Distribution of Gaussian multiplicative chaos on the unit interval

Tunan Zhu

(Joint work with Guillaume Remy)

École Normale Supérieure

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Log-correlated field

Log-correlated Gaussian field X on $[0,1]$:

$$\mathbb{E}[X(x)X(y)] = 2\ln\frac{1}{|x-y|}.$$

- X lives in the space of distributions.
- Regularization procedure X_δ.
  $X_\delta = \rho_\delta * X, \quad \rho_\delta = \frac{1}{\delta}\rho(\frac{\cdot}{\delta}),$ with smooth ρ.
- $X(x) = 2\sqrt{\ln 2}\,\alpha_0 + \sum_{n=1}^{\infty}\frac{2\alpha_n}{\sqrt{n}}T_n(2x-1)$, with α_i i.i.d $\sim \mathcal{N}(0,1)$ and T_n Chebyshev polynomials.

Gaussian multiplicative chaos (GMC)

For $\gamma \in (0,2)$, define the GMC measure $e^{\frac{\gamma}{2}X}dx$ on $[0,1]$ by the limiting procedure $e^{\frac{\gamma}{2}X_\delta - \frac{\gamma^2}{8}\mathbb{E}[X_\delta^2]}\,dx$ as $\delta \to 0$.

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- GMC on $[0, 1]$:

  Differential equations for Selberg integrals by Kaneko (1993)

  Conjecture and computations of GMC on the interval by physicists: Fyodorov-Le Doussal-Rosso, Ostrovsky (2009)

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- GMC on other geometries with Liouville field theory:

  DOZZ formula proved by Kupiainen-Rhodes-Vargas (2017)

  Fyodorov-Bouchaud formula proved by Remy (2017)

Main result

Theorem (Remy and Z., 2018)

For $0 < \gamma < 2$ *and* $a, b > -1 - \frac{\gamma^2}{4}$,

$$\int_0^1 x^a (1-x)^b e^{\frac{\gamma}{2} X(x)} dx \stackrel{law}{=} c\, e^{\mathcal{N}(0, \gamma^2 \ln 2)} Y_\gamma X_1^{-1} X_2^{-1} X_3^{-1}$$

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- $c := 2\pi 2^{-(3(1+\frac{\gamma^2}{4})+2(a+b))}$
  $Y_\gamma \sim \frac{1}{\Gamma(1-\frac{\gamma^2}{4})} \mathrm{Exp}(1)^{-\frac{\gamma^2}{4}}$ (cf. Fyodorov-Bouchaud formula)
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  $X_i :=$ generalized beta law.
- We actually compute $\mathbb{E}[(\int_0^1 x^a (1-x)^b e^{\frac{\gamma}{2} X(x)} dx)^p]$ for all $p < \frac{4}{\gamma^2} \wedge (1 + \frac{4}{\gamma^2}(1+a)) \wedge (1 + \frac{4}{\gamma^2}(1+b))$.

Applications

- Tail expansion (Rhodes-Vargas, 2017):

$$\mathbb{E}[X(x)X(y)] = 2\ln\frac{1}{|x-y|} + g(x,y)$$

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with g some continuous bounded function.
For $a \in (-1-\frac{\gamma^2}{4}, -\frac{\gamma^2}{4})$ and some $\nu > 0$,

$$\mathbb{P}\Big(\int_0^\eta x^a e^{\frac{\gamma}{2}X(x)}dx > t\Big) \overset{t\to\infty}{=} \frac{e^{\frac{\gamma^2}{8}(1+\frac{4}{\gamma^2}(1+a))^2 g(0,0)} R(\gamma, a)}{t^{1+\frac{4}{\gamma^2}(1+a)}} + O\Big(\frac{1}{t^{1+\frac{4}{\gamma^2}(1+a)+\nu}}\Big).$$

For $a \geq -\frac{\gamma^2}{4}$, the asymptotic behavior is $\asymp t^{-\frac{4}{\gamma^2}}$.

- Critical GMC (Aru-Powell-Sepúlveda 2018):

$$M' := \lim_{\gamma \to 2} \frac{1}{2-\gamma} \int_0^1 e^{\frac{\gamma}{2} X(x)} dx \stackrel{\text{law}}{=} \frac{\pi}{32} e^{\mathcal{N}(0,\gamma^2 \ln 2)} Y_1 Y_2 Y_3,$$

where Y_1^{-1} = generalized beta,
$Y_2 = \frac{2}{y^3} dy \, (y > 1)$, $Y_3 = \frac{1}{y^2} e^{-1/y} dy \, (y > 0)$.

