Introduction au domaine de recherche

Shear coordinates on the Teichmüller space of real hyperbolic surfaces with holes

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Abstract

The Teichmüller theory has been initiated in the late 19th - early 20th century by Bernhard Riemann itself, who knew that $6g-6$ real parameters were needed to describe the space of complex structures on the compact topological surface S_g of genus g, modulo diffeomorphisms of S_g. Oswald Teichmüller (1913-1943) introduced quasi-conformal mappings in the study of Riemann surfaces. In the late seventies, Thurston defined a geometric compactification of the Teichmüller space of a surface known as laminations to study the mapping class group of the latter. It led to the definition of shear coordinates on the Teichmüller space of a topological surface $S_{(g,s)}$ of genus g and with s holes. Fock and Goncharov published a paper [FG06] describing a algebraic-geometric approach to higher Teichmüller theory, showing that it is possible to define a G-Teichmüller space for any topological surface $S_{(g,s)}$ and any split reductive algebraic group G over $\mathbb{Q}$, with trivial center or simply connected. These two authors also showed that Teichmüller and laminations spaces of a surface are dual in some way ([FG05]), and that lamination spaces are tropicalisation of Teichmüller spaces. Fock and Chekov also quantised the classical Teichmüller space of a topological surface S with holes ([CF99]). In 2013, Bouschbacher constructed shear coordinates on a super-version of the Teichmüller space, or equivalently $\mathrm{SpO}(2|1)(\mathbb{R})$-Teichmüller space ([Bou13]).

Meanwhile, the development of string theory and quantum gravity since the eighties has incontestably given new motivations for those theories of Teichmüller and moduli spaces of topological surfaces.

This paper is divided in three parts. In the first one, we review the definitions of the basic objects of the study, and give some description of the mapping class group of a surface. Then, we define the Teichmüller and moduli spaces of a regular surface, and explain the construction of shear coordinates on the Teichmüller space of a hyperbolic regular surface with at least one hole. Eventually, we try to introduce two physical theories in which the classification of complex structures on a topological surface is essential: perturbative string theory on the one hand, and quantum gravity in $2+1$-dimensions on the other.

Contents

1 Regular surfaces with holes and ideal triangulations

In this section we recall the basic definitions needed in the theory of Teichmüller spaces. One starts with regular surfaces, which can in fact be generalised to ciliated surfaces (see [FG05]). The construction of shear coordinates on the Teichmüller space of a hyperbolic regular surface S with at least one hole depends on the choice of a triangulation of S, hence we will define and quickly study triangulations of such surfaces. Any triangulation of S is strongly dependent on the topology of the latter. We eventually define the mapping class group of a regular surface S, and give a description of the modular groupoid of S in terms of generators and relations.

1.1 Real hyperbolic surfaces with holes

Definition 1. *A topological oriented surface S is said to be* <u>*regular*</u> *if it is homeomorphic to:*

$$\hat{S} - \bigcup_{i=1}^{s} D_i$$

where $\hat{S}$ is a topological oriented compact surface and where the $D_i \subset S$ are closed disks which are pairwise disjoint. The boundary of S is a disjoint union of s circles. Each boundary component is called a <u>*hole*</u>.

Remark 1. *Topologically, a regular surface is completely described by the genus $\hat{g}$ of $\hat{S}$ and the number s of its holes.*

Proposition 1.1. *The Euler characteristic of any regular surface is given by:*

$$\chi(S) = 2 - 2\hat{g} - s$$

Definition 2. *If $\chi(S) < 0$, S is said to be* <u>*hyperbolic*</u>.

1.2 Ideal triangulations and topological constraints

Definition 3. *A triangulation of a regular surface with at least one hole S is a maximal family of isotopy classes of curves whose ends are contracted holes in S.*

Any triangulation of S induces a cellular decomposition of S, hence constraints of the topology of S on the possible triangulations. We have the following result:

Proposition 1.2. *Let S be a regular surface with at least one hole and let Γ be any triangulation of S. Let $V(\Gamma)$, $E(\Gamma)$ and $F(\Gamma)$ be respectively the set of vertices, edges and faces of Γ. Then:*

- $V(\Gamma) = s$
- $E(\Gamma) = 6g - 6 + 3s$
- $F(\Gamma) = 4g - 4 + 2s$

Remark 2. *The number of non-equivalent triangulations of a given regular hyperbolic surface with at least one hole is infinite (even if the topology of S fixes the cardinality of the sets of vertices, edges, and faces of any triangulation).*

However, there is a important result proven by Penner, asserting that it is always possible to find a path from one triangulation to another consisting of a finite sequence of elementary moves called *flips*.

