ALMOST SURE INVARIANCE PRINCIPLE FOR THE DOUBLING MAP

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Abstract. In this paper we prove the almost sure invariance principle for processes generated by the dynamical system $x \mapsto 2x \mod 1$. The proof relies on two steps; in the first step we divide the Birkhoff sum of the process into blocks and show that the sum over these blocks can be well approximated by a martingale. In the second step we use Skorokhod representation theorem to approximate this martingale by a Brownian motion. We also talk about some of the applications of the almost sure invariance principle.

1. Preliminaries

Definition 1.1 (Banach space V_α). For $\varphi \in L^1(m)$ and $0 < \alpha \leq 1$, we define

$$|\varphi|_\alpha = \sup_{\epsilon>0} \frac{1}{\epsilon^\alpha} \int_0^1 \text{ess sup}_{y_1,y_2\in(x-\epsilon,x+\epsilon)} |\varphi(y_1) - \varphi(y_2)| dx \,.$$

The space V_α consists of all $\varphi \in L^1(m)$ such that $|\varphi|_\alpha < \infty$. On V_α we define the norm

$$\|\varphi\|_\alpha = |\varphi|_\alpha + \|\varphi\|_{L^1} \,.$$

If φ is an $\alpha-$Hölder function, $\alpha \leq 1$, then there exists a constant C such that $\forall x, y \in \mathbb{R}$ $|\varphi(x) - \varphi(y)| \leq C|x-y|^\alpha$. Hence, it follows immediately that V_α contains all α-Hölder functions. Further, by [K] (cf. also [S]), the space V_α together with the norm $\|.\|_\alpha$ is a Banach space. For simplicity, we restrict our considerations to Hölder continuous functions.

Theorem 1.2 (Exponential decay of correlation). *There exists a real number* $0<\lambda<1$ *and a constant* C *such that for all* $g \in V_\alpha$ *and all* $h \in L^1(m)$

$$\left|\int_0^1 g(x)h(f^n(x))dx - \int_0^1 g(x)dx \int_0^1 h(x)dx\right| \leq C\|g\|_\alpha \|h\|_{L^1} \lambda^n \quad \forall n \geq 1$$

Proof. See, e.g., [AFLV, Appendix C.3 and C.4]. □

Let

$$\sigma(\varphi)^2 := \int_0^1 \varphi^2 + 2\sum_{n\geq1} \int_0^1 \varphi(x)\varphi(f^n(x))dx.$$

If φ is a α-Hölder function and $\int_0^1 \varphi = 0$ then, by theorem 1.2, we have $\sigma(\varphi)^2<\infty$. Furthermore, we can write

$$\sigma(\varphi)^2 = \lim_{n\to\infty} \frac{1}{n} \int_0^1 (\sum_{i=0}^n \varphi(f^i(x)))^2 dx,$$

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from which follows that $\sigma(\varphi)^2 \geq 0$. If $\sigma(\varphi)^2 = 0$, then φ is called a *co-boundary*. Henceforth, we will always assume that $\sigma(\varphi)^2 > 0$.

Remark 1.3. We can show that if $\int_0^1 \varphi \neq 0$, then the series $\sum_{n\geq 1}\int_0^1 \varphi(x)\varphi(f^n(x))dx$ is divergent. So from now we assume that $\int_0^1 \varphi = 0$

Example 1.4. *If we take $\varphi(x) = \cos(2\pi x)$ then it is easy to calculate $\sigma(\varphi)^2 = \frac{1}{2}$.*

We consider the function

$$f(x) = 2x \; mod \; 1,$$

which is also called the *doubling map*.
We also use the big O-notation. We say $f = O(g)$ if there exists a constant C such that $|f| \leq Cg$. Define

$$\xi_i(x) = \varphi(f^i(x)), \quad i \geq 1.$$

Definition 1.5. We say that a sequence of random variables η_i, $i \geq 1$, on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ satisfies an *almost sure invariance principle* with error exponent $\gamma < 1/2$ and variance σ>0. If there exist a sequence of random variables $\widetilde{\eta}_i$, $i \geq 1$, and a Brownian motion $B(t)$, $t \geq 0$, on an appropriate probability space $(\widetilde{\Omega}, \widetilde{\mathcal{F}}, \widetilde{\mathbb{P}})$ such that

(i) $\{\eta_i\}_{i\geq 1}$ and $\{\widetilde{\eta}_i\}_{i\geq 1}$ have the same distribution;
(ii) $\widetilde{\mathbb{P}}$ almost surely as $n \to \infty$,

$$\left| B(\sigma n) - \sum_{i=1}^{n} \widetilde{\eta}_i \right| = O(n^\gamma)\,.$$

In this article, we will prove that:

Theorem 1.6. *ξ_i satisfies the almost sure invariance principle for all error exponents $\gamma > \frac{2}{5}$ and variance $\sigma^2(\varphi)$.*

Remark 1.7. We get an error exponent $\gamma > \frac{2}{5}$. We can imporve the error exponent to γ>$\frac{1}{3}$ using similar techniques. We only have to put smaller blocks of logarithm size or of very small polynomial size between the large blocks. By applying a technique by Gouëzel [G], the error exponent could be improved to $\gamma > \frac{1}{4}$.

The almost sure invariance principle is a very strong property. It implies many other laws, e.g., the central limit theorem and the law of iterated logarithm.

