Galois side) instead through other technic (Verma modules, (ϕ, Γ) modules, eigenvarieties, etc). Although locally analytic methods are really powerful, the process of taking completion to get Banach representation is very complicated. On the other hand, modulo p representations are usually computable, and rely heavily on modular representation theory of finite group of Lie type. The importance of modulo p correspondence can also be observed through the study of Galois deformation theory.

Remark 8.0.11. *It is in the local-global compatibility* [21] *by Emerton that the Serre weights appear as* $\mathrm{soc}_{\mathrm{GL}_2(\mathbb{Z}_p)}\overline{B(f)}$.

It is necessary for us to understand how we pass from Serre's classical conjecture on optimal weight of modular cusp new form to the modern concept: Serre weight. The original conjecture by Serre predicted the optimal weight of modular cusp new form whose associated Galois representation is absolutely irreducible and has a given modulo p reduction. In other word, given an absolutely irreducible $\overline{\mathbb{F}}_p$ *representation* *of* $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$*, we need to find the minimal integer* k *such that there exist a modular cusp new form of weight* k *that gives us the given Galois representation. As the weight of modular cusp new form* f *essentially corresponds to the Hodge-Tate weight of* $\rho(f) \mid_{\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)}$*, the weight part of classical Serre conjecture essentially reduces to determine the possible Hodge-Tate weight for* $\overline{\mathbb{Q}}_p$ *points on the Galois deformation ring of* $\overline{\rho(f)} \mid_{\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)}$*. Finally, as Hodge-Tate weights are fixed in a connected potentially semi-stable Galois deformation ring, the information of possible Hodge-Tate weights is related to the possible components of (special fiber) of the potentially semi-stable locus in the big Galois deformation ring. On the other hand, which is more important, the Eichler-Shimura theorem and its generalization is compatible with p congruence in the sense of a theorem by Deligne* []*. As we can associate (indirectly)* $\overline{\mathbb{F}}_p$ *representation of* $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ *to a* $\overline{\mathbb{F}}_p$ *modular form, the optimal weight should depend only on this modulo p modular form, hence should be in a purely representation theoretic way on the automorphic side. The modern term Serre weight is basically the representation theoretic and geometric illustration of the combinatoric procedure in Serre's classical conjecture. At last, the whole picture is summarized in the Breuil-Mezard conjecture* [10] *and its geometric version* [22].

The p-adic Langlands correspondence and modulo p Langlands correspondence are expected to have their natural generalization to GL_n. Now we are going to discuss a bit about the formulation of modulo p Langlands correspondence for $\mathrm{GL}_n(\mathbb{Q}_p)$.

We need to fix certain notation and condition here. We give the notation here essentially following section 4.1 of [9].Let F be a CM field that split at p, and F^+ is the maximal totally real subfield of F. We fix a representation $\overline{r} : \mathrm{Gal}(\overline{F}/F) \to \mathrm{GL}_n(\overline{\mathbb{F}}_p)$ and denote its restriction to the decomposition group at p by $\overline{\rho} : \mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p) \to \mathrm{GL}_n(\overline{\mathbb{F}}_p)$. Let G be a unitary group over F^+ which split over F and is compact at infinity, namely $G(\mathbb{R})$ are compact for all embedding $F^+ \hookrightarrow \mathbb{R}$. Note that we have $G(\mathbb{Q}_p) = \mathrm{GL}_n(\mathbb{Q}_p)$

Fix U^p which is open compact subgroup of $\mathbb{A}_F^{f,p}$. Given a finite dimensional $\overline{\mathbb{F}}_p$ representation W of $\mathrm{GL}_n(\mathbb{Z}_p)$, consider the following space of functions

$$S(U^p, W) = \lim_{U_p} S(U^p U_p, W) \tag{19}$$

where we have

$$(20)\quad S(U^pU_p, W) = \{f : G(F^+\times)\backslash G(\mathbb{A}_F^f) \to W, u_p^{-1}\cdot f(u) = f(u_pu), u_p \in U_p\}.$$

