Lyapunov exponents in wireless comunication

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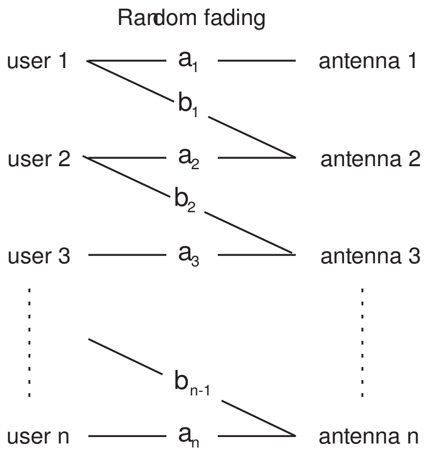
16 novembre 2007

1 Results

We consider the following communication model, n users are communicating with n antennas in such a way that user i communicates with antennas i and $i+1$ with a random fading a_i and b_i respectively.



We describe the problem in the following formal way. We consider two random sequences of complex numbers (a_n) and (b_n). The (a_n) (resp. (b_n)) are i.i.d of law π_a (resp. π_b) and the (a_n) are independent of the (b_n). We set $\Omega := ((a_n), (b_n))$. We denote by $\mathbb{P}$ the probability associated with those random sequences and by $\mathbb{E}$ the associated expectation. For a given integer n, we consider a channel H of size $n \times (n+1)$.

$$A_n = \begin{pmatrix} a_1 & b_1 & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & a_n & b_n \end{pmatrix}.$$

We consider the capacity

$$Cap_n(\rho) = \frac{1}{n} \operatorname{tr} \left\{ \log \left(I + \rho A_n A_n^* \right) \right\}.$$

Note that,

$$A_n A_n^* = \begin{pmatrix} |a_1|^2 + |b_1|^2 & \overline{a_2} b_1 & 0 & \cdots & 0 \\ a_2 \overline{b_1} & |a_2|^2 + |b_2|^2 & \overline{a_3} b_2 & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & \overline{a_n} b_{n-1} \\ 0 & \cdots & 0 & a_n \overline{b_{n-1}} & |a_n|^2 + |b_n|^2 \end{pmatrix}.$$

The results obtained so far use the following hypothesis

(H1) $\mathbb{E}_{\pi_a} (\log |x|)^2 < \infty$ and $\mathbb{E}_{\pi_b} (\log |x|)^2 < \infty$.

(H2) π_a and π_b are absolutely continuous with respect to Lebesgue measure on $\mathbb{C}$.

(H3) There exist a real M such that if x is distributed according to π_a (resp. π_b) then the density of $|x|^2$ is strictly positive on the interval $[M; \infty)$.

(H3') There exist $m_a < M_a \in \mathbb{R} \cup \{\infty\}$ (resp. $m_b < M_b \in \mathbb{R} \cup \{\infty\}$) such that if x is distributed according to π_a (resp. π_b) then the density of $|x|^2$ and the Lebesgue-measure on $[m_a; M_a]$ (resp. $[m_b; M_b]$) are mutually absolutely continuous.

(H4) $\mathbb{E}_{\pi_a} \log |x| \leq \mathbb{E}_{\pi_b} \log |x|$.

The principal result obtained so far is the following theorem.

Theorem 1.1. *Assume (H1) and (H2)*

a) For every $\rho > 0$, $Cap_n(\rho)$ converges as n goes to infinity. We call the limit $Cap(\rho)$.

b) Further assume [(H3) or (H3')]. As ρ goes to infinity,

$$Cap(\rho) = \log \rho + 2 \max \left(\mathbb{E}_{\pi_a} \log |x| \, ; \mathbb{E}_{\pi_b} \log |x| \right) + o(1).$$

Note that Theorem 1.1 continues to be true in the real set up, that is, if instead of (H2), we assume

(H2') π_a and π_b are supported on $\mathbb{R}$ and are absolutely continuous with respect to Lebesgue measure on $\mathbb{R}$.