Definition 4. *Let Γ be a triangulation of a ciliated surface, and let α be any edge of Γ. The flip of α refers to to the modification of Γ depicted in figure 1.*

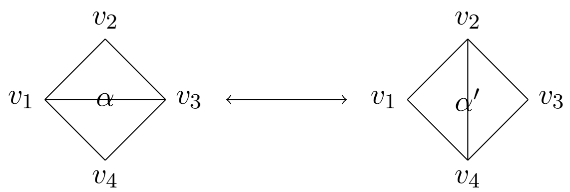


Figure 1: Flip of the internal edge α.

Theorem 1.3 ([Pen87])**.** *Two triangulations of S can always be linked by a finite sequence of flips.*

1.3 The mapping class group of a topological oriented surface

Recall that two topological smooth surfaces are homeomorphic if and only if they are diffeomorphic.

Definition 5. *Let S be an oriented regular surface, and let $\mathrm{Diff}(S)$ be the group of diffeomorphisms of any smooth surface homeomorphic to S. Let $\mathrm{Diff}_0(S)$ be the connected component of identity in $\mathrm{Diff}(S)$. The mapping class group $\mathcal{D}(S)$ (or modular group) of S if defined as:*

$$\mathcal{D}(S) = \mathrm{Diff}(S)/\mathrm{Diff}_0(S)$$

Equivalently, $\mathcal{D}(S)$ is the group of homotopy classes of diffeomorphisms of S.

The description of the mapping class group itself is rather difficult. It's much easier to describe the modular groupoid of S (a category which is closely related to $\mathcal{D}(S)$) since in can be done in terms of generators and relations.

Definition 6. *Let S be a regular surface and let $\Gamma(S)$ be the set of isotopy classes of marked triangulations of S. By "marking" of $\Gamma \in \Gamma(S)$ we mean a numeration of its edges. The marking prevents the coincidences which can arise from the potentially non-trivial symmetries of the graphs. The mapping class group $\mathcal{D}(S)$ of S acts freely on $\Gamma(S)$. Let:*

$$|\Gamma|(S) = \Gamma(S)/\mathcal{D}(S)$$

be the set of combinatorial types of marked triangulations of S.

Definition 7. *The modular groupoid of S is the category defined by:*

- $\mathrm{Ob} = |\Gamma|(S)$
- $\mathrm{Hom}(|\Gamma|, |\Gamma_1|)$ *is the set of equivalence classes of couples* (Γ, Γ_1) *where* Γ *and* Γ_1 *are triangulations of S with respective combinatorial type* $|\Gamma|$ *and* $|\Gamma_1|$*, and such that* (Γ, Γ_1) *and* (Γ', Γ'_1) *are equivalent if there exists* $g \in \mathcal{D}(S)$ *such that* $(\Gamma, \Gamma_1) = (g \cdot \Gamma', g \cdot \Gamma'_1)$*. Let* $|\Gamma, \Gamma_1|$ *be the element of* $\mathrm{Hom}(|\Gamma|, |\Gamma_1|)$ *corresponding to the equivalence class of the couple* (Γ, Γ_1)*.*

Remark 3. *This category is a groupoid in the sense that every morphism is invertible and that there exists an arrow between any two objects. It implies that for all* $|\Gamma|, |\Gamma'| \in \mathrm{Ob}$*, the set* $\mathrm{Hom}(|\Gamma|, |\Gamma|)$ *is a group and that* $\mathrm{Hom}(|\Gamma|, |\Gamma|) \simeq \mathrm{Hom}(|\Gamma'|, |\Gamma'|)$ *in the category of groups.*

Let Γ be a triangulation of S and let α be an edge of Γ. Let Γ_α be the triangulation obtained by the flip of α. We still call the morphism $|\Gamma, \Gamma_\alpha|$ a flip. One has the following proposition:

Proposition 1.4.