Corollary 1.8. *If $\xi_i, i \geq 1$, satisfy an almost sure invariance principle then $\frac{1}{\sqrt{n}}\sum_{i=1}^n \xi_i$ converges in distribution to $\mathcal{N}(0, \sigma^2)$ and*

$$\limsup_{n\to\infty} \frac{1}{\sqrt{2n\log\log n}} \sum_{i=1}^{n} \xi_i(x) = \sigma, \quad for \; \mathbb{P} - a.e. \; x,$$

where σ^2 is the variance of the related Brownian motion.
In other word, the sequence ξ_i satisfies the central limit theorem and the law of iterated logarithm.

Proof. See [PS]. □

Example 1.9. *As the following example by Erdös and Fortet shows, we have to be careful when leaving the setting of dynamical systems:*
If $\varphi(x) = \cos(2\pi x) + \cos(4\pi x)$*, then the sequence* $\xi_i = \varphi((2^i - 1)x \mod 1)$ *does not satisfy the central limit theorem (and hence not either the almost sure invariance principle).*

Suppose that r_k is the largest integer such that

$$2^{r_k} \le k^{2/\alpha} 2^k, \quad k \ge 1,$$

and let $\mathcal{F}_k$ be the σ-field generated by all intervals of the form

$$U_{i,k} = [i2^{-r_k}, (i+1)2^{-r_k}), \quad 0 \le i \le 2^{r_k} - 1, \quad k \ge 1.$$

We define

$$\chi_k = \mathbb{E}[\xi_k | \mathcal{F}_k], \quad k \ge 1.$$

Lemma 1.10. *There exists a constant* C *such that* $|\chi_k - \xi_k| \le Ck^{-2}$ *for all* $x \in [0,1), k \ge 1$.

Proof. Let $x \in U_{i,k}$,

$$\begin{aligned}
|\chi_k(x) - \xi_k(x)| &= \frac{1}{|U_{i,k}|} | \int_{U_{i,k}} \varphi(f^k(y)) - \varphi(f^k(x)) dy| \\
&\le \frac{1}{|U_{i,k}|} \int_{U_{i,k}} |\varphi(f^k(y)) - \varphi(f^k(x))| dy \\
&\le \frac{C}{|U_{i,k}|} \int_{U_{i,k}} |f^k(y) - f^k(x)|^\alpha dy \le C \sup_{x,y \in U_{i,k}} |f^k(y) - f^k(x)|^\alpha \\
&\le C(2^{-r_k} 2^k)^\alpha \le C(\frac{1}{k^{2/\alpha}})^\alpha = \frac{C}{k^2}.
\end{aligned}$$

□

Observe that Lemma 1.3 implies that

$$\sum_{k=1}^{\infty} |\chi_k - \xi_k| \le C \sum_{k=1}^{\infty} k^{-2} = O(1).$$

Lemma 1.11. *We have for* $2^j \ge 2k^{2/\alpha}$ *and* $k \ge 0$

$$\mathbb{E}[\xi_{k+j} | \mathcal{F}_k] = \mathbb{E}[\chi_{k+j} | \mathcal{F}_k] = 0.$$

Proof. Since $2^j \ge 2k^{2/\alpha}$, it follows that $2^{k+j} \ge 2^{r_k}$, i.e. $k + j \ge r_k$. Let $x \in U_{i,k}$, hence

$$\begin{aligned}
\mathbb{E}[\mathbb{E}[\xi_{k+j} | \mathcal{F}_{k+j}] \mathcal{F}_k](x) = \mathbb{E}[\xi_{k+j} | \mathcal{F}_k](x) &= \frac{1}{|U_{i,k}|} \int_{U_{i,k}} \varphi(f^{k+j}(y)) dy \\
&= \int_0^1 \varphi(f^{k+j-r_k}(y)) dy = 0.
\end{aligned}$$

And

$$\mathbb{E}[\chi_{k+j} | \mathcal{F}_k] = \mathbb{E}[\mathbb{E}[\xi_{k+j} | \mathcal{F}_{k+j}] \mathcal{F}_k].$$

Therefore we have

$$\mathbb{E}[\xi_{k+j} | \mathcal{F}_k] = \mathbb{E}[\chi_{k+j} | \mathcal{F}_k] = 0.$$

□

Fix $0 < \kappa < 1$. Let $I_j, j \geq 1$, be the block of integers containing $[j^\kappa]$ consecutive positive integers and such that there are no gaps between consecutive block.

Set $h_j := \min\{\nu \mid \nu \in I_j\}$, so that $h_j = \sum_{\nu=1}^{j-1}[\nu^\kappa] + 1$.

By integration, we obtain that

$$h_j = O(j^{1+\kappa}).$$

We define the blocks

$$y_j := \sum_{\nu \in I_j} \chi_\nu \textit{ and } \omega_j := \sum_{\nu \in I_j} \xi_\nu.$$

Let M_N be the index of y_j containing χ_N, and we use M short for M_N.