Since the argument is identical, we do not discuss this case further.
Without loss of generality, we can assume (H4), indeed, exchanging the entries a_i and b_i for $1 \leq i \leq n$, we substitute $\overline{A_n A_n^*}$ to $A_n A_n^*$ and the eigenvalues, which are real, do not change, hence the capacity does not change.
Note that only (H1) and (H2) are needed for part a) of Theorem 1.1. The proof of this part uses only the theory of products of random matrices and we give it in Section 5.
Part b) uses the theory of Markov chains and is specific to the particular matrix A_n, as a by product of this proof, we obtain a second proof of part a), however under the additional assumption [(H3) or (H3')].
The structure of the paper is as follows. In the development of the paper, we give a proof assuming [(H3) or (H3')]. In Section 2, we introduce an auxiliary sequence which allows us to reformulate the problem in terms of certain random sequences. In Section 3 we reduce the problem to the analysis of a special Markov chain, whose study we carry out in Section 4.

2 Auxiliary sequence

We begin with a technical lemma that will not be proved here.

Lemma 2.1. *Assume (H2). $\mathbb{P}$-a.s, $A_nA_n^*$ does not have multiple eigenvalues.*

In the sequel, we denote by $\lambda_1, \ldots, \lambda_n$ the ordered eigenvalues of $A_nA_n^*$. For a given λ, we consider the following sequence of complex numbers (the dependence in λ will only be mentioned when it is relevant) : $x_1 = 1$, $x_2 = x_1 \frac{\lambda - |a_1|^2 - |b_1|^2}{\overline{a_2} b_1}$, and for $n \geq 2$,

$$a_n \overline{b_{n-1}} x_{n-1} + (|a_n|^2 + |b_n|^2) x_n + \overline{a_{n+1}} b_n x_{n+1} = \lambda x_n,$$

that is

$$x_{n+1} = \frac{\lambda - |a_n|^2 - |b_n|^2}{\overline{a_{n+1}} b_n} x_n - \frac{a_n \overline{b_{n-1}}}{\overline{a_{n+1}} b_n} x_{n-1}. \tag{2.1}$$

Note that $x_{n+1}(\lambda) = 0$ if and only if λ is an eigenvalue of $A_nA_n^*$. Moreover, x_{n+1} is a polynomial in λ of degree n with highest coefficient $1/\prod_{i=1}^{n}(\overline{a_{i+1}} b_i)$. One can thus write using Lemma 2.1

$$x_{n+1}(\lambda) = \prod_{i=1}^{n} (\overline{a_{i+1}} b_i)^{-1} \prod_{i=1}^{n} (\lambda - \lambda_i) \qquad \mathbb{P} - \text{a.s,}$$

Hence, $\mathbb{P}$-a.s,

$$Cap_n(\rho) = \log(\rho) + \frac{1}{n} \log |x_{n+1}(\lambda)| + \frac{1}{n} \sum_{i=1}^{n} \log |a_{i+1} b_i|, \tag{2.2}$$

where $\lambda = -1/\rho$. By the Law of Large Numbers (LLN),

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \log |a_{i+1} b_i| = \mathbb{E}_{\pi_a} \log |x| + \mathbb{E}_{\pi_b} \log |x| \qquad \mathbb{P} - \text{a.s.}$$

Because of (2.2), to prove part b) of Theorem 1.1, we only need to show the following lemma.

Lemma 2.2. *Assume (H1), (H2) and [(H3) or (H3')]*

a) For every $\lambda < 0$, $\frac{1}{n} \log |x_{n+1}(\lambda)|$ converges $\mathbb{P}$-a.s as n goes to infinity. The limit is $\gamma(\lambda)$, the Lyapunov exponent defined by (5.1).

b) Assume further (H4). Then $\gamma(\lambda)$ converges $\mathbb{P}$-a.s to $\mathbb{E}_{\pi_b} \log |x| - \mathbb{E}_{\pi_a} \log |x|$ as λ goes to 0.