- *(The square of a flip is identity) Let α be an edge of Γ. Then:*

$$|\Gamma_\alpha, \Gamma| \circ |\Gamma, \Gamma_\alpha| = \mathrm{id}_{\Gamma,\Gamma}$$

- *(Flips in disjoint edges commute) Let α and β be two edges of Γ sharing no vertex. Then:*

$$|\Gamma_\alpha, \Gamma_{\alpha\beta}| \circ |\Gamma, \Gamma_\alpha| = |\Gamma_\beta, \Gamma_{\alpha\beta}| \circ |\Gamma, \Gamma_\beta|$$

- *(Flips satisfy the pentagon relation) Let α and β be two edges of Γ having a vertex in common. Then:*

$$|\Gamma_\alpha, \Gamma| \circ |\Gamma_{\beta\alpha}, \Gamma_\alpha| \circ |\Gamma_{\alpha\beta}, \Gamma_{\beta\alpha}| \circ |\Gamma_\beta, \Gamma_{\alpha\beta}| \circ |\Gamma, \Gamma_\beta| = \mathrm{id}_{\Gamma,\Gamma}$$

2 Teichmüller spaces

This section is devoted to the definition of the Teichmüller space $\mathcal{T}(S)$ and of the moduli space $\mathcal{M}(S)$ of a regular surface S, and to the construction of shear coordinates on the Teichmüller $\mathcal{X}$-space $\mathcal{T}^x(S)$ of a regular hyperbolic surface with at least one hole S. Eventually, we investigate some properties of shear coordinates and of Teichmüller spaces, and some links between $\mathcal{T}(S)$ and $\mathcal{T}^x(S)$.

2.1 Definition of Teichmüller and moduli spaces

Definition 8. *Let S be a regular surface.*

- *The Teichmüller space of S denoted by $\mathcal{T}(S)$ is the set of complex structures on S modulo* $\mathrm{Diff}_0(S)$*.*
- *The moduli space of S denoted by $\mathcal{M}(S)$ is the set of complex structures on S modulo* $\mathrm{Diff}(S)$*.*

Hence one has the following relation:

$$\mathcal{M}(S) = \mathcal{T}(S)/\mathcal{D}(S)$$

Proposition 2.1. *Let S be a regular hyperbolic surface with at least one hole. We have a one-to-one correspondence between:*

- *The Teichmüller space $\mathcal{T}(S)$*
- *The set of hyperbolic riemannian metrics on S with geodesic boundary or hyperbolic cusps, modulo $\mathrm{Diff}_0(S)$.*
- *The set of discrete injective group morphisms:*

$$\pi(S) \to \mathrm{PSL}_2(\mathbb{R})$$

  modulo overall conjugation by $\mathrm{PSL}_2(\mathbb{R})$.

Proof. Riemann's uniformisation theorem states that the only simply connected Riemann surfaces are either conformally equivalent to the sphere $\mathbb{P}^1(\mathbb{C})$, the plane $\mathbb{C}$ or the Poincaré half-plane $\mathbb{H}$. The requirement that S is hyperbolic implies that its universal cover (which is endowed with a complex structure as soon as S is) is conformally equivalent to $\mathbb{H}$. Now since $\mathbb{H}$ has a canonical hyperbolic metric (i.e a metric of constant curvature -1), this metric descends to a hyperbolic metric on S. Eventually, since $\mathbb{H}$ is the universal cover of S, S is isomorphic to a quotient of $\mathbb{H}$ by a discrete subgroup Δ of its automorphisms group. Recall that the automorphisms group of $\mathbb{H}$ is isomorphic to $\mathrm{PSL}_2(\mathbb{R})$ acting as:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot z = \frac{az+b}{cz+d}$$

for all $z \in \mathbb{H}$. Δ is canonically isomorphic to $\pi_1(S)$, and the proposition follows. Any discrete subgroup of $\mathrm{PSL}_2(\mathbb{R})$ which is finitely generated is called a <u>Fuchsian</u> group. $\square$

