THE CANONICAL HEIGHT ON ABELIAN VARIETIES

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Contents

Intorduction

We will introduce the height theory for projective varieties, the canonical height on abelian varieties. There are some estimations of them and we can use the height theory to prove some finiteness theorems.

1. Height and canonical height

We begin with the height theory. The main references are [HS] and [BG]. Most of the details in this section can be found in the two books. Some basic results in algebraic geometry and algebraic number theory that we will use can be found in [Ha] and [La].

Recall that the set of standard absolute values on the rational number field $\mathbf{Q}$ is the set $M_{\mathbf{Q}}$ consisting of the archimedean absolute value $|\cdot|_\infty$ and the $p-$adic absolute values $|\cdot|_p$ for every prime p. The set of standard absolute values on a number field k is the set M_k consisting of all absolute values on k whoes restriction to $\mathbf{Q}$ is one of the standard absolute values on $\mathbf{Q}$. We write M_k^∞ for the set of archimedean absolute values in M_k, and similarly M_k^o denotes the set of nonarchimedean absolute values on k. For $v \in M_k$, the local degree of v is the number

$$n_v = [k_v : \mathbf{Q}_v],$$

where $\mathbf{Q}_v$ is the completion of $\mathbf{Q}$ at the restriction of v to $\mathbf{Q}$. The normalized absolute value associated to v is

$$\|x\|_v = |x|_v^{n_v}.$$

Definition 1.1. For an element $\alpha \in k$, put the height of α relative to k is

$$H_k(\alpha) = \prod_{v \in M_K} \max\{1, \|\alpha\|_v\}.$$

The absolute logarithmic height is

$$h(\alpha) = \frac{1}{[k:\mathbf{Q}]} \log H_k(\alpha).$$

For $\alpha \in \bar{\mathbf{Q}}$, observe that $h(\alpha)$ does not depend on the choice of the number field k containing α.

For example, let $\frac{m}{n}$ be a rational number, with m and n coprime. Then

$$H_{\mathbf{Q}}\left(\frac{m}{n}\right) = \prod_{v \in M_{\mathbf{Q}}} \max\left\{1, \left|\frac{m}{n}\right|_v\right\} = \max\{|m|_\infty, |n|_\infty\}.$$

It is easy to prove that the set of rational numbers with fixed bounded height is a finite set. When we prove that the set of rational number is countable, we used similar argument. We will generalize it to the case of projective variety.

Theorem 1.2 (Kronecker)**.** *For an algebraic number $\alpha \in \bar{\mathbf{Q}}$ nonzero, one has $h(\alpha) = 0$ iff α is a root of unity.*

For an algebraic number $\alpha \in \bar{\mathbf{Q}}$ which is not 0 and not a root of unity, one has $h(\alpha) > 0$. The Lehmer's conjecture is the following question:

can we find a positive constant C such that $h(\alpha) \geqslant \frac{C}{[\mathbf{Q}(\alpha):\mathbf{Q}]}$ for any $\alpha \in \bar{\mathbf{Q}}$ which is not 0 and not a root of unity?

The conjecture is still open in its full generality. The next result is due to Dobrowolski [Do].

Theorem 1.3 (Dobrowolski)**.** *Let α be a nonzero algebraic number which is not a root of unity. Then*

$$h(\alpha) \geq \frac{C}{d}\left(\frac{\log\log d}{\log d}\right)^3.$$

where C is a constant not depending on α and $d = [\mathbf{Q}(\alpha):\mathbf{Q}]$.

Definition 1.4. Let k be a number field, for a point $P = (x_0 : \cdots : x_n) \in \mathbf{P}^n(k)$, the height of P relative to k is the quantity

$$H_k(P) = \prod_{v \in M_k} \max\{\|x_0\|_v, \cdots, \|x_0\|_v\}.$$

The absolute logarithmic height of P is

$$h(P) = \frac{1}{[k:\mathbf{Q}]} \log H_k(P).$$

It's not difficult to check that H_k is well-defined over $\mathbf{P}^n(k)$ and for all $P \in \mathbf{P}^n(\bar{\mathbf{Q}})$, that $h(P)$ does not depend on the choice of k such and $\mathbf{P}^n(k)$ contains P.