3 Reduction to a Markov chain

We take $c_n := x_n/x_{n-1}$, for $n \geq 3$. Note that by (2.1) and (H2), $\mathbb{P}$-a.s, $x_n \neq 0$, hence c_n is well defined and non-zero. By (2.1), we get

$$c_{n+1} = \frac{\lambda - |a_n|^2 - |b_n|^2}{\overline{a_{n+1}} b_n} - \frac{a_n \overline{b_{n-1}}}{c_n \overline{a_{n+1}} b_n}.$$

Let $d_n = c_n \overline{a_n} b_{n-1}$. Then,

$$d_{n+1} = \lambda - |a_n|^2 - |b_n|^2 - \frac{|a_n|^2 |b_{n-1}|^2}{d_n} = \lambda - |b_n|^2 - |a_n|^2 \left(1 + \frac{|b_{n-1}|^2}{d_n}\right).$$

Let $e_n = \left(1 + \frac{|b_{n-1}|^2}{d_n}\right)$. Then $d_{n+1} = \lambda - |b_n|^2 - |a_n|^2 e_n$, and

$$e_n = \frac{-\lambda + |a_{n-1}|^2 e_{n-1}}{-\lambda + |b_{n-1}|^2 + |a_{n-1}|^2 e_{n-1}}, \tag{3.1}$$

with the initial conditions,

$$c_3 = \frac{\lambda - |a_2|^2 - |b_2|^2}{\overline{a_3} b_2} - \frac{|a_2|^2 |b_1|^2}{\overline{a_3} b_2 (\lambda - |a_1|^2 - |b_1|^2)};$$

$$d_3 = \lambda - |b_2|^2 - |a_2|^2 \left(1 - \frac{|b_1|^2}{-\lambda + |a_1|^2 + |b_1|^2}\right).$$

$d_3 \in \mathbb{R}$ and $d_3 < -|b_2|^2$, hence, $0 < e_3 < 1$. From (3.1) we conclude that for all n, $e_n \in \mathbb{R}$ and $0 < e_n < 1$. Now, for all n,

$$c_n = \frac{d_n}{\overline{a_n} b_{n-1}} = \frac{\overline{b_{n-1}}}{\overline{a_n}} \frac{1}{e_n - 1}.$$

Then,

$$\begin{aligned} \frac{1}{n} \log |x_{n+1}| &= \frac{1}{n} \sum_{i=3}^{n+1} \log |c_i| + \frac{1}{n} \log |x_2| \\ &= \frac{1}{n} \sum_{i=3}^{n+1} \left(\log \left| \frac{b_{i-1}}{a_i} \right| - \log(1 - e_i) \right) + \frac{1}{n} \log |x_2| \end{aligned} \tag{3.2}$$

$\frac{1}{n} \sum_{i=3}^{n+1} \log \left| \frac{b_{i-1}}{a_i} \right|$ converges to $\mathbb{E}_{\pi_b} \log |x| - \mathbb{E}_{\pi_a} \log |x|$ by the LLN. We now study in details the Markov chain e_n.

4 Study of the Markov chain e_n

For simplicity, we write $\delta := -\lambda$ and we re-index the chain so that it starts from e_0. As in (3.1),

$$e_n = \frac{\delta + |a_{n-1}|^2 e_{n-1}}{\delta + |b_{n-1}|^2 + |a_{n-1}|^2 e_{n-1}}. \tag{4.1}$$

We denote by f_n the random function s.t. $e_n = f_n(e_{n-1})$. We denote by $\mathbb{P}_{e_0}$ the law of the sequence starting from e_0 and by $\mathbb{E}_{e_0}$ the associated expectation.

