THE COMBINATORIAL CHOW RING OF PRODUCTS OF GRAPHS

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ABSTRACT. In this paper we study the structure of a Chow ring associated to a product of graphs. This ring naturally arises from the Gross-Schoen desingularization of a product of regular proper semi-stable curves over discrete valuation rings, for the corresponding family of dual graphs, and can be viewed as the universal combinatorial part of the Chow ring of products of semistable curves, when the family of dual graphs is fixed. Moreover, Johannes Kolb conjectured remarkable vanishing properties in this combinatorial Chow ring, which he showed to control the behavior of the non-Archimedean height pairing on products of smooth proper curves over non-Archimedean fields.

Our aim in this paper is to give a fairly complete description of these Chow rings.

- We prove the localization theorem, which describes the Chow ring of the product of any family of graphs as an inverse limit of the Chow ring of hypercubes (i.e., products of edges). This shows that the Chow rings of products of graphs form a sheaf for the topology generated by the products of subrgaphs.

- We provide a complete description of the degree map, leading to a complete description of the degree of intersection between divisors in the special fiber of the regular semistable model of any product of curves.

- We prove vanishing theorems in the Fourier dual description of the Chow ring of the hypercube which confirms the above mentioned vanishing conjectures of Kolb. Combined with his work, this leads to an analytic formula for the arithmetic intersection number of adelic metrized line bundles on products of curves over complete discretely valued fields, which generalizes a previous result of Shou-Wu Zhang in his work on Gross-Schoen cycles and dualizing sheaves.

1. INTRODUCTION

Let R be a complete discrete valuation ring with an algebraically closed residue field k and fraction field K, and let X be a smooth proper curve over K. By semi-stable reduction theorem, replacing K with a finite extension if necessary, we can find a regular proper strict semi-stable model $\mathfrak{X}$ of X over the valuation ring. Denote by $\mathfrak{X}_s$ the special fiber of $\mathfrak{X}$, and let $G = (V, E)$ be the dual graph of $\mathfrak{X}_s$. For each vertex $v \in V$, denote by X_v the corresponding irreducible component of $\mathfrak{X}_s$. The intersection products $X_v.X_u$ are described by the graph G as follows:

$$(1) \qquad \forall\, u, v \in V, \qquad X_u.\,X_v = \begin{cases} \text{number of edges } \{u, v\} \text{ in } G & \text{if } u \neq v, \\ -\mathrm{val}_G(v) & \text{if } u = v, \end{cases}$$

where $\mathrm{val}_G(v)$ is the valence of v in G. The intersection products satisfy the following two sets of relations:

($\mathscr{A}1$) For all $u, v \in V$, $X_v X_u = 0$ if $\{u, v\} \notin E$;
($\mathscr{A}2$) For all $u \in V$, $X_u(\sum_{v \in V} X_v) = 0$.

Consider the polynomial ring $Z(G) = \mathbb{Z}[C_v \,|\, v \in V]$ on variables C_v, and define the ideal $\mathscr{I}_{\mathrm{rat}} \subseteq Z(G)$ of elements *rationally equivalent to zero* as the ideal generated by the polynomials C_uC_v, for $\{u,v\} \notin E$, and $C_u(\sum_{v\in V} C_v)$ for $u \in V$. Define the Chow ring $\mathrm{Chow}_{\mathrm{GS}}(G)$ of the graph G by $\mathrm{Chow}_{\mathrm{GS}}(G) := Z(G)/\mathscr{I}_{\mathrm{rat}}$, and note that we have a morphism of graded rings $\mathrm{Chow}_{\mathrm{GS}}(G) \to \mathrm{Chow}^c_{\mathfrak{X}_s}(\mathfrak{X})$, where $\mathrm{Chow}^c_{\mathfrak{X}_s}(\mathfrak{X})$ denotes the subring of the Chow ring with support $\mathrm{Chow}_{\mathfrak{X}_s}(\mathfrak{X})$ of $\mathfrak{X}$ with support in $\mathfrak{X}_s$ generated by the irreducible components of the special fiber (see [12, Chapter 17] or [13, Section 8] for the definition of Chow rings with support).