Recall that the field of definition of a point $P = (x_0 : \cdots : x_n) \in \mathbf{P}^n(k) \subset \mathbf{P}^n(\bar{\mathbf{Q}})$ is the field

$$\mathbf{Q}(P) = \mathbf{Q}(\frac{x_0}{x_i}, \cdots, \frac{x_n}{x_i})$$

for any i with $x_i \neq 0$.

One has

Theorem 1.5 (Northcott's property)**.** *For any numbers B and $D \geqslant 0$, the set*

$$\{P \in \mathbf{P}^n(\bar{\mathbf{Q}}) \mid h(P) \leq B \text{ and } [\mathbf{Q}(P) : \mathbf{Q}] \leq D\}$$

is finite.

This is one of the fundamental finiteness theorem in Diophantine geometry. One direct corollary of this property is the following

Corollary 1.6. *Let k be a number field. For any number B, the set*

$$\{P \in \mathbf{P}^n(k) \mid h(P) \leq B\}$$

is finite.

One problem is counting points for projective varieties over number fields. We will introduce it in the third section.

We will generalize the definition of height to all projective varieties over a number field.

Let V be a projective variety defined over $\bar{\mathbf{Q}}$. Let $\varphi\colon V \to \mathbf{P}^n$ be a morphism. The absolute logarithmic height on V relative to φ can be defined as

$$h_\varphi(P) = h(\varphi(P))$$

for all $P \in V(\bar{\mathbf{Q}})$.

The following result will be one of crucial ingredients in the construction of the height machine later.

Theorem 1.7. *Let V be a projective variety defined over $\bar{\mathbf{Q}}$. Let $H \subset \mathbf{P}^n$ and $H' \subset \mathbf{P}^m$ be hyperplanes. Let $\varphi\colon V \to \mathbf{P}^n$ and $\psi\colon V \to \mathbf{P}^m$ be morphisms such that φ^*H and ψ^*H' are linearly equivalent (meaning that the morphisms φ and ψ are associated with the same complete linear system). Then one has*

$$h_\varphi(P) = h_\psi(P) + O(1).$$

Here the constant $O(1)$ will depend on V, φ, ψ but not on $P \in V(\bar{\mathbf{Q}})$.

Theorem 1.8 (Weil's height machine)**.** *For any smooth projective variety V defined over a number field k. Let $\mathrm{Div}(V)$ be the group of divisors of V. There exists a map*

$$h_V\colon \mathrm{Div}(V) \to \{\textit{functions } V(\bar{k}) \to \mathbf{R}\}$$

$$D \mapsto h_{V,D}$$

with the following properties:

- *Normalization: for any hyperplane of codimension one $H \subset \mathbf{P}^n$ one has for all $P \in \mathbf{P}^n(\bar{k})$*
$$h_{\mathbf{P}^n,H}(P) = h(P) + O(1),$$
where on the right hand side is the usual height on $\mathbf{P}^n(\bar{k})$.
- *Functoriality: for a morphism of varieties $\varphi\colon V \to W$ inducing a map $\varphi^*\colon \mathrm{Div}(W) \to \mathrm{Div}(V)$, and a divisor $D \in \mathrm{Div}(W)$ one has for all $P \in V(\overline{K})$*
$$h_{V,\varphi^*D}(P) = h_{W,D}(\varphi(P)) + O(1).$$