Then we have

$$C^{-1}M^{1+\kappa} \leq N \leq CM^{1+\kappa}.$$

Lemma 1.12. *There exists a constant C such that $|y_j^2 - \omega_j^2| \leq Cj^{-2}$ for all $x \in [0, 1)$.*

Proof. Since $|\xi_\nu| = |\varphi(f^\nu(x))| \leq \|\varphi\|_\infty$ and $|\chi_\nu| = |\mathbb{E}[\xi_\nu|\mathcal{F}_\nu]| \leq \|\xi_\nu\|_\infty \leq \|\varphi\|_\infty$, we have

$$|y_j + \omega_j| = |\sum_{\nu \in I_j} \chi_\nu + \xi_\nu| \leq \sum_{\nu \in I_j} (|\chi_\nu| + |\xi_\nu|) \leq 2|I_j|\|\varphi\|_\infty \leq 2j^\kappa\|\varphi\|_\infty.$$

Due to Lemma 1.3

$$|y_j^2 - \omega_j^2| = |y_j - \omega_j||y_j + \omega_j| \leq 2j^\kappa\|\varphi\|_\infty \sum_{\nu \in I_j} |\chi_\nu - \xi_\nu| \leq 2j^\kappa\|\varphi\|_\infty \sum_{\nu \in I_j} \nu^{-2}$$
$$\leq 2j^\kappa\|\varphi\|_\infty|I_j|h_j^{-2} \leq 2Cj^\kappa\|\varphi\|_\infty j^\kappa j^{-2-2\kappa} = 2C\|\varphi\|_\infty j^{-2}.$$

□

2. Main Estimate

In this section we prove a law of large numbers for the random processes y_i. This allows us to approximate this processes by a martingales in the next section.

Lemma 2.1. *For all δ>0 there exists a constant C such that $\int_0^1 w_j^4 \leq C|I_j|^{2+\delta}$.*

Proof. We begin by expanding the sum and use the inequality $|\{(v_1, v_2, v_3, v_4) \in S^4 : v_1 \leq v_2 \leq v_3 \leq v_4, v_2 - v_1 \leq |S|^\delta \;\&\; v_4 - v_3 \leq |S|^\delta\}| \leq |S|^{2+\delta}$. So

$$\int_0^1 w_j^4 = \sum_{v_1,v_2,v_3,v_4 \in I_j} \int_0^1 \xi_{v_1}\xi_{v_2}\xi_{v_3}\xi_{v_4}$$
$$\leq 4 \sum_{\substack{v_1 \leq v_2 \leq v_3 \leq v_4 \in I_j \\ v_2 - v_1 \geq |I_j|^\delta \text{ or } v_4 - v_3 \geq |I_j|^\delta}} |\int_0^1 \xi_{v_1}\xi_{v_2}\xi_{v_3}\xi_{v_4}| + 4\|\varphi\|_\infty^4|I_j|^{2+\delta}.$$

Since the Lebesgue measure is f invariant, it follows that $|\int_0^1 \xi_{v_1}\xi_{v_2}\xi_{v_3}\xi_{v_4}| = |\int_0^1 \varphi(x)\varphi(f^{v_2-v_1}x)\varphi(f^{v_3-v_1}x)\varphi(f^{v_4-v_1}x)dx|$. If $v_2 - v_1 \geq |I_j|^\delta$ then by using theorem 1.2, we deduce

$$\begin{aligned}
&\left|\int_0^1 \varphi(x)\varphi(f^{v_2-v_1}(x))\varphi(f^{v_3-v_1}(x))\varphi(f^{v_4-v_1}(x))\right| \\
&\qquad \leq \left|\int_0^1 \varphi(x)dx \int_0^1 \varphi(x)\varphi(f^{v_3-v_2}(x))\varphi(f^{v_4-v_2}(x))dx\right| \\
&\qquad\quad + C \cdot \|\varphi\|_\alpha \|\varphi\varphi \circ f^{v_3-v_2}\varphi \circ f^{v_4-v_2}\|_\infty \lambda^{v_2-v_1} \\
&\qquad = O(\lambda^{v_2-v_1}) = O(\lambda^{I_j^\delta}),
\end{aligned}$$

where we used that $\int_0^1 \varphi = 0$. If $v_4 - v_3 \geq |I_j|^\delta$ then let $r = v_3 - v_1$, so we have

$$\int_0^1 \varphi \cdot \varphi \circ f^{v_2-v_1} \cdot \varphi \circ f^r \cdot \varphi \circ f^{r+v_4-v_3} = \sum_{i=0}^{2^r-1} \int_{\frac{i}{2^r}}^{\frac{i+1}{2^r}} \varphi \cdot \varphi \circ f^{v_2-v_1} \cdot \varphi \circ f^r \cdot \varphi \circ f^{r+v_4-v_3}.$$

The function f^r is invertible on the interval $[\frac{i}{2^r}, \frac{i+1}{2^r}]$ so there exist functions $g_i : [0,1] \to [\frac{i}{2^r}, \frac{i+1}{2^r}]$ such that $g_i \circ f^r = id, f^r \circ g_i = id$. So by a change of variables and theorem 1.2

$$\begin{aligned}
&\sum_{i=0}^{2^r-1} \int_{\frac{i}{2^r}}^{\frac{i+1}{2^r}} \varphi \cdot \varphi \circ f^{v_2-v_1} \cdot \varphi \circ f^r \cdot \varphi \circ f^{r+v_4-v_3} \\
&= \sum_{i=0}^{2^r-1} \frac{1}{2^r} \int_0^1 \varphi \circ g_i \cdot \varphi \circ f^{v_2-v_1} \circ g_i \cdot \varphi \cdot \varphi \circ f^{v_4-v_3} \\
&\leq C\lambda^{I_j^\delta} \sup_i \|\varphi \circ g_i \cdot \varphi \circ f^{v_2-v_1} \circ g_i \cdot \varphi\|_\alpha.
\end{aligned}$$