Proposition 4.1. *Assume (H2) and [(H3) or (H3')]. The Markov chain e_n has a unique stationary probability, say, μ_δ and for $s \in L^1(\mu_\delta)$, for every starting point $e_0 \in [0, 1]$, $\mathbb{P}_{e_0}$-a.s,*

$$\frac{1}{n} \sum_{i=0}^{n} s(e_i) \xrightarrow[n \to \infty]{} \int s d\mu_\delta.$$

The proof is omitted. We also omit the proof of the following fact,

$$-\log(1-\cdot) \in L^1(\mu_\delta). \tag{4.2}$$

With Proposition 4.1, we get

$$\frac{1}{n}\sum_{k=3}^{n+1} -\log(1-e_k) \xrightarrow[n\to\infty]{} \int_0^1 -\log(1-x)d\mu_\delta(x) \qquad \mathbb{P}_{e_3} - \text{a.s.} \tag{4.3}$$

With (3.2), it gives another proof of Lemma 2.2 a). Let us prove Lemma 2.2 b). Take $\eta > 0$ and $\varepsilon > 0$ small.

$$\begin{aligned}
&\int_0^1 -\log(1-x)d\mu_\delta(x) \\
&= \int_0^\varepsilon -\log(1-x)d\mu_\delta(x) + \int_\varepsilon^{1-\eta} -\log(1-x)d\mu_\delta(x) + \int_{1-\eta}^1 -\log(1-x)d\mu_\delta(x) \\
&\leq -\varepsilon\log(1-\varepsilon) - \log\eta\mu_\delta([\varepsilon,1]) + \int_{1-\eta}^1 -\log(1-x)d\mu_\delta(x).
\end{aligned} \tag{4.4}$$

By (4.2), the last term converges to 0 as η goes to 0. By (3.2), (4.3) and (4.4), to prove Lemma 2.2 b), we only have to prove that for any given $\varepsilon > 0$,

$$\mu_\delta([\varepsilon,1]) \xrightarrow[\delta\to 0]{} 0.$$

For that, by Proposition 4.1, we need to show that the proportion of the time that the chain e_n spends above ε converges to 0 as δ goes to infinity. We take $\varepsilon < \varepsilon_0 < 1$, where ε_0 will be chosen later. We consider the Markov chain $z_n := \log e_n$ and the random function g_n such that $z_n = g_n(z_{n-1})$. It is enough to show that the proportion of the time that z_n spends above $\log\varepsilon$ goes to 0 as δ goes to 0. Let us couple z_n with another Markov chain w_n, such that $w_n \geq z_n$ a.s. and that the time that w_n spends above $\log\varepsilon$ goes to 0 as δ goes to 0.
For that, we need good information on the jumps of z_n.

Lemma 4.2. *Assume (H1) and (H4). Set*

$$\begin{aligned}
j_n(z_{n-1}) &:= z_n - z_{n-1} \\
&= \log\left(\frac{\delta}{e^{z_{n-1}}} + |a_{n-1}|^2\right) - \log\left(\delta + |b_{n-1}|^2 + |a_{n-1}|^2 e^{z_{n-1}}\right).
\end{aligned}$$

$(\forall\delta > 0)\,(\exists\varepsilon' > 0)\,(\forall x \geq \log\varepsilon')$

a) $\mathbb{E}j_n(x) \leq 0$,

b) $\mathbf{V}ar j_n(x) \leq V := \mathbb{E}\left(\left(\log(|a_{n-1}|^2 + |b_{n-1}|^2)\right)^2 + \left(\log(|a_{n-1}|^2)\right)^2\right) + C.$