Remark 4. *Even if an annulus and a punctured disk are topologically equivalent, the are not biholomorphic. If S is endowed with a complex structure, one has to distinguish:*

- *the <u>holes</u> which are the boundary components of S whose neighbourhood is conformally equivalent to an annulus.*
- *the <u>punctures</u> which are the boundary components of S whose neighbourhood is conformally equivalent to a punctured disk.*

We have a one-to-one correspondence between the complex structures on S modulo $\mathrm{Diff}_0(S)$ and the hyperbolic structures on S modulo $\mathrm{Diff}_0(S)$ in the sense that every hole of the complex structure corresponds to a geodesic boundary for the hyperbolic structure, and that every puncture corresponds to a hyperbolic cusp.

Let's define a slightly different version of the Teichmüller space of S:

Definition 9. *Let S be a regular hyperbolic surface with at least one hole. The Teichmüller $\mathcal{X}$-space of S denoted by $\mathcal{T}^x(S)$ is the set of all complex structures on S modulo $\mathrm{Diff}_0(S)$, together with an orientation of all holes but the punctures.*

2.2 Construction of shear coordinates on $\mathcal{T}^x(S)$

Let S be a regular hyperbolic surface with at least one hole. The complete construction of coordinates on $\mathcal{T}^x(S)$ is done in two steps: first, one has to assign a tuple of real numbers to any point in $\mathcal{T}^x(S)$, and then show that any tuple of coordinates indeed corresponds to a point in $\mathcal{T}^x(S)$. The reconstruction step (the latter one) is at the same time intuitive and technical, hence we will refer to [FG05] for a complete description of this reconstruction, and will only develop the construction of coordinates from a hyperbolic structure on S. Let's recall an important property of riemannian hyperbolic manifolds first:

Proposition 2.2. *Let M be a riemannian closed hyperbolic manifold with constant negative curvature. Then every non-trivial free homotopy class of closed curves on S contains exactly one geodesic.*

Let Γ be a triangulation of S, and suppose that S is endowed with a complex structure (or equivalently, with a hyperbolic metric with geodesic boundary or hyperbolic cusps). Choose an orientation for all holes of S but the punctures.

- Every edge of the triangulation connecting two punctures can be made geodesic in a canonical way.
- For an edge connecting a puncture with a hole, choose an arbitrary point of the geodesic boundary component corresponding to the hole (preserving the isotopy class of the triangulation) and consider the geodesic connecting the puncture with that point. Move that point in the direction specified by the orientation of the hole. The desired ray is the limiting one of this process.
- Any edge connecting two holes can undergo the same procedure for each of its ends.

Eventually, one has a triangulation of S which is in the isotopy class of Γ, and such that its edges are geodesic and of infinite hyperbolic length. Thus the triangles it defines are standard, in the sense that they are isometric to a triangle in $\mathbb{H}$ with vertices lying on $\partial\mathbb{H} \simeq \mathbb{RP}^1$ and with geodesic edges. Moreover, one has the following result:

Proposition 2.3. *Any two standard hyperbolic triangles in $\mathbb{H}$ are isometric.*

Proof. Let (z_1, z_2, z_3) be a triple of points in $\mathbb{RP}^1$ such that the natural orientation of the triple coincides with the standard one on $\mathbb{RP}^1$ (one says that it is positively oriented). There exists a unique matrix in $\mathrm{PSL}_2(\mathbb{R}) = \mathrm{Aut}(\mathbb{H})$ such that the image of the triple (z_1, z_2, z_3) is the triple $(\infty, -1, 0)$. □

The fact that the matrix used is the proof is unique implies that:

Proposition 2.4. *Let (z_1, z_2, z_3, z_4) be a quadruple of points in $\mathbb{RP}^1$ such that (z_1, z_2, z_3) and (z_3, z_4, z_1) are positively oriented. Let $\mathrm{Conf}_4^{>}(\mathbb{RP}^1)$ be the set of orbits of such quadruples under the diagonal action of $\mathrm{PSL}_2(\mathbb{R})$:*

$$M \cdot (z_1, z_2, z_3, z_4) = (M \cdot z_1, M \cdot z_2, M \cdot z_3, M \cdot z_4)$$

for all $M \in \mathrm{PSL}_2(\mathbb{R})$. Then:

$$\mathrm{Conf}_4^{>}(\mathbb{RP}^1) \simeq \mathbb{R}_+^*$$

Proof. Let M be the unique matrix sending (z_1, z_2, z_3) to $(\infty, -1, 0)$. Since matrices in $\mathrm{PSL}_2(\mathbb{R})$ are invertible, the image of z_4 is a real number different from 0 and -1. Moreover, since $\mathrm{PSL}_2(\mathbb{R})$ preserves the orientation of triples in $\mathbb{RP}^1$, z_4 is strictly positive. □