For a graph consisting of a single edge $e = \{u,v\}$ on two vertices, we have $\mathrm{Chow}_{\mathrm{GS}}(e) = \mathbb{Z}[C_u,C_v]/(C_u^2 + C_uC_v, C_v^2 + C_uC_v) \simeq \mathbb{Z} \oplus \mathbb{Z}C_u \oplus \mathbb{Z}C_v \oplus \mathbb{Z}C_uC_v$. For a general graph G, the structure of $\mathrm{Chow}_{\mathrm{GS}}(G)$ is completely described by an exact sequence of the form

$$0 \to \mathrm{Chow}_{\mathrm{GS}}(G) \to \prod_{e\in E} \mathrm{Chow}_{\mathrm{GS}}(e) \to \prod_{\{e_1,e_2\}\in L(G)} \mathbb{Z}, \tag{2}$$

where $L(G)$ denotes the line graph of G (see Definition 1.2).

We have a (local) degree map $\deg : \mathrm{Chow}_{\mathrm{GS}}(G) \to \mathbb{Z}$ which is defined as follows. For any edge $e = \{u,v\} \in E$, define the map $\deg_e : \mathrm{Chow}_{\mathrm{GS}}(e) \to \mathbb{Z}$ by sending an element x of $\mathrm{Chow}_{\mathrm{GS}}(e)$ to the coefficient of C_uC_v in x. The degree map deg is then the composition of the embedding $\mathrm{Chow}_{\mathrm{GS}}(G) \hookrightarrow \prod_{e\in E} \mathrm{Chow}_{\mathrm{GS}}(e)$ with the map $\sum_{e\in E} \deg_e$. By definition, the degree map coincides with the intersection pairing (1), i.e., for all $u,v \in V$, we have $\deg(C_uC_v) = X_uX_v$.

Our aim in this paper is to provide a generalization of the above picture for the products of (an arbitrary number) of proper smooth curves over a complete discretely valued field.

So let $X_1, \dots, X_d$ be proper smooth curves over K, and, replacing K with a finite extension if necessary, consider a regular strict semi-stable model $\mathfrak{X}_i$ of X_i over the valuation ring for each i. Starting from the product $\mathfrak{X}_1 \times_{\mathrm{Spec}(R)} \cdots \times_{\mathrm{Spec}(R)} \mathfrak{X}_d$, the Gross-Schoen desingularization procedure [14] provides a regular proper semi-stable model $\mathfrak{X}$ of the product $X = X_1 \times \cdots \times X_d$ over the valution ring R. The desingularization depends on the choice of a total order on the components of the special fiber of each $\mathfrak{X}_i$.

Denote by $G_1 = (V_1,E_1), \dots, G_d = (V_d,E_d)$ the dual graphs of the special fibers of $\mathfrak{X}_1 \dots, \mathfrak{X}_d$, respectively, and suppose that a total order $\leq_i$ on the vertex set V_i is given for each i. The dual complex of the special fiber of the Gross-Schoen model $\mathfrak{X}$ of X is a triangulation of the product $\mathscr{G} = G_1 \times \cdots \times G_d$. When $\mathscr{G}$ is given by its natural cubical structure with cubes corresponding to the elements of the product $\mathscr{E} = E_1 \times \cdots \times E_d$, the triangulation consists of the union of the standard triangulation of these d-dimensional cubes, compatible with the fixed total orders on the vertex set of each graph G_i, see Section 2 for the precise definition. The vertex set $\mathscr{G}_0$ of the simplicial set $\mathscr{G}$ is the product $\mathscr{G}_0 = V_1 \times \cdots \times V_d$ of the vertex sets, whose elements are in bijection with irreducible components of the special fiber $\mathfrak{X}_s$ of $\mathfrak{X}$: for an element $\mathbf{v} \in \mathscr{G}_0$, we denote by $X_{\mathbf{v}}$ the corresponding irreducible component of $\mathfrak{X}_s$.