C is a constant independent of everything. ε' is a function of δ but we will not write it to keep the notation clear. Moreover,

$$\lim_{\delta\to 0}\varepsilon' = 0.$$

The proof is omitted. We continue with the proof of Lemma 2.2 b)
We take $\delta > 0$ such that $0 < \varepsilon' < \varepsilon < \varepsilon_0 < 1$. We define w_n in a way that it stays between $\log \varepsilon'$ and 0. Set $w_0 = z_0$, for δ small enough, $w_0 > \log \varepsilon'$. For $x \in [\log \varepsilon'; 0]$, denote

$$h_n(x) = g_n(x) - \mathbb{E} j_n(x) \geq g_n(x).$$

That is

$$\begin{aligned} h_n(x) = x + \log \left(\frac{\frac{\delta}{e^x} + |a_{n-1}|^2}{\delta + |b_{n-1}|^2 + |a_{n-1}|^2 e^x} \right) - \\ \mathbb{E} \log \left(\frac{\frac{\delta}{e^x} + |a_{n-1}|^2}{\delta + |b_{n-1}|^2 + |a_{n-1}|^2 e^x} \right). \end{aligned} \tag{4.5}$$

Notice that

$$\mathbb{E}(h_n(z_{n-1}) - z_{n-1} | z_{n-1}) = 0. \tag{4.6}$$

– If $h_n(w_{n-1}) > 0$, set $w_n = 0$.
– If $h_n(w_{n-1}) < \log \varepsilon'$, set $w_n = \log \varepsilon'$.
– Otherwise, set $w_n = h_n(w_{n-1})$.
In the first two case, we say that the chain is *truncated*. Note that for all n, $w_n \geq z_n$. Indeed, either $w_n = 0 \geq z_n$ or $w_n \geq h_n(w_{n-1}) \geq g_n(w_{n-1}) \geq g_n(z_{n-1}) = z_n$, by induction and using the fact that g_n is a.s. non-decreasing. Therefore, the proportion of the time that the chain w_n spends above $\log \varepsilon$ is larger that the proportion of the time that chain z_n spends above $\log \varepsilon$.

Proposition 4.3. *Assume (H2).*

a) The Markov chain w_n has a unique stationary probability, say, ν_δ and for $s \in L^1(\nu_\delta)$, for every starting point $w_0 \in [\log \varepsilon', 0]$, $\mathbb{P}_{w_0}$-a.s,

$$\frac{1}{n} \sum_{i=0}^{n} s(w_i) \xrightarrow[n \to \infty]{} \int s d\nu_\delta.$$

b) We denote T the return time to 0, starting from 0. Then $\nu_\delta(0) = 1/\mathbb{E}_0 T$.

The proof is omitted. We continue with the proof of Lemma 2.2 b).
By Proposition 4.3 a), to prove that the proportion of the time that w_n spends above $\log \varepsilon$ goes to 0 as δ goes to 0, we only need to prove that

$$\nu_\delta([\log \varepsilon, 0]) \xrightarrow[\delta \to 0]{} 0.$$

Let us first prove that $\mathbb{E} T \xrightarrow[\delta \to 0]{} \infty$, which by Proposition 4.3 b) will prove that

$$\nu_\delta(0) \xrightarrow[\delta \to 0]{} 0.$$

We use the following lemma.

Lemma 4.4. *Assume (H2).*

a) There exist $u > 0$ and $\alpha > 0$ dependent on ε and independent of δ such that for all $x \in [2\log\varepsilon; 0]$,

$$\mathbb{P}\left(h_n(x) \geq x + u\right) > \alpha.$$

b) There exist $v > 0$ and $\beta > 0$ dependent on ε and independent of δ such that

$$\mathbb{P}\left(\log\varepsilon < h_1(0) < -v\right) > \beta.$$

The proof is omitted. We continue with the proof of Lemma 2.2 b).
We denote $\mathcal{A}$ the event $\log\varepsilon < h_1(0) < -v$. On $\mathcal{A}$, we define the stopping time

$$\tilde{T} = 1 + \inf\{n \geq 1; h_{n+1}(w_n) > 0 \text{ or } h_{n+1}(w_n) < \log\varepsilon'\}.$$

By martingale arguments, we get

$$\mathbb{E}(\tilde{T}) \geq \beta\frac{v(-\log\varepsilon') - (\log\varepsilon)^2}{V}.$$