Observe that g_i and $f^{v_2-v_1} \circ g_i$ are contractions. Hence using the definition of the norm $\|.\|_\alpha$ one can easily see that $\|\varphi \circ g_i \cdot \varphi \circ f^{v_2-v_1} \circ g_i \cdot \varphi\|_\alpha$ is bounded uniformly in i, j. Since $|I_j|^4 \lambda^{|I_j|^\delta} \to 0$ as $j \to \infty$ and $\|\varphi \circ g_i \cdot \varphi \circ f^{v_2-v_1} \circ g_i \cdot \varphi\|_\alpha$ is bounded. we get $\int_0^1 w_j^4 = O(|I_j|^4 \lambda^{|I_j|^\delta}) + O(|I_j|^{2+\delta}) = O(|I_j|^{2+\delta})$. □

Lemma 2.2. *There exists a constant C such that*

$$\left|\int_0^1 w_j^2 dx - \sigma(\varphi)^2 |I_j|\right| \leq C, \qquad \forall j \geq 1.$$

Proof.

$$\begin{aligned}
&\sigma(\varphi)^2|I_j| - \int_0^1 w_j^2 dx \\
&\quad = \sigma(\varphi)^2|I_j| - \sum_{k,l\in I_j}\int_0^1 \xi_k\xi_l dx \\
&\quad = \sigma(\varphi)^2|I_j| - \sum_{k\in I_j}\int_0^1 \xi_k\xi_k dx - \sum_{k,l\in I_j, k\neq l}\int_0^1 \xi_k\xi_l dx \\
&\quad = \sigma(\varphi)^2|I_j| - |I_j|\int_0^1 \varphi^2 dx - 2\sum_{0<i\leq |I_j|-1}(|I_j|-i)\int_0^1 \varphi(x)\varphi(f^i(x))dx \\
&\quad = 2\sum_{i=1}^{|I_j|-1}(i-|I_j|)\int_0^1 \varphi(x)\varphi(f^i(x))dx + 2|I_j|\sum_{n\geq 1}\int_0^1 \varphi(x)\varphi(f^i(x))dx \\
&\quad = 2\sum_{i=1}^{|I_j|-1} i\int_0^1 \varphi(x)\varphi(f^i(x))dx + 2|I_j|\sum_{i=|I_j|}^{\infty}\int_0^1 \varphi(x)\varphi(f^i(x))dx.
\end{aligned}$$

Using theorem 1.2 we deduce that the first term is a convergent series and the second term is bounded as $j\to\infty$. □

Theorem 2.3 (Gal-Koksma Strong Law of Large Numbers)**.** *Let $\{Z_n\}$ be a sequence of random variables such that $\forall n\geq 1$ $E(Z_n)=0,\ E(Z_n^2)<\infty$. Suppose that there exists constants $\sigma, C>0$ such that for all integers $n>0,\ m\geq 0$*

$$E\left(\sum_{j=m}^{m+n} Z_j\right)^2 \leq C((m+n)^\sigma - m^\sigma).$$

Then for each $\delta\geq 0$ and a.e $\omega\in\Omega$

$$\left|\sum_{j\leq N} Z_j\right| = O(N^{\frac{1}{2}\sigma}\log^{2+\delta} N)$$

Proof. See [PS, Theorem A1] □

Lemma 2.4.

$$\left|\int_0^1 w_i^2 w_j^2 - \int_0^1 w_i^2\int_0^1 w_j^2\right| = O(i^{-2}j^\kappa)\quad \forall i<j-1$$

.

Proof. We claim that for j big enough we can find k such that y_i is F_k-measurable and $2^r\int_{s2^{-r}}^{(s+1)2^{-r}} w_j^2 dx = \int_0^1 w_j^2$ for $r=r_k$. If $n<m\in\mathbb{N}$ then using the change of variables formula we have $2^n\int_{s2^{-n}}^{(s+1)2^{-n}}\varphi\circ f^m dx = 2^n\int_{s2^{-n}}^{(s+1)2^{-n}}\varphi\circ f^{m-n}\circ f^n dx = \int_0^1\varphi\circ f^m, \forall 0\leq s\leq 2^n-1$. So if $r<h_j$ then $2^r\int_{s2^{-r}}^{(s+1)2^{-r}} w_j^2 dx = \int_0^1 w_j^2$. Since y_i is $F_{h_{i+1}}$ measurable we need to make sure that $r_{h_{i+1}}<h_j$. since $r_{h_{i+1}}\leq r_{h_{j-1}}\leq h_{j-1}+\frac{2\log h_{j-1}}{\alpha\log 2},\ h_j = O(j^{1+\kappa})$, it follows that $r_{h_{i+1}}\leq r_{h_{j-1}}<h_j$ for j big enough.

Hence we can choose $k = h_{i+1}$. To prove the lemma we have

$$\begin{aligned}\int_0^1 (y_i^2 - \int_0^1 y_i^2)(w_j^2 - \int_0^1 w_j^2)dx \\ &= \sum_{s=0}^{s=2^r-1} \int_{s2^{-r}}^{(s+1)2^{-r}} (y_i^2 - \int_0^1 y_i^2)(w_j^2 - \int_0^1 w_j^2)dx \\ &= \sum_{s=0}^{s=2^r-1} 2^r \int_{s2^{-r}}^{(s+1)2^{-r}} (y_i^2 - \int_0^1 y_i^2)dx \int_{s2^{-r}}^{(s+1)2^{-r}} (w_j^2 - \int_0^1 w_j^2) \\ &= \sum_{s=0}^{s=2^r-1} \int_{s2^{-r}}^{(s+1)2^{-r}} (y_i^2 - \int_0^1 y_i^2)dx (2^r \int_{s2^{-r}}^{(s+1)2^{-r}} w_j^2 dx - \int_0^1 w_j^2) = 0.\end{aligned}$$