Remark 5. *One can parametrise the set* $\mathrm{Conf}_4^{>}(\mathbb{RP}^1)$ *by the opposite of the cross-ratio of the quadruple* (z_1, z_2, z_3, z_4) *which is given by:*

$$\chi(z_1, z_2, z_3, z_4) = \frac{\det(e_1, e_2)\det(e_3, e_4)}{\det(e_2, e_3)\det(e_4, e_1)}$$

for any (e_1, e_2, e_3, e_4) *in* $\mathbb{R}^4$ *representing* (z_1, z_2, z_3, z_4)*.*

Now consider the lift $\tilde{\Gamma}$ of Γ in $\mathbb{H}$ specified by the hyperbolic structure. The vertices of $\tilde{\Gamma}$ are on the boundary of $\mathbb{H}$ since the edges of Γ are of infinite length. Consider a quadrilateral in $\tilde{\Gamma}$ formed by two adjacent triangles of vertices (z_1, z_2, z_3) and (z_3, z_4, z_1), and let $x = -\chi(z_1, z_2, z_3, z_4)$. The image of this quadrilateral by the projection map:

$$\pi : \mathbb{H} \to S$$

is again a quadrilateral formed by two adjacent triangles. Now observe that in fact any of its lifts to $\tilde{\Gamma}$ has the same value of x. Hence one can assigns a value x_i to each edge e_i of Γ.

This construction together with the reconstruction step gives:

Theorem 2.5. *Let* Γ *be a triangulation of a regular hyperbolic surface with at least one hole. The data of a strictly positive real number for each edge of* Γ *is a parametrisation of* $\mathcal{T}^x(S)$*. These numbers are called shear coordinates associated to* Γ*.*

Hence $\mathcal{T}^x(S)$ can be endowed with a structure of real smooth manifold of dimension $6g - 6 + 3s$. In fact:

$$\mathcal{T}^x(S) \simeq \mathbb{R}^{6g-6+3s}$$

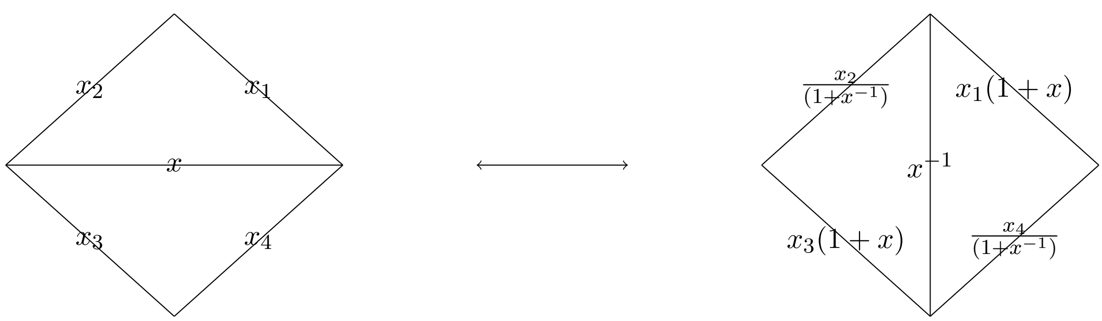
and is in particular topologically trivial.

2.3 Some properties of Teichmüller spaces

In the remaining of this section S refers to a hyperbolic regular surface with at least one hole.