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- Max of X (Madaule, Ding-Roy-Zeitouni 2016):

$$\max_{x \in [0,1]} X_\delta(x) - 2 \ln \frac{1}{\delta} + \frac{3}{2} \ln \ln \frac{1}{\delta} \xrightarrow{\delta \to 0} \mathcal{G}_1 + \ln M' + cst$$

$$\overset{\text{law}}{=} \mathcal{G}_1 + \mathcal{G}_2 + \mathcal{N}(0, 4 \ln 2) + \ln Y_1 + \ln Y_2 + \tilde{cst}$$

where $\mathcal{G}_1, \mathcal{G}_2$ are independent Gumbel laws ($\ln Y_3 = \mathcal{G}_2$), and $\tilde{cst}$ depends only on the regularization.

Idea of proof (I)

For $t \in (-\infty, 0)$, introduce the auxiliary functions:

- $U(t) = \mathbb{E}[(\int_0^1 (x-t)^{\frac{\gamma^2}{4}} x^a (1-x)^b e^{\frac{\gamma}{2}X(x)} dx)^p]$
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$U(t)$, $\tilde{U}(t)$ solutions to hypergeometric equations

$$t(1-t)F''(t) + (C - (A+B+1)t)F'(t) - ABF(t) = 0,$$

with parameters

$$A = -\frac{p\gamma^2}{4}, B = -(a+b+1) - (2-p)\frac{\gamma^2}{4}, C = -a - \frac{\gamma^2}{4}.$$
$$\tilde{A} = -p, \tilde{B} = -\frac{4}{\gamma^2}(a+b+2) + p - 1, \tilde{C} = -\frac{4}{\gamma^2}(a+1).$$

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$\diamond$ Technical difficulties since we go beyond the Lebesgue integrable bound: $a, b > -1 - \frac{\gamma^2}{4}$.

- We want to compute $M(p,a,b) := \mathbb{E}[(\int_0^1 x^a (1-x)^b e^{\frac{\gamma}{2}X(x)} dx)^p]$.

We will need four shift equations:

$$\frac{M(p, a+\frac{\gamma^2}{4}, b)}{M(p,a,b)} = \frac{\Gamma(1+a+\frac{\gamma^2}{4})\Gamma(2+a+b-(2p-2)\frac{\gamma^2}{4})}{\Gamma(1+a-(p-1)\frac{\gamma^2}{4})\Gamma(2+a+b-(p-2)\frac{\gamma^2}{4})} \quad (1)$$

$$\frac{M(p, a+1, b)}{M(p,a,b)} = \frac{\Gamma(\frac{4}{\gamma^2}(1+a)+1)\Gamma(\frac{4}{\gamma^2}(2+a+b)-(2p-2))}{\Gamma(\frac{4}{\gamma^2}(1+a)-(p-1))\Gamma(\frac{4}{\gamma^2}(2+a+b)-(p-2))} \quad (2)$$

Remark: $M(p,a,b) = M(p,b,a)$.

- We want to compute $M(p,a,b) := \mathbb{E}[(\int_0^1 x^a(1-x)^b e^{\frac{\gamma}{2}X(x)}dx)^p]$.

$$\frac{M(p,a,0)}{M(p-1,a,0)} = \frac{\Gamma(1-\frac{p\gamma^2}{4})\Gamma(1-(p-1)\frac{\gamma^2}{4})}{\Gamma(1-\frac{\gamma^2}{4})} \times \frac{\Gamma(1+a-(p-1)\frac{\gamma^2}{4})\Gamma(2+a-(p-2)\frac{\gamma^2}{4})}{\Gamma(2+a-(2p-3)\frac{\gamma^2}{4})\Gamma(2+a-(2p-2)\frac{\gamma^2}{4})} \tag{3}$$

$$\frac{M(p,-\frac{(k+1)\gamma^2}{4},0)}{M(p-\frac{4}{\gamma^2},-\frac{(k+1)\gamma^2}{4},0)} = f(\gamma)\Gamma(\frac{4}{\gamma^2}-p)\Gamma(\frac{4}{\gamma^2}+1-p) \times \frac{\Gamma(\frac{4}{\gamma^2}-k-p)\Gamma(\frac{8}{\gamma^2}-k+1-p)}{\Gamma(\frac{12}{\gamma^2}-k+1-2p)\Gamma(\frac{8}{\gamma^2}-k+1-2p)} \tag{4}$$

In (3), $0 < a < \frac{\gamma^2}{4}$; In (4), we work with integer k and $\frac{4}{k+1} < \gamma^2 < \frac{4}{k}$.