- *Additivity: for $D, E \in \mathrm{Div}(V)$ one has for all $P \in V(\bar{k})$*
$$h_{V,D+E}(P) = h_{V,D}(P) + h_{V,E}(P) + O(1).$$
- *Linear equivalence: if $D, E \in \mathrm{Div}(V)$ are two linearly equivalent divisors, then*
$$h_{V,D} = h_{V,E} + O(1).$$
- *Positivity: If D is an effective divisor on V, let B be the set of base points of the associated linear system $|D|$. Then for all $P \in (V \smallsetminus B)(\bar{k})$*
$$h_{V,D}(P) \geq O(1).$$
- *Algebraic equivalence: if $D, E \in \mathrm{Div}(V)$, where D is an ample divisor and E is algebraically equivalent to zero, then*
$$\lim_{\substack{P \in V(\bar{k}) \\ h_{V,D}(P) \to \infty}} \frac{h_{V,E}(P)}{h_{V,D}(P)} = 0.$$
- *Finiteness: if $D \in \mathrm{Div}(V)$ is an ample divisor, then for any finite field extension k'/k and any constant B the set*
$$\{P \in V(k') \mid h_{V,D}(P) \leq B\}$$
is finite.
- *Uniqueness The height functions $h_{V,D}$ are determined, up to $O(1)$, by normalization, functoriality just for embeddings $\varphi \colon V \hookrightarrow \mathbf{P}^n$, and additivity.*

Remark 1.9. The "$O(1)$" constants that appear in the height machine depend on the varieties, divisors, and morphisms, but not on the points on the varieties.

The following theory was developed by Néron and Tate, for a smooth projective variety with a 'nice' morphism to itself, we can define a particular height function with some nice properties.

Theorem 1.10 (Néron–Tate)**.** *Let V/k be a smooth projective variety defined over a number field k. Let $\varphi \colon V \to V$ be a morphism such that $\varphi^* D \sim \alpha D$ with $\alpha > 1$ for some divisor D. Then there exists a unique function $\widehat{h}_{V,\varphi,D} \colon V(\bar{k}) \to \mathbf{R}$, called the canonical height on V, with the following properties:*

(1) $\widehat{h}_{V,\varphi,D}(P) = \lim_{n\to\infty} \alpha^{-n}\, h_{V,D}(\varphi^n(P))$, *where* $\varphi^n = \varphi \circ \cdots \circ \varphi$.
(2) $\widehat{h}_{V,\varphi,D}(P) = h_{V,D}(P) + O(1)$.
(3) $\widehat{h}_{V,\varphi,D}(\varphi(P)) = \alpha\, \widehat{h}_{V,\varphi,D}(P)$.

The canonical height depends only on the linear equivalence class of D.

For some projective varieties, it is easy to find such a divisor and construct a morphism with this property. For the projective space, the hyperplan divisor and any morphism with degree bigger than one will satisfy the condition of this theorem. For an abelian variety, a symmetric divisor or an anti-symmetric divisor and the multiplication by n map will work for any integer n bigger than one. The main object in the next section is the Néron–Tate height or the canonical height on an abelian variety. Now we introduce a property that used some technique in dynamical systems.

The canonical height has the following properties.

Proposition 1.11. *Let V/k be a smooth projective variety defined over a number field. Let $\varphi\colon V \to V$ be a morphism such that $\varphi^* D \sim \alpha D$ with $\alpha > 1$ for some divisor D. Assume that D is an ample divisor. Then*

(1) $\widehat{h}_{V,\varphi,D}(P) \geqslant 0$, *moreover,* $\widehat{h}_{V,\varphi,D}(P) = 0$ *iff φ is preperiodic at P, meaning that the set*
$$\{P, \varphi(P), \varphi^2(P), \ldots\}$$
is finite.

(2) *The set*
$$\{P \in V(k) \mid \varphi \text{ is preperiodic at } P\}$$
is finite.

2. Canonical height on abelian varieties

In this part, I will introduce the canonical heights on abelian varieties.

Recall Mumford's formula about divisors on an abelian variety. Let A be an abelian variety and let D be a divisor on A. One has
$$[n]^* D \sim \frac{n^2+n}{2} D + \frac{n^2-n}{2}[-1]^* D.$$
Where $[n]$ is multiplication by n map. A divisor D is called symmetric if $[-1]^* D \sim D$. Let A be an abelian variety and D be a symmetric divisor, one has
$$[n]^* D \sim n^2 D.$$

Applying the Néron-Tate theorem in the case, we obtain the following.