Here the limit is the inverse limit over all open compact subgroup of $\mathrm{GL}_n(\mathbb{Z}_p)$. Then $S(U^p, W)$ is equipped with a natural action of $\mathrm{GL}_n(\mathbb{Q}_p)$. Let Σ be the set of places of F^+ over which F splits and that are prime to U^p. There are exactly two places of F over each places of F^+ in Σ. We choose one from each pair of places of F and denote this set by $\tilde{\Sigma}$, and then we denote its conjugate by $\tilde{\Sigma}^c$. Therefore we can construct a universal Hecke algebra $\mathbb{T} := \mathbb{Z}[T^i_{\tilde{w}}, T^i_{\tilde{w}^c}]$ for $w \in \Sigma$ and $1 \le i \le n-1$. We observe that this Hecke algebra has a natural action on $S(U^p, W)$. On the other hand, by the local compatibility property (Hecke eigenvalues correspond to Frobenius eigenvalues), we can naturally define a maximal ideal $\mathfrak{m}_{\overline{r}}$ of the Hecke algebra $\mathbb{Z}$ by $\overline{r}$ using local-global compatibility (See section 4.1 [9]).
Now we consider the $\mathfrak{m}_{\overline{r}}$ isotypic component of $S(U^p, W)$ which is $S(U^p, W)[\mathfrak{m}_{\overline{r}}] \subset S(U^p, W)$ consisting of elements in $S(U^p, W)$ killed by the maximal ideal $\mathfrak{m}_{\overline{r}}$. In the following we restrict to the case $W = \overline{\mathbb{F}}_p$ which is the trivial representation.

The representation $S(U^p, \overline{\mathbb{F}}_p)[\mathfrak{m}_{\overline{r}}]$ of $\mathrm{GL}_n(\mathbb{Q}_p)$ is actually smooth and admissible, and is the conjectural object on the automorphic side corresponding to $\overline{\rho}$ in the mod p Langlands correspondence.

Therefore we have a natural conjecture here

Conjecture 8.0.3. *$S(U^p, \overline{\mathbb{F}}_p)[\mathfrak{m}_{\overline{r}}]$ depends only on $\overline{\rho}$, namely independent of the choice of $\overline{r}$.*
On the other hand, $S(U^p, \overline{\mathbb{F}}_p)[\mathfrak{m}_{\overline{r}}]$ determines $\overline{\rho}$.

A first step towards the conjecture above is the so called Serre weight conjecture, it relate the $\mathrm{GL}_n(\mathbb{Z}_p)$ socle on the automorphic side to the restriction to the inertial subgroup of the representation on the Galois side.

Conjecture 8.0.4. $\mathrm{soc}_{\mathrm{GL}_n(\mathbb{Z}_p)}S(U^p,\overline{\mathbb{F}}_p)[\mathfrak{m}_{\overline{r}}]$ *can be determined from* $\overline{\rho}\mid_{I_{\mathbb{Q}_p}}$ *through a explicit procedure in* [33].

Remark 8.0.12. *The Serre weight conjectures are vast generalization of Serre's classical conjecture on optimal weight of modular forms. The first modern form of Serre weight conjectures might be the Buzzard-Diamond-Jarvis conjecture on Serre weights in the case of* $\mathrm{GL}_2(K)$ *(See* [12]*). There have been tons of work around these conjectures. One may prove the modularity of certain Serre weight through automorphic lifting theorems (See* [29] *for example) or through the so called weight cycling (See* [23]*). On the other hand, one can manage to eliminate Serre weights using methods from integral p-adic Hodge theories, for example Breuil modules and Strong divisible modules (See* [8] [34] *for example). Finally, there is recent paper* [41] *that is able to prove Serre weight conjectures in many cases through calculation of the deformation space of Kisin modules. We also need to mention that the famous (geometric) Breuil-Mezard conjecture (*[10], [22]*) lies behind these Serre weight conjectures.*

In fact, as illustrated in Herzig's thesis [33], the procedure divides into three steps. Firstly, we construct a Deligne Lusztig representation of $\mathrm{GL}_n(\mathbb{F}_p)$ from $\overline{\rho}\mid_{I_{\mathbb{Q}_p}}$. Secondly, we take the mod p reduction of this Deligne Lusztig representation and apply a reflection functor to the the set of constituents. Thirdly, we need more information to eliminate Serre weight if $\overline{\rho}$ is not semi-simple. There have been tons of work around the Serre weight conjecture, and we will not go any further along this line.