Hence,

$$\begin{aligned}|\int_0^1 (w_i^2 - \int_0^1 w_i^2)(w_j^2 - \int_0^1 w_j^2)dx| \\ &\leq |\int_0^1 (y_i^2 - \int_0^1 y_i^2)(w_j^2 - \int_0^1 w_j^2)dx| \\ &\quad + |\int_0^1 (w_i^2 - \int_0^1 w_i^2 - y_i^2 + \int_0^1 y_i^2)(w_j^2 - \int_0^1 w_j^2)dx| \\ &\leq 2 \sup_{x\in[0,1]} |y_i^2(x) - w_i^2(x)| \int_0^1 |w_j^2 - \int_0^1 w_j^2| dx = O(i^{-2} j^{\kappa}).\end{aligned}$$

In the last inequality we used lemma 2.2 and lemma 1.6. □

Remark 2.5. To use the change of variables formula we extend φ to the real line with period 1 and then $\varphi(2^m x \ mod\ 1) = \varphi(2^m x)$.

Proposition 2.6 (Main Estimate)**.** *For each* $\delta>0$ *and for a.e* $x \in [0,1]$, *there exists a constant* C *such that*

$$|\sigma^2 N - \sum_{j=1}^{M(N)} y_j^2| \leq C N^{\max(1-\frac{1}{2+2\kappa}, \frac{1}{1+\kappa})+\delta}, \qquad \forall N \geq 1.$$

Where $M = M(N), C^{-1} M^{1+\kappa} \leq N \leq C M^{1+\kappa}$.

Proof. By using Cauchy-Schwartz inequality and lemma 2.1, we have

$$\int_0^1 w_{j-1}^2 w_j^2 \leq \sqrt{\int_0^1 w_{j-1}^4} \sqrt{\int_0^1 w_j^4} \leq C |I_{j-1}|^{\frac{2+\frac{\delta}{\kappa}}{2}} |I_j|^{\frac{2+\frac{\delta}{\kappa}}{2}} = O(|I_j|^{2+\frac{\delta}{\kappa}}) = O(j^{2\kappa+\delta}).$$

Hence, by lemma 2.4 and for some constant C, we have

$$\begin{aligned}\sum_{m\leq i\leq j\leq n+m} \left| \int_0^1 w_i^2 w_j^2 - \int_0^1 w_i^2 \int_0^1 w_j^2 dx \right| &\leq C \sum_{j=m}^{n+m} j^{2\kappa+\delta} \\ &= O((n+m)^{1+2\kappa+\delta} - m^{1+2\kappa+\delta}).\end{aligned}$$

So we apply theorem 2.3 with $Z_i = w_i^2 - E(w_i^2)$ and $\sigma = 2\kappa + \delta + 1$ which implies for a.e $x \in [0,1]$

$$\left| \sum_{j \le M(N)} w_j^2 - Ew_j^2 \right| = O(M(N)^{\frac{1+2\kappa+\delta}{2}} \log^{2+\delta} M(N)).$$

By lemma 2.2 and lemma 1.6

$$\left| \sigma(\varphi)^2 N + \sum_{j=1}^{M(N)} (w_j^2 - y_j^2 - Ew_j^2) \right| \le \sum_{j=1}^{\infty} j^{-2} + O(M(N)).$$

To finish the proof we note that $M(N) = O(N^{\frac{1}{1+\kappa}})$, and we conclude,

$$\left| \sum_{j \le M(N)} y_j^2 - \sigma(\varphi)^2 N \right| = O(N^{1-\frac{1-\delta}{2+2\kappa}} \log^{2+\delta} N) + O(N^{\frac{1}{1+\kappa}}).$$

□

3. The martingale representation

Lemma 3.1. *Let $\{y_j\}_{j=1}^{\infty}$ be an arbitrary sequence of random variables and let $\{\mathcal{L}_j\}_{j=0}^{\infty}$ be a nondecreasing sequence of σ-fields such that y_j is $\mathcal{L}_j$-measurable($\mathcal{L}_0$ is the trivial σ-field). Suppose that*

$$\sum_{k=0}^{\infty} \mathbb{E}|\mathbb{E}[y_{j+k}|\mathcal{L}_j]| < \infty$$

for each $j \ge 1$. Then for each $j \ge 1$

$$y_j = Y_j + u_j - u_{j+1}$$

where $\{Y_j\}_{j=1}^{\infty}$ is a $\{\mathcal{L}_j\}_{j=1}^{\infty}$ martingale difference sequence and

$$u_j = \sum_{k=0}^{\infty} \mathbb{E}[y_{j+k}|\mathcal{L}_{j-1}].$$

Proof. See [PS, Lemma 2.1]. □

Lemma 3.2. *Let $\mathcal{L}_j$ be the σ-field generated by $(y_1, y_2, ..., y_j)$. Then we can represent y_j by*

$$y_j = Y_j + u_j - u_{j+1}$$

where $\{Y_j\}_{j=1}^{\infty}$ is a $\{\mathcal{L}_j\}_{j=1}^{\infty}$ martingale difference sequence and

$$|u_j| = O(\log j) \ a.s.$$

Proof. Firstly, we will proof that $\sum_{k=0}^{\infty} \mathbb{E}|\mathbb{E}[y_{j+k}|\mathcal{L}_j]| < \infty$, then we can use Lemma 3.1 to represent y_j.
Since