Theorem 2.1. *Let A/k be an abelian variety defined over a number field, and let $D \in \operatorname{Div}(A)$ be a symmetric divisor. There is a height function*
$$\widehat{h}_{A,D}\colon A(\bar{k}) \to \mathbf{R},$$
called the canonical height on A relative to D, with the following properties:

(1) $\widehat{h}_{A,D}(P) = h_{A,D}(P) + O(1)$ *for all* $P \in A(\bar{k})$.

(2) $\widehat{h}_{A,D}([m]P) = m^2 \widehat{h}_{A,D}(P)$ *for all integers m and all* $P \in A(\bar{k})$.

(3) $\widehat{h}_{A,D}(P+Q) + \widehat{h}_{A,D}(P-Q) = 2\widehat{h}_{A,D}(P) + 2\widehat{h}_{A,D}(Q)$ *for all* $P, Q \in A(\bar{k})$.

(4) *The canonical height map $\widehat{h}_{A,D}\colon A(\bar{k}) \to \mathbf{R}$ is a quadratic form. The associated pairing $\langle \cdot, \cdot \rangle_D\colon A(\bar{k}) \times A(\bar{k}) \to \mathbf{R}$ defined by*
$$\langle P, Q \rangle_D = \frac{\widehat{h}_{A,D}(P+Q) - \widehat{h}_{A,D}(P) - \widehat{h}_{A,D}(Q)}{2}$$
is bilinear and satisfies $\langle P, P \rangle_D = \widehat{h}_{A,D}(P)$.

(5) *The canonical height $\widehat{h}_{A,D}$ depends only on the divisor class of D. It's uniquely determined by (1) and (2) for any one integer $m \geq 2$.*

One can apply the Néron-Tate theorem to obtain
$$\widehat{h}_{A,D}(P) = \lim_{n \to \infty} \frac{1}{4^n} h_{A,D}([2^n]P).$$

By the property (2), one has

$$\widehat{h}_{A,D}(P+Q) = \widehat{h}_{A,D}(P)$$

and

$$\widehat{h}_{A,D}(Q) = 0$$

for all $P, Q \in A(\bar{k})$ such that Q is a point of finite order.

Moreover, if D is ample, one has $\widehat{h}_{A,D}(P) \geq 0$, with equality if and only if P is a point of finite order. In the case, the canonical height is well-defined over $A(\bar{k})/(torsion)$ and it becomes a positive quadratic form over $A(\bar{k})/(torsion)$. In fact, we can use the finiteness property of Northcott to prove a stronger result.

Theorem 2.2. *Let A be an abelian variety over a number field k. let D be a symmetric ample divisor on A, we can extend the canonical height function $\mathbf{R}-$linear to a quadratic form*

$$\widehat{h}_{A,D} \colon A(\bar{k}) \otimes \mathbf{R} \to \mathbf{R},$$

then the quadratic form is positive.

Remark 2.3. A positive quadratic form over $\mathbf{Q}$ can be not positive after a $\mathbf{R}-$linear extension. The proof of this theorem can be found in [La2] [HS].

The Mordell-Weil theorem tells us that the group of rational points for an abelian variety A over a number field k is finitely generated. One has $A(k)/\text{torsion}$ is $\mathbf{Z}-$module of finite rank. The last theorem implies that the canonical height function on a relative to an ample symmetric divisor D has a positive lower bound for non-torsion points in $A(k)$.

One of the main goal for my thesis is finding a lower bound of the canonical height function of abelian varieties. There is an abelian Lehmer problem (see [DH]).

Conjecture 2.4 (Abelian Lehmer Problem)**.** Let A be an abelian variety defined over a number field k. Let D be an ample symmetric divisor on A, there is a constant $C(A, D) > 0$, such that for all P of infinite order modulo all proper abelian sub-variety of A, one has

$$\widehat{h}_{A,D}(P) \geq C(A, D)\delta(P)^{-1},$$

where $\delta(P) = \min\{\deg(V)^{\frac{1}{\operatorname{codim} V}}, P \in V\}$, and V can be any irreducible proper sub-varieties of A defined over k. There $\deg(v)$ means the degree of V relative to D.

It's an analogue of Lehmer conjecture for abelian varieties over a number field.