9. Beyond Serre weights

We are going to introduce some recent results ([34]) for $\mathrm{GL}_3(\mathbb{Q}_p)$. Finally, we mention that the author is trying to generalize [34] to $\mathrm{GL}_n(\mathbb{Q}_p)$ with Chol Park. Our first step is the generalization to $\mathrm{GL}_4(\mathbb{Q}_p)$, which is already technically difficult.

Following the notation of the last section, if the representation $\overline{\rho}$ is semi-simple, then we do not lose much information by restricting to the inertia subgroup. Hence in the semi-simple case, the fact that $S(U^p,\overline{\mathbb{F}}_p)[m_{\overline{r}}]$ determines $\overline{\rho}$ almost reduces to the Serre weight conjecture. However, if the representation $\overline{\rho}$ is not semi-simple, then new phenomena appears.

[8] is make the first step to consider a similar problem in the setting of $\mathrm{GL}_2(K)$ where K is an unramified extension of $\mathbb{Q}_p$. More recently [34] generalize the result of [8] in the setting of $\mathrm{GL}_3(\mathbb{Q}_p)$. The target of the author and Chol Park is to generalize such kind of result to $\mathrm{GL}_4(\mathbb{Q}_p)$ and even general $\mathrm{GL}_n(\mathbb{Q}_p)$.

More precisely, let us consider an ordinary $\overline{\rho}$, which by definition is the extension of several $\overline{\mathbb{F}}_p$ characters. Then the data of Serre weight conjecture can tell us information of the diagonal, namely semi-simplification of $\overline{\rho}$, but shed no light the extension parameters between these $\overline{\mathbb{F}}_p$ characters.

The question is, how can we find the information of the extension parameter of the $\overline{\mathbb{F}}_p$ ordinary representation $\overline{\rho}$ inside $S(U^p, \overline{\mathbb{F}}_p)[\mathfrak{m}_{\overline{r}}]$? The paper [8] and [34] point out a general philosophy to solve this problem.

Firstly, there are very few known bridge to connect the automorphic side and the Galois side. One of them is the classical local Langlands correspondence for GL_n which is fully proven by Harris-Taylor [32] through global methods. The crucial point of [8] [34] is Galois deformation theory. One studies the Galois deformation ring of $\overline{\rho}$ in various settings, and tries to pass the invariant of $\overline{\rho}$ to certain lifting ρ, and then to smooth irreducible characteristic 0 representation of $\mathrm{GL}_n(K)$ and then mod p. Basically, one need to keep track of the invariant through each step. This procedure is much more difficult than I said above as there are many involved technical calculations. I just explain the main steps of the story and stop right there.

The diagram of the story is as follows:

$\overline{\rho} \rightleftharpoons$ Fontaine Laffaille modules $\rightleftharpoons$ Breuil modules $\leftarrow$ strong divisible modules $\rightarrow$ filtered ϕ, N modules $\rightleftharpoons$ tamely ramified Weil Deligne representations $\rightleftharpoons$ tamely ramified principal series of $\mathrm{GL}_n(\mathbb{Q}_p) \rightarrow$ char 0 principal series of $\mathrm{GL}_n(\mathbb{F}_p) \leftarrow$ lattice of the principal series $\rightarrow$ mod p principal series $\rightarrow$ invariant hiding inside pair of extensions of Serre weights

References

[1] Ahmed Abbes. Réduction semi-stable des courbes d'après artin, deligne, grothendieck, mumford, saito, winters. *PROGRESS IN MATHEMATICS-BOSTON-*, 187:59–110, 2000.