Consider the Chow ring with support $\mathrm{Chow}_{\mathfrak{X}_s}(\mathfrak{X})$. The intersection products between $X_{\mathbf{v}}$, for $\mathbf{v} \in \mathscr{V}$, in $\mathrm{Chow}_{\mathfrak{X}_s}(\mathfrak{X})$ verify three types of equations (see $(\mathscr{R}1), (\mathscr{R}2), (\mathscr{R}3)$ below), given by Kolb in [18, 19], two of which are higher dimensional analogous of the relations $(\mathscr{A}1)$ and $(\mathscr{A}2)$. This leads to the definition of a Chow ring for the product of graphs, that we describe next.

1.1. **Definition of the Chow ring** $\mathrm{Chow}_{\mathrm{GS}}(\mathscr{G})$**.** Denote by $Z(\mathscr{G})$ the polynomial ring with coefficients in $\mathbb{Z}$ generated by the vertices of $\mathscr{G}$, namely,

$$Z(\mathscr{G}) := \mathbb{Z}[C_{\mathbf{v}} \,|\, \mathbf{v} \in \mathscr{G}_0],$$

where the variables $C_{\mathbf{v}}$ are associated to the vertices (0-simplices) of $\mathscr{G}$. We view $Z(\mathscr{G})$ as a graded ring where each variable $C_{\mathbf{v}}$ is of degree one.

The graded ideal $\mathscr{I}_{\mathrm{rat}}$ of all the elements of $Z(\mathscr{G})$ which are *rationally equivalent to zero* is defined as the ideal generated by the following three types of generators:

($\mathscr{R}1$) $C_{\mathbf{v}_1}C_{\mathbf{v}_2}\dots C_{\mathbf{v}_k}$ for $k \in \mathbb{N}$ and elements $\mathbf{v}_j \in \mathscr{G}_0$ such that $\mathbf{v}_1, \dots, \mathbf{v}_k$ do not form a simplex in $\mathscr{G}$;

($\mathscr{R}2$) $C_{\mathbf{u}}\Big(\sum_{\mathbf{v}\in\mathscr{G}_0} C_{\mathbf{v}}\Big)$ for any vertex $\mathbf{u} \in \mathscr{G}_0$; and

($\mathscr{R}3$) $C_{\mathbf{u}}C_{\mathbf{w}}\Big(\sum_{\mathbf{v}\in\mathscr{G}_0 : v_i = u_i} C_{\mathbf{v}}\Big)$ for any pair of vertices $\mathbf{u}, \mathbf{w} \in \mathscr{G}_0$ and any index $1 \le i \le d$ with $u_i \neq w_i$.

Definition 1.1. The combinatorial Chow ring of $\mathscr{G}$ is the graded ring $\mathrm{Chow}_{\mathrm{GS}}(\mathscr{G}) := Z(\mathscr{G})/\mathscr{I}_{\mathrm{rat}}$.

The ring $\mathrm{Chow}_{\mathrm{GS}}(\mathscr{G})$ is the universal graded commutative ring with generators indexed by vertices of $\mathscr{G}$ and verifying relations ($\mathscr{R}1$), ($\mathscr{R}2$), ($\mathscr{R}3$) above; in particular we get a well-defined map

$$\alpha_{\mathfrak{X}} : \mathrm{Chow}_{\mathrm{GS}}(\mathscr{G}) \to \mathrm{Chow}_{\mathfrak{X}_s}(\mathfrak{X}), \tag{3}$$

for $\mathfrak{X}$ the Gross-Schoen desingularization of the products of curves $\mathfrak{X}_1 \times \cdots \times \mathfrak{X}_d$ (where for each i, the dual graph of the special fiber of $\mathfrak{X}_i$ is isomorphic to G_i).