We have proved that $\mathbb{E}T \xrightarrow[\delta\to 0]{} \infty$, which proves that $\nu_\delta(0) \xrightarrow[\delta\to 0]{} 0$.
Using Lemma 4.4, let us prove by induction that for $\mathcal{N} \leq \left\lceil \frac{-\log\varepsilon}{u} \right\rceil$,

$$\nu_\delta\left([-\mathcal{N}u; 0]\right) \leq \alpha^{-\mathcal{N}}\nu_\delta(0).$$

$$\begin{aligned}
\nu_\delta([-(\mathcal{N}-1)u;0]) &\geq \int \nu_\delta(dw_0)\mathbb{P}_{w_0}(w_1 \in [-(\mathcal{N}-1)u;0]) \\
&\geq \int_{[-\mathcal{N}u;0]} \nu_\delta(dw_0)\mathbb{P}_{w_0}(w_1 \in [-(\mathcal{N}-1)u;0]) \\
&\geq \int_{[-\mathcal{N}u;0]} \nu_\delta(dw_0)\mathbb{P}_{w_0}(h_1(w_0) \geq u + w_0) \\
&\geq \alpha\nu_\delta([-\mathcal{N}u;0]).
\end{aligned}$$

Therefore,

$$\nu_\delta\left([\log\varepsilon; 0]\right) \leq \alpha^{\left\lceil \frac{-\log\varepsilon}{u} \right\rceil}\nu_\delta(0).$$

So,

$$\nu_\delta([\log\varepsilon, 0]) \xrightarrow[\delta\to 0]{} 0.$$

That concludes the proof of Lemma 2.2 b).

5 Product of random matrices

We prove Lemma 2.2 b) assuming (H1) and (H2). We use the theory of product of random matrices theory. It will be proved again in Section 4. For a general introduction to the aspects of the theory we use here, the reader may consult [5], [6], [8], [9] and [14].
Let us take $|\cdot|$ any norm on $\mathbb{C}^2$ and $\|\cdot\|$ the associated operator norm on $M_2(\mathbb{C})$. For a given λ,

$$\begin{pmatrix} x_{n+1} \\ x_n \end{pmatrix} = \begin{pmatrix} \frac{\lambda - |a_n|^2 - |b_n|^2}{\overline{a_{n+1}} b_n} & -\frac{a_n \overline{b_{n-1}}}{\overline{a_{n+1}} b_n} \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x_n \\ x_{n-1} \end{pmatrix}$$

For $a, a', b, b' \in \mathbb{C}$
0, we define

$$g(\lambda, a, a', b, b') := \begin{pmatrix} \frac{\lambda - |a|^2 - |b'|^2}{\overline{a'} b'} & -\frac{a\overline{b}}{\overline{a'} b'} \\ 1 & 0 \end{pmatrix} \in GL(2, \mathbb{C}).$$

Finally, we define

$$g_n(\lambda) := g(\lambda, a_n, a_{n-1}, b_{n-1}, b_n) = \begin{pmatrix} \frac{\lambda - |a_n|^2 - |b_n|^2}{\overline{a_{n+1}} b_n} & -\frac{a_n \overline{b_{n-1}}}{\overline{a_{n+1}} b_n} \\ 1 & 0 \end{pmatrix},$$

$$M_n := g_n \dots g_2.$$

Set $\mathcal{E} = (\mathbb{C} - 0)^4$ which is a borel set of a separable and complete metric space. $X_n := (a_{n+1}, a_n, b_n, b_{n-1})$ is a Markov chain on $\mathcal{E}$, with invariant measure $\Pi := \pi_a \times \pi_a \times \pi_b \times \pi_b$. With (H1),

$$\mathbb{E}_\Pi \left(\log^+ \| g(\lambda, a, a', b, b') \| + \log^+ \left\| g(\lambda, a, a', b, b')^{-1} \right\| \right) < \infty.$$