3. Some applications and estimations of height theory

The Northcott property shows that the set of $k-$points $X(k)$ of a projective variety X over a number field k is finite. One problem in diophantine geometry is counting rational points on varieties. There are some famous results about it.

Theorem 3.1 (Mordell-Weil)**.** *Let A be an abelian variety over a number field k. The group of $k-$ points on A is finitely generated.*

Theorem 3.2 (Mordell-Faltings)**.** *Let C be a smooth projective curve of genus g over a number field k. If $g \geqslant 2$, the set $C(k)$ is finite.*

Now we introduce some results about the counting function.

Let C/k be a smooth projective curve C/k of genus g over a number field k, if g is 0, then either $C(k)$ is empty or $C(k)$ is isomorphic with $\mathbf{P}^1(k)$. The following result about the counting function for a projective space over number field is due to Schanuel (see [Sc]).

Theorem 3.3. *Let k be a number field of degree d, one has*

$$\sharp\{P \in \mathbf{P}^n(k) \mid H_k(P) \leqslant B\} = CB^{n+1} + O(B^{n+1-\frac{1}{d}})\}$$

for all B positive and d or n is not 1, where C is a constant which only depends on k and n. In the case of $\mathbf{P}^1(\mathbf{Q})$, one has

$$\sharp\{P \in \mathbf{P}^1(\mathbf{Q}) \mid H_k(P) \leqslant B\} = \frac{12}{\Pi^2}B^2 + O(B\log B).$$

Remark 3.4. In Schanuel's article, he precises the constant in the theorem.

In the case g equals to 1, $C(k)$ is either empty or a finitely generated abelian group. Since the torsion part is finite and the canonical height is a positive quadratic form positive defined on the free part, one obtains the following.

Theorem 3.5. *Let A be an abelian variety over a number field k and let D be an ample symmetric divisor on A, one has*

$$\sharp\{p \in A(k) \mid \widehat{h}_{A,D}(P) \leqslant B\} = CB^{\frac{r}{2}} + O(B^{\frac{r-1}{2}})$$

where C is a constant which only depends on A/k and D and r is the rank of $A(k)$.

For the Mordell's conjecture, Mumford proved a weaker result for it, see [Mu].

Theorem 3.6 (Mumford)**.** *Let C be a smooth curve of genus g over a number field k. If $g \geqslant 2$, for a fixed embedding from C to a projective space, one has*

$$\sharp\{P \in C(k) | H_k(P) \leqslant B\} \leqslant R \log\log B$$

where R is a constant, depending on C and the embedding.

Vojta gives a proof of Mordell's conjecture which starts with Mumford's ideal of the proof of this thoerem, see [Vo].

Now we introduce some applications of height theory.

There is a weak Mordell-Weil theorem.

Theorem 3.7. *Let A be an abelian variety defined over a number field k and let $m \geqslant 2$ be an integer. Then the group $A(k)/mA(k)$ is finite.*

Obviously, the Mordell-Weil theorem implies this weaker version. We can use the height theory to show that the weaker version also implies the stronger version.

Because the Northcott property, we can use the method of infinite descent. Let S be the set of representatives for $A(k)/mA(k)$. Let B be upper bound of the canonical heights of points in S. Then $A(k)$ can be generated by the finite set

$$\{P \in A(k) \mid \widehat{h}(P) \leqslant B\},$$

where $\widehat{h}$ is the canonical height function relative to a symmetric ample divisor on A.

When we define the Weil height machine, one has a map

$$h_V\colon \operatorname{Pic}(V) \to \frac{\{\text{functions } V(\bar{k}) \to \mathbf{R}\}}{\{\text{bounded functions } V(\bar{k}) \to \mathbf{R}\}}.$$

The map is a homomorphism of abelian group. Since the right side is torsion free, the torsion elements are in the kernel of h_V. One can show that the kernel is exactly the set of torsion elements in $\operatorname{Pic}(V)$. The idea of the proof is to use the intersection pairing between divisors and curves of V, we can reduce the problem to prove the elements in the kernel of the map in $\operatorname{Pic}^0(V)$ is torsion. Then we can use the theory of Albanese variety and the canonical height pair to solve this problem. In the case of abelian variety, it's easier to prove it (See [**?**]). The similar idea can get a better estimation of heights relative to a divisor algebraically equivalent to 0. One has

Theorem 3.8. *Let V be a smooth projective variety defined over a number field k. Let D, E be divisors with D ample and E algebraically equivalent to* 0*. Then there is a positive constant C such that*

$$h_{V,E}(P) \leqslant C\sqrt{h_{V,D}+1}$$

for all $P \in V(\bar{k})$.