$$y_j = \sum_{\nu \in I_j} \chi_\nu = \sum_{\nu \in I_j} \mathbb{E}[\xi_\nu|\mathcal{F}_\nu],$$

then y_j is $\mathcal{F}_{h_{j+1}-1}$ measurable. Due to lemma 1.11, when k is great enough, i.e. $2^{h_{j+k}-h_{j+1}+1} \ge 2(h_{j+1}-1)^{2/\alpha}$, we have

$$\mathbb{E}[\chi_\nu|\mathcal{F}_{h_{j+1}-1}] = 0, \quad \nu \in I_{j+k}.$$

And we obtain

$$\mathbb{E}[y_{j+k}|\mathcal{L}_j] = \sum_{\nu\in I_{j+k}} \mathbb{E}[\chi_\nu|\mathcal{L}_j] = \sum_{\nu\in I_{j+k}} \mathbb{E}[\mathbb{E}[\chi_\nu|\mathcal{F}_{h_{j+1}-1}]|\mathcal{L}_j] = 0.$$

Then $\sum_{k=0}^{\infty} \mathbb{E}|\mathbb{E}[y_{j+k}|\mathcal{L}_j]|$ just has finitely many items, so the sum is finite.
Let J be the smallest number that satisfies $2^J \geq 2h_j^{2/\alpha}$. Then we have $J \leq 2 + (2/(\alpha \log 2)) \log h_j$, i.e. $J = O(\log h_j) = O(\log j)$. For u_j defined in Lemma 3.1 we get

$$\begin{aligned}|u_j| &= \sum_{k=0}^{\infty} |\mathbb{E}[y_{j+k}|\mathcal{L}_{j-1}]| \leq \sum_{k=0}^{\infty} \sum_{\nu\in I_{j+k}} |\mathbb{E}[\mathbb{E}[\chi_\nu|\mathcal{F}_\nu]|\mathcal{L}_{j-1}]| = \sum_{\nu\geq h_j} |\mathbb{E}[\mathbb{E}[\chi_\nu|\mathcal{F}_\nu]|\mathcal{L}_{j-1}]| \\ &= \sum_{\nu=h_j}^{h_j+J-1} |\mathbb{E}[\mathbb{E}[\chi_\nu|\mathcal{F}_\nu]|\mathcal{L}_{j-1}]| + \sum_{\nu\geq h_j+J} |\mathbb{E}[\mathbb{E}[\chi_\nu|\mathcal{F}_\nu]|\mathcal{L}_{j-1}]| \\ &= \sum_{\nu=h_j}^{h_j+J-1} |\mathbb{E}[\mathbb{E}[\chi_\nu|\mathcal{F}_\nu]|\mathcal{L}_{j-1}]| \leq J\|\varphi\|_\infty = O(\log j).\end{aligned}$$

□

Set $v_j = u_j - u_{j-1}$. By Lemma 3.2 we have

$$|v_j| = O(\log j) \ a.s.$$

and $y_j = Y_j + v_j$.

Lemma 3.3. *If* $\gamma > \max(\frac{2+\kappa}{2+2\kappa}, \frac{1}{1+\kappa})$, *then*

$$|\sum_{j=1}^{M} Y_j^2 - \sigma^2 N| = O(N^\gamma) \ a.s. \quad as \quad N \to \infty.$$

Proof. We have

$$Y_j^2 = (y_j - v_j)^2 = y_j^2 - 2y_j v_j + v_j^2.$$

For each δ we find C such that

$$\sum_{j=1}^{M} v_j^2 \leq C \sum_{j=1}^{M} \log^2 j \leq CM \log^2 M = O(N^{\frac{1}{1+\kappa}} \log^2 N) = O(N^{\frac{1}{1+\kappa}+\delta}).$$

Using Cauchy's inequality and Proposition 2.6. $\forall\delta$ and *a.e.* $x \in [0, 1)$ we can find C so that

$$|\sum_{j=1}^{M} 2y_j v_j| \leq (\sum_{j=1}^{M} y_j^2)^{1/2} (\sum_{j=1}^{M} v_j^2)^{1/2} \leq CN^{1/2} N^{\frac{1}{2+2\kappa}+\delta} \leq CN^{\frac{2+\kappa}{2+2\kappa}+\delta}.$$

Hence, we have

$$|\sum_{j=1}^{M} y_j^2 - Y_j^2| \leq \sum_{j=1}^{M} (|2y_j v_j| + |v_j^2|) \leq CN^\gamma.$$

Then by Proposition 2.6, we get the result. □

Lemma 3.4. *For each* $\gamma > \frac{1+2\kappa}{2+2\kappa}$,

$$|\sum_{j=1}^{M} \mathbb{E}[Y_j^2|\mathcal{L}_{j-1}] - Y_j^2| = O(N^\gamma).$$

Proof. Let $R_j = \mathbb{E}[Y_j^2|\mathcal{L}_{j-1}] - Y_j^2$. Since $\mathbb{E}[R_j|\mathcal{L}_{j-1}] = 0$, then R_j is a $\mathcal{L}_{j-1}$ martingale difference sequence. Using Lemma 2.1, we obtain $\forall\delta$