[2] Thomas Barnet-Lamb, Toby Gee, David Geraghty, and Richard Taylor. Local-global compatibility for l= p, i. *arXiv preprint arXiv:1105.2240*, 2011.

[3] Laurent Berger. Représentations p-adiques et équations différentielles. *Inventiones mathematicae*, 148(2):219–284, 2002.

[4] Armand Borel, William Casselman, et al. *Automorphic forms, representations, and L-functions*, volume 2. American Mathematical Soc., 1979.

[5] Armand Borel and Nolan R Wallach. *Continuous cohomology, discrete subgroups, and representations of reductive groups*, volume 67. American Mathematical Soc., 2013.

[6] Christophe Breuil. Sur quelques représentations modulaires et p-adiques de $\mathrm{GL}_2(\mathbb{Q}_p)$: I. *Compositio Mathematica*, 138(2):165–188, 2003.

[7] Christophe Breuil, Brian Conrad, Fred Diamond, and Richard Taylor. On the modularity of elliptic curves over q: wild 3-adic exercises. *Journal of the American Mathematical Society*, 14(4):843–939, 2001.

[8] Christophe Breuil and Diamond Fred. Formes modulaires de hilbert modulo p et valeurs d'extensions entre caracteres galoisiens. *Ann.Sci.éc.Norm. Supér.(4)*, 47(5):905–974, 2014.

[9] Christophe Breuil, Florian Herzig, et al. Ordinary representations of $g(\mathbb{Q}_p)$ and fundamental algebraic representations. *Duke Mathematical Journal*, 164(7):1271–1352, 2015.

[10] Christophe Breuil, Ariane Mézard, et al. Multiplicités modulaires et représentations de $\mathrm{GL}_2(\mathbb{Q}_p)$ et de $\mathrm{Gal}(\bar{\shortmid}/\mathbb{Q})$ en $\ell = p$. appendice par guy henniart. sur l'unicité des types pour GL_2. *Duke Mathematical Journal*, 115(2):205–310, 2002.

[11] Colin J Bushnell and Guy Henniart. *The local Langlands conjecture for* GL(2), volume 335. Springer Science & Business Media, 2006.

[12] Kevin Buzzard, Fred Diamond, Frazer Jarvis, et al. On serre's conjecture for mod ℓ galois representations over totally real fields. *Duke Mathematical Journal*, 155(1):105–161, 2010.

[13] Frank Calegari and Matthew Emerton. Completed cohomology a survey. *Non-abelian fundamental groups and Iwasawa theory*, 393:239–257, 2011.

[14] Pierre Colmez. Représentations de $\mathrm{GL}_2(\mathbb{Q}_p)$ et $(\phi,\ \gamma)$-modules. *Astérisque*, 330(281):509, 2010.

[15] Pierre Colmez and Jean-Marc Fontaine. Construction des représentations p-adiques semistables. *Inventiones mathematicae*, 140(1):1–43, 2000.

[16] Gary Cornell, Joseph H Silverman, and Glenn Stevens. *Modular forms and Fermat's last theorem*. Springer Science & Business Media, 2013.

[17] P Deligne. Formes modulaires et representations£-adiques. séminaire bourbaki, 1968/1969, expose n 355. 139-172. *Lecture Notes in Math*, 179.

[18] Pierre Deligne and Jean-Pierre Serre. Formes modulaires de poids 1. In *Annales scientifiques de l'École Normale Supérieure*, volume 7, pages 507–530, 1974.

[19] Fred Diamond and Jerry Shurman. *A first course in modular forms*, volume 228. Springer Science & Business Media, 2006.

[20] Matthew Emerton. On the interpolation of systems of eigenvalues attached to automorphic hecke eigenforms. *Inventiones mathematicae*, 164(1):1–84, 2006.