The proof of the theorem can be found in [HS]. In the case of curves, a divisor is ample if and only if its degree is positive. A divisor is algebraically equivalent to 0 if and only if its degree is 0. Applying the theorem to curves, we can say the "main part" of the height function depends on the degree.

Theorem 3.9. *Let C/k be a smooth curve defined over a number field k. Let D, E be divisors with D ample, then*

$$\lim_{\substack{P\in C(\bar{k})\\ h_{C,D}(P)\to\infty}} \frac{h_{C,E}(P)}{h_{C,D}(P)} = \frac{\deg E}{\deg D}.$$

Masser uses his counting theorem and gets a lower bound of the canonical height on elliptic curves over number field. (see [Ma]) The counting theorem can be generalized to the case of abelian varieties.

Let k be a number field, let g_2, g_3 be elements of k with $g_2^3 \neq 27g_3^2$, and let E be the elliptic curve defined by

$$y^2 = 4x^3 - g_2x - g_3.$$

We write $q(P)$ for the associated canonical height

$$\widehat{h}(P) = \lim_{n\to\infty} \frac{1}{4^n} h([2^n]P).$$

Let $\varpi \geq 1$ be an upper bound for the absolute logarithmic height of the point in projective space with coordinates 1, g_2, g_3, one has

Theorem 3.10 (Masser)**.** *There is a positive effective constant C, depending only on the degree of $k/\mathbf{Q}$, such that for any $D \geq 1$ and any extension K of k of degree at most D, the number of points P in $E(K)$ with*

$$\widehat{h}(P) < \frac{1}{CD}$$

is at most $C\sqrt{\varpi}D(\varpi + \log D)$.

One of the consequences of the theorem is the following.

Corollary 3.11. *There is a positive effective constant C_1 depending only on the degree of k, such that*

$$\widehat{h}(P) \geq \frac{1}{C_1 \varpi D^3 (\varpi + \log D)^2}$$

for all non-torsion pints P in $E(K)$.

When we consider the height on elliptic curves we often use the canonical height for its nice properties. For the difference between Weil height and the canonical height on the elliptic curve E, one has the following.

Theorem 3.12.

$$|h(P) - \widehat{h}(P)| \leqslant C\varpi$$

for all $P \in E(k)$, where C is a absolute constant.

The proof of this result can be found in [Zi].

In the case of Galois extensions, there are better estimations of heights in the Lehmer problem which are due to Francesco Amoroso and David Masser (see [AM]).

Theorem 3.13. *For any integer $r \geqslant 1$ and any $\varepsilon > 0$ there is a positive effective constant $c(r, \varepsilon)$ with the following property. Let $\mathbf{F}/\mathbf{Q}$ be a Galois extension of degree D and $\alpha \in \mathbf{F}^{\times}$. We assume that there are r conjugates of α over $\mathbf{Q}$ that are multiplicatively independent. Then*

$$h() \geqslant c(r, \varepsilon) D^{\frac{-1}{r+1} - \varepsilon}.$$

For the canonical height on elliptic curves, there are also some better results in the Galois case.

Theorem 3.14. *For all $P \in E(\bar{k})$ of infinite order such that $k(P)/k$ is Galois of degree D, one has*

$$\widehat{h}(P) \geqslant \frac{C}{D}$$

for some positive constant C.

The proof can be found in [GM]

Remark 3.15. We introduced the height theory in the case of number fields. We can also consider the height theory in the case of function fields, see [La3]. Many works on elliptic curves can be generalized to the case of abelian varieties. For Masser's counting theorem. The abelian variety version of it is also due to Masser, see [Ma2].

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