$$\begin{aligned}
\mathbb{E}[R_j^2] &= \mathbb{E}[Y_j^4] - \mathbb{E}[\mathbb{E}^2[Y_j^2|\mathcal{L}_{j-1}]] \leq \mathbb{E}[Y_j^4] = \mathbb{E}[(y_j + v_j)^4] \\
&\leq \mathbb{E}[y_j^4] + \mathbb{E}[v_j^4] \leq \mathbb{E}[\omega_j^4] + \mathbb{E}[|\omega_j^4 - y_j^4|] + \mathbb{E}[v_j^4] \\
&\leq C|I_j|^{2+\delta} + \mathbb{E}[|\omega_j^2 - y_j^2||\omega_j^2 + y_j^2|] + C\log^4 j \\
&\leq Cj^{\kappa(2+\delta)} + \mathbb{E}[Cj^{-2}(|\omega_j^2| + |y_j^2|)] + Cj^\delta \\
&\leq Cj^{2\kappa+\delta} + \mathbb{E}[Cj^{-2}|I_j|^2] + Cj^\delta \\
&\leq Cj^{2\kappa+\delta} + Cj^{-2}j^{2\kappa} + Cj^\delta = Cj^{2\kappa+\delta} + Cj^{2\kappa-2} + Cj^\delta \leq Cj^{2\kappa+\delta}.
\end{aligned}$$

Then

$$\sum_{j=1}^{\infty} j^{-1-2\kappa-2\delta}\mathbb{E}[R_j^2] \leq \sum_{j=1}^{\infty} j^{-1-2\delta} < \infty,$$

Since R_j is a martingale difference sequence, we conclude (see e.g. [C])

$$\sum_{j=1}^{\infty} j^{-\frac{1+2\kappa}{2}-\delta} R_j$$

converges a.s. By Kronecker's lemma, for a.e. $x \in [0,1)$ we find a constant C such that

$$\sum_{j=1}^{\infty} R_j(x) \leq CM^{\frac{1+2\kappa}{2}+\delta} \leq CN^{\frac{1+2\kappa}{2+2\kappa}+\delta}.$$

□

4. Skorokhod representation theorem

We now apply Skorokhod representation theorem.

Theorem 4.1 (Skorokhod representation theorem)**.** *Let* $\{Y_i\}_{i=i}^\infty$ *be a sequence of random variables on a probability space* $(\Omega, \mathcal{F}, \mathbb{P})$ *satisfying:*

(i) $\mathbb{E}[Y_i^2] < \infty$

(ii) $\mathbb{E}[Y_i|\sigma(Y_1, ...Y_{i-1})] = 0$ $\mathbb{P} - a.s.$ *for all* $i \geq 1$.

Then there exists a sequence of random variable $\{\widetilde{Y}_i\}_{i=1}^\infty$ *and a Brownian motion* $\{B(t)\}_{i\in[0,\infty)}$ *together with a sequence of nonnegative random variable* $\{T_i\}_{i=i}^\infty$ *on an appropriate probability space* $(\widetilde{\Omega}, \widetilde{\mathcal{F}}, \widetilde{\mathbb{P}})$ *with the following properties.*

(1) $\{Y_i\}_{i=1}^\infty$ *and* $\{\widetilde{Y}_i\}_{i=1}^\infty$ *have the same distribution.*

(2)
$$\sum_{i=1}^{n}\widetilde{Y}_i = B(\sum_{i=1}^{n} T_i), \qquad \widetilde{\mathbb{P}} - a.s.\ for\ any\ n \in \mathbb{N}.$$

(3) *T_n is $\tilde{\mathcal{F}}_n$-measurable and*
$$\mathbb{E}[T_n|\widetilde{\mathcal{F}}_{n-1}] = \mathbb{E}[\tilde{Y_n}^2|\widetilde{\mathcal{F}}_{n-1}]\ , \quad \widetilde{\mathbb{P}} - a.s.\ n = 1, 2, 3...,$$
where $\widetilde{\mathcal{F}}_0 = \phi, \widetilde{\Omega}$ and $\widetilde{\mathcal{F}}_n$ is defined as the σ algebra generated by $\tilde{Y_1}, \tilde{Y_2}, ..., \tilde{Y_n}$ and $\{B(t)\}_{0 \le t \le \sum_{i=1}^{n} T_i}$ for $n \ge 1$.

Proof. See [HH, Theorem A.1]. □

In what follows, we will use the same notation as in the statement of Skorokhod representation theorem. But we will skip the $\sim$ in the notation. So $\mathcal{L}_n \subset \mathcal{F}_n$ and we have
$$E(T_j|\mathcal{F}_{j-1}) = E(Y_j^2|\mathcal{F}_{j-1}) = E(Y_j^2|\mathcal{L}_{j-1})a.s$$

Lemma 4.2. *If*
$$\gamma > \max(\frac{2+\kappa}{2+2\kappa}, \frac{1}{1+\kappa})$$
then
$$\sum_{j=1}^{M(N)} T_j - \sigma(\varphi)^2 N = O(N^\gamma)\ \ a.s.$$

Proof. Using Skorokhod representation theorem, we have
$$\sum_{i=0}^{M} T_i - \sigma(\varphi)^2 N$$
$$= \sum_{i=0}^{M}[T_i - E(T_i|\mathcal{F}_{j-1})] + \sum_{i=0}^{M}[E(Y_j^2|\mathcal{L}_{j-1}) - Y_j^2] + \sum_{i=0}^{M} Y_j^2 - \sigma(\varphi)^2 N \quad a.s.$$
The last two terms are bounded by $O(N^\gamma)$ because of Lemma 3.3, 3.4. Write $R_j = T_j - E(T_i|\mathcal{F}_{j-1})$ then we can see that this is a martingale difference sequence satisfying $ER_j^2 = O(EY_j^4)$. So doing a similar proof that was used in lemma 3.3 we deduce that the first term is also bounded by $O(N^\gamma)$. □