[21] Matthew Emerton. Local-global compatibility in the p-adic langlands programme for $GL_2/\mathbb{Q}$. *preprint*, 2011.

[22] Matthew Emerton and Toby Gee. A geometric perspective on the breuil–mézard conjecture. *Journal of the Institute of Mathematics of Jussieu*, 13(01):183–223, 2014.

[23] Matthew Emerton, Toby Gee, Florian Herzig, et al. Weight cycling and serre-type conjectures for unitary groups. *Duke Mathematical Journal*, 162(9):1649–1722, 2013.

[24] Gerd Faltings. Crystalline cohomology and p-adic galois representations. *Algebraic analysis, geometry and number theory (J. Igusa, ed.)*, pages 25–80, 1989.

[25] Laurent Fargues and Jean-Marc Fontaine. Courbes et fibrés vectoriels en théorie de hodge p-adique. *preprint*, 2011.

[26] Jean Marc Fontaine. Le corps des périodes p-adiques. *Astérisque*, 223(59):111, 1994.

[27] Jean-Marc Fontaine and Barry Mazur. Geometric galois representations. *Elliptic curves, modular forms, Fermat's last theorem (Hong Kong, 1993), Ser. Number Theory, I*, pages 41–78, 1995.

[28] Jean-Marc Fontaine and Yi Ouyang. Theory of p-adic galois representations. *preprint*, 2008.

[29] Toby Gee and Mark Kisin. The breuil-mézard conjecture for potentially barsotti-tate representations. *arXiv preprint arXiv:1208.3179*, 2012.

[30] Stephen S Gelbart. *Automorphic forms on adele groups*. Number 83. Princeton University Press, 1975.

[31] Benedict H Gross. On the satake isomorphism. *Galois representations in arithmetic algebraic geometry*, 254:223, 1998.

[32] Michael Harris and Richard Taylor. *The Geometry and Cohomology of Some Simple Shimura Varieties. (AM-151)*, volume 151. Princeton university press, 2001.

[33] Florian Herzig et al. The weight in a serre-type conjecture for tame n-dimensional galois representation. *Duke Mathematical Journal*, 149(1):37–116, 2009.

[34] Florian Herzig, Daniel Le, and Stefano Morra. On mod p local-global compatibility for GL_3 in the ordinary case. *arXiv preprint arXiv: 1602.02073*, 2016.

[35] Luc Illusie. Autour du théoreme de monodromie locale. *Astérisque*, 223:9–57, 1994.

[36] Hervé Jacquet and Robert P Langlands. *Automorphic forms on GL (2)*, volume 278. Citeseer, 1972.

[37] Chandrashekhar Khare and Jean-Pierre Wintenberger. Serre's modularity conjecture (i). *Inventiones mathematicae*, 178(3):485–504, 2009.

[38] Chandrashekhar Khare and Jean-Pierre Wintenberger. Serre's modularity conjecture (ii). *Inventiones mathematicae*, 178(3):505–586, 2009.

[39] Mark Kisin. Crystalline representations and f-crystals. In *Algebraic geometry and number theory*, pages 459–496. Springer, 2006.

[40] Michel Lazard. Groupes analytiques p-adiques. *Publications Mathématiques de l'IHÉS*, 26:5–219, 1965.

[41] Daniel Le, Bao V Le Hung, Brandon Levin, and Stefano Morra. Potentially crystalline deformation rings and serre weight conjectures. *arXiv preprint arXiv:1512.06380*, 2015.

[42] Colette Moeglin and Jean-Loup Waldspurger. *Spectral Decomposition and Eisenstein Series: A Paraphrase of the Scriptures*, volume 113. Cambridge University Press, 1995.

[43] Jean-Pierre Serre et al. Sur les représentations modulaires de degré 2 de Gal($\bar{\text{ıı}}/\mathbb{Q}$). *Duke Mathematical Journal*, 54(1):179–230, 1987.

[44] John Tate. Number theoretic background. In *Proc. Symp. Pure Math*, volume 33, pages 3–26, 1979.