1.2. **Statement of the main results.** The main contributions of this paper are the structure Theorem 1.5 which describes the additive structure of the graded pieces of the Chow ring, the localization Theorem 1.4, which is a generalization of the exact sequence (2) to products of graphs, the calculation of the degree map, which is a generalization of (1) to higher dimension, and a vanishing theorem confirming a conjecture of Kolb. We now discuss these results.

1.2.1. *Localization.* We prove the localization theorem 1.4, a generalization of the exact sequence (2) to products of graphs, which shows that the calculations in the ring $\mathrm{Chow}_{\mathrm{GS}}(\mathscr{G})$ can be reduced to calculation in the Chow ring of the hypercubes of dimension d, namely, the products of d copies of the complete graph K_2 on two vertices.

Recall first that a homomorphism of graphs $f : H \to G$ is a map $f : V(H) \to V(G)$ such that for any edge $\{u, v\} \in E(H)$, either $f(u) = f(v)$ or $\{f(u), f(v)\} \in E(G)$.

Let $H_1, \dots, H_d$ be d simple connected graphs. Define $\mathscr{H} = H_1 \times \cdots \times H_d$ with the induced simplicial structure. Suppose that for each $i = 1, \dots, d$, a homomorphism of graphs $f_i : H_i \to G_i$ is given (such that f_i respects also the two fixed orderings on the vertex set of H_i and G_i). The product of f_i leads to a morphism of simplicial sets $f : \mathscr{H} \to \mathscr{G}$, and induces a morphism of graded rings $f^* : Z(\mathscr{G}) \to Z(\mathscr{H})$, which is defined on the level of generators by sending $C_{\mathbf{v}}$ for $\mathbf{v} \in \mathscr{G}_0$ to

$$f^*(C_{\mathbf{v}}) = \sum_{\substack{\mathbf{u}\in\mathscr{H}_0 \\ f(\mathbf{u})=\mathbf{v}}} C_{\mathbf{u}}.$$

It is not hard to see that the map f^* sends $\mathscr{I}_{\mathrm{rat}}(\mathscr{G})$ to $\mathscr{I}_{\mathrm{rat}}(\mathscr{H})$ and induces a well-defined map of Chow rings $f^* : \mathrm{Chow}_{\mathrm{GS}}(\mathscr{G}) \to \mathrm{Chow}_{\mathrm{GS}}(\mathscr{H})$, c.f., Proposition 2.4.

Let now $G_1 = (V_1, E_1), \dots, (G_d, E_d)$ be a collection of d simple connected graphs, and $\mathscr{G} = \prod_i G_i$, as above. For $\mathbf{e} \in \mathscr{E} = E_1 \times \cdots \times E_d$, let $\square_{\mathbf{e}} = e_1 \times \cdots \times e_d$. Regarding each edge e_i as a subgraph of G isomorphic to K_2, and applying the functoriality to the inclusions of the subgraph $e_i \hookrightarrow G_i$, we get a map $\iota_{\mathbf{e}}^* : \mathrm{Chow}_{\mathrm{GS}}(\mathscr{G}) \to \mathrm{Chow}_{\mathrm{GS}}(\square_{\mathbf{e}}) \simeq \mathrm{Chow}_{\mathrm{GS}}(\square^d)$ associated to the inclusion map of simplicial sets

$$\iota_{\mathbf{e}} : \square_{\mathbf{e}} \hookrightarrow \mathscr{G}.$$

By definition, the map $\iota_{\mathbf{e}}^*$ is identity on the generators associated to the vertices of $\square_{\mathbf{e}}$, and is zero otherwise.