Change of coordinates under a flip The construction above gives a set of coordinates on $\mathcal{T}^x(S)$ for each triangulation Γ of S. These shear coordinates of course depend on Γ, but since one knows that any two triangulations of S are always linked by a finite sequence of flips, one is able to compute changes of coordinates as soon as one knows how the coordinates change under a flip. A simple calculation shows:

Proposition 2.6. *Let* Γ *be a triangulation of* S *and let* T_1, T_2 *be two triangles of* Γ *which are adjacent along exactly one edge. Then the change of coordinates under the flip of their common edge is given in the following figure, with all edges keeping the same coordinate:*



This formula indicates the *cluster structure* of $\mathcal{T}^x(S)$.

Link between $\mathcal{T}^x(S)$ and $\mathcal{T}(S)$

Proposition 2.7. *Let Γ be a triangulation of S, and let $x = (x_i)_{i\in E(\Gamma)}$ be a point in $\mathcal{T}^x(S)$. Consider the surface S together with its boundary components corresponding to holes. In the free homotopy class of closed curves homotopic to any hole h there is a unique geodesic which is the hole itself. The hyperbolic length of the latter is given by:*

$$l_h = \prod_{i\in H} x_i$$

where the sum runs over the set H of all edges which are incident to h. Moreover from [FG05] one has that:

- *If $l_h > 1$, the orientation of h coincides with the one induced by the orientation of S.*
- *If $l_h < 1$, the orientation of h does not coincide with the one induced by the orientation of S.*
- *If $l_h = 1$, the hole is in fact a puncture.*

Proposition 2.8. *$\mathcal{T}^x(S)$ is a ramified cover of $\mathcal{T}(S)$ of degree 2^s, where s is the number of boundary components of S. The covering map is the map which forgets the orientation of holes. Moreover, the branching points of index k are the complex structures on S for which exactly k boundary components are not holes but punctures. Thus $\mathcal{T}(S)$ has a structure of manifold with corners of dimension $(6g - 6 + 3s)$, that is, is locally modelled on:*

$$[0, \infty[^k \times] - \infty, \infty[^{6g-6+3s-k}$$

Proof. Let $x \in \mathcal{T}(S)$ and h any hole of S. Choose an orientation for all holes of S but h. There are exactly two points in $\mathcal{T}^x(S)$ above x with prescribed orientations, each of them corresponding to one of the two possible orientations of h. □

2.4 Weil-Peterson form on Teichmüller spaces

Definition 10. *Let Γ be a triangulation of S, and $(x^\alpha)_{\alpha\in E(\Gamma)}$ the set of shear coordinates associated to Γ. The ϵ-matrix associated to Γ is a $(\#E(\Gamma) \times \#E(\Gamma))$-skew-symmetric matrix given by:*

$$\epsilon_{\alpha\beta} = \sum_{i\in F(\Gamma)} < \alpha, i, \beta >$$

where $\alpha, \beta \in E(\Gamma)$ and where $< \alpha, i, \beta >$ equals 1 (resp. -1) if α and β are sides of the triangle i and α is in the clockwise (resp. counterclockwise) direction from β, otherwise it equals 0.

Definition 11. *Let M be a smooth manifold. A Poisson bracket $\{.,.\}$ on M is a smooth bilinear map:*

$$\{.,.\} : \mathcal{C}^\infty(M) \times \mathcal{C}^\infty(M) \to \mathcal{C}^\infty(M)$$

satisfying for all $f, g, h \in \mathcal{C}^\infty(M)$:

1. *$\{f, g\} = -\{g, f\}$ (skew-symmetry)*
2. *$\{f, \{g, h\}\} + \{g, \{h, f\}\} + \{h, \{f, g\}\} = 0$ (Jacobi identity)*
3. *$\{fg, h\} = f\{g, h\} + g\{f, h\}$ (Leibniz's rule)*

Definition 12. *The map*

$$\{.,.\}: \mathcal{C}^\infty(\mathcal{T}^x(S)) \times \mathcal{C}^\infty(\mathcal{T}^x(S)) \to \mathcal{C}^\infty(\mathcal{T}^x(S))$$

given by:

$$\{f, g\} = \sum_{\alpha,\beta \in E(\Gamma)} \epsilon_{\alpha\beta} x^\alpha x^\beta \frac{\partial f}{\partial x^\alpha} \frac{\partial g}{\partial x^\beta}$$

is a Poisson bracket on $\mathcal{T}^x(S)$ called the Weil-Peterson form. Interestingly, it does not depend on the triangulation.