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$$U(t) = \mathbb{E}[(\int_0^1 (x-t)^{\frac{\gamma^2}{4}} x^a (1-x)^b e^{\frac{\gamma}{2}X(x)} dx)^p]$$
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U solutions to hypergeometric equations:

$$\begin{aligned} U(t) &= C_1 F_1(t) + C_2 |t|^{1-C} F_2(t) && \text{basis at } 0 \\ &= D_1 |t|^{-A} F_3(t^{-1}) + D_2 |t|^{-B} F_4(t^{-1}) && \text{basis at } -\infty \end{aligned}$$

where F_i are hypergeometric functions ($F_i(0) = 1$).

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$$\left. \begin{aligned} C_1 &= M(p, a + \tfrac{\gamma^2}{4}, b) \\ D_1 &= M(p,a,b),\ D_2 = 0 \end{aligned} \right\} \Rightarrow \text{shift equation (1)}$$

Idea of proof (II)

$\tilde{U}(t) = \mathbb{E}[(\int_0^1 (x-t)x^a(1-x)^b e^{\frac{\gamma}{2}X(x)}dx)^p]$

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$$\left.\begin{aligned} \tilde{C}_1 &= M(p, a+1, b) \\ \tilde{D}_1 &= M(p,a,b),\ \tilde{D}_2 = 0 \end{aligned}\right\} \Rightarrow \text{shift equation (2)}$$

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For $0 < a < 1 - \frac{\gamma^2}{4}$ and $b = 0$,

$$C_2 = p \frac{\Gamma(a+1)\Gamma(-a-\frac{\gamma^2}{4}-1)}{\Gamma(-\frac{\gamma^2}{4})} M(p-1, a-\frac{\gamma^2}{4}, 0) \Rightarrow \text{shift equation (3)}$$

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Or for $-1-\frac{\gamma^2}{4} < a < -1$ and $b = 0$,

$$C_2 = f(\gamma, a)\frac{\Gamma(-p+1+\frac{4}{\gamma^2}(a+1))}{\Gamma(-p)} M(p-1-\frac{4}{\gamma^2}(a+1), -2-a-\frac{\gamma^2}{4}, 0)$$

$$a = -\frac{(k+1)\gamma^2}{4} \Rightarrow \text{shift equation (4)}$$

Another statement

Theorem

For $\gamma \in (0,2)$ and for p, a, b satisfying appropriate bounds, $M(p,a,b)$ is given by

$$\frac{(2\pi)^p(\frac{2}{\gamma})^{p\frac{\gamma^2}{4}}}{\Gamma(1-\frac{\gamma^2}{4})^p}\frac{\Gamma_\gamma(\frac{2}{\gamma}(a+1)-(p-1)\frac{\gamma}{2})\Gamma_\gamma(\frac{2}{\gamma}(b+1)-(p-1)\frac{\gamma}{2})\Gamma_\gamma(\frac{2}{\gamma}(a+b+2)-(p-2)\frac{\gamma}{2})\Gamma_\gamma(\frac{2}{\gamma}-p\frac{\gamma}{2})}{\Gamma_\gamma(\frac{2}{\gamma})\Gamma_\gamma(\frac{2}{\gamma}(a+1)+\frac{\gamma}{2})\Gamma_\gamma(\frac{2}{\gamma}(b+1)+\frac{\gamma}{2})\Gamma_\gamma(\frac{2}{\gamma}(a+b+2)-(2p-2)\frac{\gamma}{2})},$$

where the function $\Gamma_\gamma(x)$ is defined for $x>0$ and $Q=\frac{\gamma}{2}+\frac{2}{\gamma}$ by:

$$\ln\Gamma_\gamma(x)=\int_0^\infty\frac{dt}{t}\left[\frac{e^{-xt}-e^{-\frac{Qt}{2}}}{(1-e^{-\frac{\gamma t}{2}})(1-e^{-\frac{2t}{\gamma}})}-\frac{(\frac{Q}{2}-x)^2}{2}e^{-t}+\frac{x-\frac{Q}{2}}{t}\right].$$

Perspectives

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- Joint law of GMC on the interval
- Other geometries
- Differential equations for other observables

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Thank you!