Write $e_i = u_i v_i$ for vertices $u_i < v_i$ of G_i with respect to the total order of G_i, and define the two corresponding facets $\square_{\mathbf{e},0_i}$ and $\square_{\mathbf{e},1_i}$ of $\square_{\mathbf{e}}$ associated to u_i and v_i, respectively, by

$$\square_{\mathbf{e},0_i} := e_1 \times \cdots \times e_{i-1} \times \{u_i\} \times e_{i+1} \times \cdots \times e_d, \text{ and}$$
$$\square_{\mathbf{e},1_i} := e_1 \times \cdots \times e_{i-1} \times \{v_i\} \times e_{i+1} \times \cdots \times e_d.$$

Let $\iota_{\mathbf{e},0_i} : \square_{\mathbf{e},0_i} \hookrightarrow \square_{\mathbf{e}}$, and similarly, $\iota_{\mathbf{e},1_i} : \square_{\mathbf{e},1_i} \hookrightarrow \square_{\mathbf{e}}$, the inclusion maps. By functoriality, we get maps $\iota_{\mathbf{e},0_i}^* : \mathrm{Chow}_{\mathrm{GS}}(\square_{\mathbf{e}}) \to \mathrm{Chow}_{\mathrm{GS}}(\square_{\mathbf{e},0_i})$ and $\iota_{\mathbf{e},1_i}^* : \mathrm{Chow}_{\mathrm{GS}}(\square_{\mathbf{e}}) \to \mathrm{Chow}_{\mathrm{GS}}(\square_{\mathbf{e},1_i})$ on the level of Chow rings.

We next recall the definition of the line graph of a give graph.

Definition 1.2. Let $G = (V, E)$ be a given simple connected graph on vertex set V and edge set E. The *line graph* of G denoted by $L(G)$ is the graph on vertex set E and with edge set consisting of all the pairs $\{e, e'\} \subset E$ with e and e' incident edges in G.

Let $G_1 = (V_1, E_1), \dots, G_d = (V_d, E_d)$ be a collection of d simple connected graphs as before. For each $i = 1, \dots, d$, define the set $\mathscr{E}_i := E_1 \times \cdots \times E_{i-1} \times E(L(G_i)) \times E_{i+1} \times \cdots \times E_d$. Thus, an element $\mathbf{x}$ of $\mathscr{E}_i$ is a collection $e_j \in E_j$, for $j \neq i$, and $\{e_{i,1}, e_{i,2}\}$ with $e_{i,1}, e_{i,2} \in E_i$ and $e_{i,1} \cap e_{i_2} \neq \emptyset$. The element $\mathbf{x} \in \mathscr{E}_i$ therefore gives two hypercubes $\square_{\mathbf{e}_1}$ and $\square_{\mathbf{e}_2}$ in $\mathscr{G}$, with $\mathbf{e}_k = e_1 \times \cdots \times e_{i-1} \times e_{i,k} \times \cdots \times e_d$, for $k = 1, 2$. Note that $\square_{\mathbf{e}_1}$ and $\square_{\mathbf{e}_2}$ share the facet $\square_{\mathbf{x}} := e_1 \times \cdots \times e_{i-1} \times (e_1 \cap e_2) \times e_{i+1} \times \cdots \times e_d$. Denote by $\iota_{\mathbf{x},1}$ and $\iota_{\mathbf{x},2}$ the inclusion of $\square_{\mathbf{x}}$ in $\square_{\mathbf{e}_1}$ and $\square_{\mathbf{e}_2}$, respectively. Denoting by $j_{\mathbf{x}}^* : \mathrm{Chow}_{\mathrm{GS}}(\square_{\mathbf{e}_1}) \times \mathrm{Chow}_{\mathrm{GS}}(\square_{\mathbf{e}_2}) \to \mathrm{Chow}_{\mathrm{GS}}(\square_{\mathbf{x}})$ the map which sends the pair (α, β) to $\iota_{\mathbf{x},1}^*(\alpha) - \iota_{\mathbf{x},2}^*(\beta)$, we get a map

$$j : \prod_{\mathbf{e} \in \mathscr{E}} \mathrm{Chow}_{\mathrm{GS}}(\square_{\mathbf{e}}) \longrightarrow \prod_{i=1}^{d} \prod_{\mathbf{x} \in \mathscr{E}_i} \mathrm{Chow}_{\mathrm{GS}}(\square_{\mathbf{x}}).$$