Remark 6. *Even if $\mathcal{T}(S)$ is a manifold with corners and not a smooth manifold, the Weil-Peterson form on $\mathcal{T}^x(S)$ induces a well-defined Poisson bracket on $\mathcal{T}(S)$ for the following reason. The neighbourhood of a corner is diffeomorphic to $[0, +\infty[^k \times] - \infty, +\infty[^{n-k}$. Let $(y_1, ..., y_k, y_{k+1}, ..., y_n)$ be a set of coordinates consistent with this decomposition. Then one can see that $\{y_i, f\} = 0$ for all $i \in [|1, k|]$ and all $f \in \mathcal{C}^\infty(\mathcal{T}^x(S))$, hence the Poisson bracket on $\mathcal{T}(S)$ is well defined.*

One can quantise the Poisson bracket on the classical Teichmüller space (see [CF99]). It gives a lot of information about the possible representations of the mapping class group of a regular surface.

3 Interactions with physics

In this last section we are going to give two examples of physical theories in which Teichmüller and moduli spaces of topological surfaces are essential. The first one is perturbative string theory, in which one has to integrate over the moduli space of the world-sheet surface in order to take into account all possible "paths" from an initial string configuration to a final one. Every path has a weight given by e^{-S} where S is the action functional of the theory (with an euclidean theory). We recall the definitions of quantum field theories and study some formulas describing amplitudes of scattering processes in perturbative bosonic string theory. Then following [Wit89], we give some characteristic features of general relativity in $2+1$ dimensions, which classical phase space should be defined as the Teichmüller space of its boundary, which may be quantised, leading to quantum gravity in $2+1$-dimensions.

3.1 Moduli spaces in amplitudes calculations of perturbative string theory

Definition 13. *A d-dimensional quantum field theory (QFT) is the data of:*

- *A d-dimensional smooth manifold M.*
- *A set of objects over M, for example the set of sections of a vector bundle over M, or the set of maps:*

  $$\phi_\alpha : M \to N$$

  for some target manifold N (sigma-model), or the set of smooth riemannian metrics on M (quantum gravity).
- *The choice of an action functional $S[\phi_\alpha, ...]$ on the space of fields. In QFTs one has to integrate over the space of fields against some measure on it weighted by e^{-S}. This integration is called path integration, and depicts the quantum nature of the theory.*

A conformal field theory is a QFT which is invariant under conformal transformations of the base manifold M.

Bosonic string theory is a 2-dimensional field theory in which the base manifold M is called the world-sheet. Naively, fields in bosonic string theory split in two sets:

- the set of matter fields ϕ_α, which can be though of as the coordinates of an embedding $M \to N$. N is called target space (and can be though of as the space-time).
- the set of smooth metrics g on M, since the introduction of those as fields allows the use of a linear action (the Polyakov action) instead of the more intuitive but more complicated to handle Nambu-Goto action.

Since we want to study scattering amplitudes, incoming and outgoing string states has to be specified. In the world-sheet theory, they correspond to local vertex operators $\mathcal{V}_j(k)$ where $k \in \mathrm{T^*N}$ is the target momentum and j the internal state. For example, an n-particle scattering amplitude in the closed oriented bosonic string theory is given by:

$$S_{j_1 \dots j_n}(k_1, ..., k_n) = \sum_{g=0}^{\infty} \int \frac{[d\phi\ dg]}{V_{\text{diff} \times \text{Weyl}}} \exp(-S_m - \lambda(2-2g)) \prod_{i=1}^{n} \int_{M_g} d^2\sigma_i g(\sigma_i)^{1/2} \mathcal{V}_j(k_i, \sigma_i)$$

where the first integral denotes path integration, and the second one is the standard integration over the topological compact oriented surface of genus g. S_m denotes the action for matter fields, and the action of the global theory is modified by a factor which is proportional to the Euler characteristic $\chi(M_g) = 2 - 2g$ of M_g.