Notice that $g_n(\lambda)$ is a continuous function of X_n, therefore $((X_n, M_n), \Pi)$ is a multiplicative Markovian process. By [3, Proposition 2.2], $1/n \log \|M_n(\lambda)\|$ converges $\mathbb{P}$-almost surely and in $L_1(\Omega)$, we set

$$\gamma(\lambda) = \lim_{n \to \infty} \frac{1}{n} \log \|M_n(\lambda)\|. \tag{5.1}$$

$\gamma(\lambda)$ is the first Lyapunov exponent. The $L_1(\Omega)$ convergence already gives an easy upper bound for $\gamma(\lambda)$. By the property of operator norm,

$$\gamma(\lambda) \leq \mathbb{E}_\pi \log \|g_2(\lambda)\|.$$

Moreover, we can refine that bound into a whole family of upper bounds, for $k \in \mathbb{N}$,

$$\gamma(\lambda) \leq \frac{1}{k} \mathbb{E}_\pi \log \|g_2(\lambda) ... g_{k+1}(\lambda)\|. \tag{5.2}$$

Note that this upper bound is getting better as k increases and tight as $k \to \infty$. Let us now prove that

$$\frac{1}{n} \log |x_{n+1}(\lambda)| \xrightarrow[n \to \infty]{} \gamma(\lambda).$$

Definition 5.1. *The multiplicative system* $((X_n, M_n), \Pi)$ *is irreducible if there is no measurable non-random family* $\{V(X), X \in E\}$ *of proper subspaces of* $\mathbb{C}^2$ *s.t.*

$$M_n V(X_0) = V(X_n), \qquad \mathbb{P}\text{-}a.s., \ \forall n \in \mathbb{N}.$$

Lemma 5.2. *The multiplicative system* $((X_n, M_n), \Pi)$ *is irreducible*

The proof is an adaptation of the proof of [3, Proposition 6.1.1], it is omitted. By [4, Lemma 2.6], irreducibility implies that

$$\lim_{n \to \infty} \frac{1}{n} \log \left| \begin{pmatrix} x_{n+2} \\ x_{n+1} \end{pmatrix} \right| = \gamma.$$

The following lemma completes the proof.

Lemma 5.3.

$$\lim_{n\to\infty} \frac{1}{n}\left(\log\left|\begin{pmatrix} x_{n+2} \\ x_{n+1} \end{pmatrix}\right| - \log|x_{n+1}|\right) = 0.$$

The proof is omitted.

6 Further questions

6.1 Continuity results

In [6], there are several results of continuity of Lyapunov exponents for Schrödinger operators. We would like to use those results to get a proof of Theorem 1.1 without the assumption (H3). Nevertheless, there are two main obstacles. [6] deals with random Schrödinger which is a slightly different model, most of all because the off-diagonal coefficients of $A_n A_n^*$ are 1 in that case. The most important obstacle is that the continuity requires irreducibility, among other conditions. Unfortunately, the multiplicative system $((X_n, M_n), \Pi)$ is not irreducible for $\lambda = 0$, which is precisely the point at which we need irreducibility. Indeed, $(-\overline{b_{n-1}}/\overline{a_n}, 1)$ is a fixed direction. Since we have irreducibility for all $\lambda \neq 0$, we may get a continuity result. There exists a counter example that shows that a weak form of irreducibility for all $\lambda \neq 0$ is not enough for a continuity theorem [12]. Because of that, we will have to use the full extent of our hypothesis.

6.2 Higher dimension

Our result deals with A_n bidiagonal, that is $A_n A_n^*$ tridiagonal. We would like to get a result for A_n tridiagonal, imagine for example a model where every user i would communicate with antennas $i-1$, i and $i+1$. For that case, we first need to derive a formula like (2.2) and then, we need to study the resulting chain. The problem is that the chain x_n will no longer be one dimensional, therefore the reduction to a Markov chain will be much more complicated. We should be able to get existence results by tools coming from the products of random matrices theory but the behavior in high-SNR will be much more difficult to study.
Finally, the long term goal is to get results for general k-diagonal matrices.

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