We will define the two new random processes by
$$S(t) = \sum_{k \le t} \xi_k, \quad \text{and} \quad S^* = \sum_{k=1}^{M([t])} Y_k.$$

Lemma 4.3. *$S(t) - S^*(t) = O(t^{\frac{\kappa}{\kappa+1}})$ a.s.*

Proof. By the definition of $S(t), S^*(t)$ we have
$$S(t) - S^*(t) = \sum_{k \le t}(\xi_k - \chi_k) - \sum_{t<k<h_{M+1}} \chi_k + \sum_{k=1}^{M(t)} y_k - Y_k.$$
By lemma 1.3 the first term is bounded. The second term contains at most $M[t]^\kappa$ terms hence it is bounded by $\|\varphi\|_\infty t^{\frac{\kappa}{\kappa+1}}$. The last sum is equal to $u_1 - u_{M+1}$ which is equal to $O(\log t)$ using lemma 3.3, 3.4. □

Theorem 4.4. *Let* $\gamma > \max(\frac{2+\kappa}{2+2\kappa}, \frac{1}{1+\kappa})$*. Then for each* $\delta>0$ *we have*

$$S^*(t) - B(\sigma^2(\varphi)t) = O(t^{\frac{\gamma}{2}+\delta}) \text{ a.s.}$$

Proof. Let $\sigma^2 = \sigma^2(\varphi)$ and $P_n = n^{\delta + \frac{1}{1-\gamma}}$. Then by Skorokhod representation theorem

$$\max_{P_n \le t \le P_{n+1}} \left| S^*(t) - B(\sigma^2 t) \right| = \max_{P_n \le t \le P_{n+1}} \left| B(\sum_{k=1}^{M([t])} T_j) - B(\sigma^2 t) \right|. \tag{1}$$

By lemma 4.2 and the mean value theorem we have for $P_n \le t \le P_{n+1}$, n large enough

$$\sigma^2 P_{n-1} \le \sigma^2 P_n + O(P_{n+1}^\gamma) \le \sum_{k=1}^{M([t])} T_k \le \sigma^2 P_{n+1} + O(P_{n+1}^\gamma) \le \sigma^2 P_{n+2} \quad \text{a.s.} \tag{2}$$

For $a \le b$, let

$$R(a,b) = \max_{a \le s,t \le b} |B(s) - B(t)|.$$

So for n large enough using (1),(2), we obtain

$$\max_{P_n \le t \le P_{n+1}} \left| S^*(t) - B(\sigma^2 t) \right| \le R(\sigma^2 P_{n-1}, \sigma^2 P_{n+2}) \quad \text{a.s.}$$

Now, again using the mean value theorem and the basic properties of Brownian motion we get for n large enough

$$\mathbb{P}\left(R(\sigma^2 P_{n-1}, \sigma^2 P_{n+2}) \ge P_n^{\frac{1}{2}(\gamma+\delta)} \right) = \mathbb{P}\left(R(0,1) \ge \left(\frac{P_n^{\frac{1}{2}(\gamma+\delta)}}{\sigma^2(P_{n+2} - P_{n-1})} \right)^{\frac{1}{2}} \right)$$

We will denote $\left(\frac{P_n^{\frac{1}{2}(\gamma+\delta)}}{\sigma^2 P_{n+2} - \sigma^2 P_{n-1}} \right)^{\frac{1}{2}}$ by a to simplify the next equations. So since $R(0,1) \le \max_{0 \le t \le 1} |B(t)|$. Hence we can continue the chain of inequalities by $\mathbb{P}(R(0,1) \ge a) \le \mathbb{P}\left(\max_{0 \le t \le 1} |B(t)| \ge \frac{1}{2} a \right)$. Using Levy's theorem on Brownian motion we get

$$\mathbb{P}\left(R(\sigma^2 P_{n-1}, \sigma^2 P_{n+2}) \ge P_n^{\frac{1}{2}(\gamma+\delta)} \right) \le \mathbb{P}\left(|B(1)| \ge \frac{1}{2} a \right) = O(\exp(-n^c)),$$

where $c \le 1$ is a constant. So by Borel Cantelli lemma it follows that a.s.

$$R(\sigma^2 P_{n-1}, \sigma^2 P_{n+2}) \ge P_n^{\frac{1}{2}(\gamma+\delta)}$$

happens for finitely many n only. So this proves the lemma. □

It follows from lemma 4.3 and 4.4 that we need to minimize $\max(\frac{2+\kappa}{4+4\kappa}, \frac{1}{2+2\kappa}, \frac{\kappa}{1+\kappa})$. The minimum value is at $\kappa = \frac{2}{3}$. Lemma 4.3 says that for $\delta>0$, we have $S^*(t) - B(\sigma^2(\varphi)t) = O(t^{\frac{2}{5}+\delta})$ and using lemma 4.3. We deduce that

$$S(t) - B(\sigma^2(\varphi)t) = O(t^{\frac{2}{5}+\delta}) \qquad \forall \delta>0.$$

This concludes the proof of theorem 1.6.

Remark 4.5. We can also deduce a slightly better estimate by doing double block partitioning and using similar arguments like above. Where each two consecutive blocks, the first is of size $[j^\kappa]$ and the second block is of size $\log j$. This gives us less correlation between big blocks which gives better estimates. Recall remark 1.7.

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