Remark 7. *Once again, the replacement of the Nambu-Goto action by the Polyakov action can ony be made at the price of supplementary dynamics fields, namely the smooths metrics on M_g. However, diffeomorphisms of M_g and Weyl transformation of the metrics correspond to transformation of the fields corresponding to the same physical content, hence the term $V_{\text{diff} \times \text{Weyl}}$ in the action. It stands for the volume of the gauge group. In other words, one has to divide by this volume to avoid the over-counting due to the gauge freedom of this theory. This volume is of course infinite, but there is a procedure called the Fadeev-Popov gauge fixing, which replaces this term by a path integral over new anti-commuting fields called the ghosts.*

The Fadeev-Popov procedure leads to the following expression for $S_{j_1 \dots j_n}(k_1, ..., k_n)$:

$$S_{j_1 \dots j_n}(k_1, ..., k_n) = \sum_{g=0}^{\infty} \int_F \frac{d^\mu t}{n_R} \int [d\phi\ db\ dc] \exp(-S_m - S_g - \lambda(2-2g)) \mathcal{A}[t, \phi, b, c]$$

where $\mathcal{A}$ is some functional of the fields, and where F is the moduli space of complex structures on M_g as well as of the tuple of insertion points on M_g. To put it in a nutshell, over-counting due to gauge freedom is replaced by another conformal field theory together with an integral in a moduli space.

Hence in perturbative string theory integration over moduli spaces arise in the path integral as the way to take into account all the different possible ways to go from a given asymptotic initial string configuration to a given final asymptotic string configuration.

3.2 Quantum gravity in dimension $2+1$

In a cornerstone paper of 1988 [Wit89], Witten showed that the Hilbert space of wave functions of quantum gravity in 2+1 space-time dimensions can be seen as the space of functions on the Teichmüller space of the boundary of the space-time. More precisely, one looks at flat space-times of the form:

$$M = \Sigma \times \mathbb{R}$$

where Σ is a genus g Riemann surface representing space, and where $\mathbb{R}$ represents time. Let X be the $2+1$ Minkowski space with coordinates t, x, y and metric $ds^2 = -(dt)^2 + (dx)^2 + (dy)^2$. The

$(2+1)$-dimensional Lorentz group is the group $\mathrm{SO}(2,1)$, which is isomorphic to $\mathrm{SL}_2(\mathbb{R})$. Moreover, the hypersurface H' defined by:

$$t^2 - x^2 - y^2 = 1$$

is isometric to the hyperbolic Poincaré half-plane $\mathbb{H}$. Hence any Fuchsian group $\Delta \subset \mathrm{SL}_2(\mathbb{R})$ can be seen as a subgroup of $\mathrm{SO}(2,1)$ acting on H', and the quotient H'/Δ is a Riemann surface as well. Now one can consider the action of Δ on the whole future light-cone which is defined as the subset in X such that $t > 0$ and $t^2 - x^2 - y^2 > 1$. Since the metric on X is flat, the quotient X/Δ is flat as well, and each hypersurface defined by:

$$t^2 - x^2 - y^2 = \tau$$

and quotiented by Δ is a Riemann surface isomorphic to H'/Δ if $\tau > 0$. In that description τ plays the role of time, hence one has a description of an expanding universe with a singularity at $\tau = 0$. Equivalently, the quotient of the whole past light-cone under the action of Δ give rise to a shrinking universe with a final singularity. The discussion in [Wit89] explains the reasons why one should consider the Teichmüller space of X/Δ as the classical phase space of general relativity in $2+1$ dimensions.

References

[Bou13] F. Bouschbacher. Shear coordinates on the super Teichmüller space. 06 2013.

[CF99] L. Chekov and V.V. Fock. Quantum Teichmüller space. *ArXiv*, 9908165v2, 1999.

[FG05] V.V. Fock and A.B. Goncharov. Dual Teichmüller and lamination spaces. *ArXiv*, 0510312v1, 2005.

[FG06] V.V. Fock and A.B. Goncharov. Moduli spaces of local systems and higher Teichmüller theory. *ArXiv*, 0311149v4, 2006.

[Pen87] R.C. Penner. The decorated Teichmüller space of punctured surfaces. *Comm. Math. Phys.*, 113:299–339, 1987.

[Wit89] E. Witten. 2+1 dimensional gravity as an exact soluble system. *Nuclear Physics*, B(311):46–78, 1988/89.