Note that the $\mathbf{x}$-coordinate of j is the composition of $j_{\mathbf{x}}^*$ with the projection from

$$\prod_{\mathbf{e} \in \mathscr{E}} \mathrm{Chow}_{\mathrm{GS}}(\square_{\mathbf{e}}) \to \mathrm{Chow}_{\mathrm{GS}}(\square_{\mathbf{e}_1}) \times \mathrm{Chow}_{\mathrm{GS}}(\square_{\mathbf{e}_2}).$$

Remark 1.3. The map $j_{\mathbf{x}}^*$, and so j, is well-defined only up to the sign consisting in changing the role of e_1 and e_2. In order the get a well-defined map, we can fix a total order on the set E_i, for $i = 1, \dots, d$, and for any edge $\{e_{i,1}, e_{i,2}\}$ of $L(G_i)$ require that $e_{i,1} < e_{i,2}$ with respect to this total order. We remark that the choice of the sign is irrelevant for what follows.

With these notations, we can state our localization theorem.

Theorem 1.4. *The map of graded rings* $\prod_{\mathbf{e}\in\mathcal{E}} \iota_{\mathbf{e}}* : \mathrm{Chow}_{\mathrm{GS}}(\mathscr{G}) \to \bigoplus_{\mathbf{e}\in\mathscr{E}} \mathrm{Chow}_{\mathrm{GS}}(\square_{\mathbf{e}})$ *is injective and identifies* $\mathrm{Chow}_{\mathrm{GS}}(\mathscr{G})$ *with the kernel of the map*

$$j : \prod_{\mathbf{e}\in\mathscr{E}} \mathrm{Chow}_{\mathrm{GS}}(\square_{\mathbf{e}}) \longrightarrow \prod_{i=1}^{d}\prod_{\mathbf{x}\in\mathscr{E}_i} \mathrm{Chow}_{\mathrm{GS}}(\square_{\mathbf{x}}).$$

In other words, the ring $\mathrm{Chow}_{\mathrm{GS}}(\mathscr{G})$ is the inverse limit of the Chow ring of cubes of $\mathscr{G}$ for the diagram of maps induced by the inclusion of cubes. Endowing the simplicial set $\mathscr{G}$ with the cubical topology with a basis of open sets consisting of the products of subgraphs of G_i, the localization theorem ensures that the Chow rings form a sheaf for the coverings of $\mathscr{G}$ with open sets whose union covers all the simplices of $\mathscr{G}$.

1.2.2. *Description of the additive structure of* $\mathrm{Chow}_{\mathrm{GS}}(\mathscr{G})$*.* The proof of the localization theorem is indirect, and is based on a structure theorem which provides a description of the Chow groups in terms of non-degenerate simplices of the product $\mathscr{G}$ and specific relations, taking into account the cubical structure of $\mathscr{G}$. We now describe this.

The total orders $\leq_i$ on the vertex sets V_i, $i = 1, \dots, d$, induce a partial order $\leq$ on $\mathscr{G}_0$ defined by saying $\mathbf{u} = (u_1, \dots, u_d) \leq \mathbf{v} = (v_1, \dots, v_d)$ if $u_i \leq_i v_i$ for each i.

Let $\mathbf{u} < \mathbf{v}$ be two elements of $\mathscr{G}_0$ such that $\{\mathbf{u}, \mathbf{v}\}$ forms a one-simplex. This means that for each i, we have $u_i = v_i$ or $\{u_i, v_i\} \in E_i$, c.f. Section 2. Denote by $I(\mathbf{u}, \mathbf{v})$ the set of all indices i with $u_i < v_i$, and let $\mathbf{e}_{\mathbf{u},\mathbf{v}}$ be the cube of dimension $|I(\mathbf{u}, \mathbf{v})|$ formed by all the vertices $\mathbf{z} = (z_1, \dots, z_d)$ in $\mathscr{G}_0$ with $z_i \in \{u_i, v_i\}$, i.e., $\mathbf{e}_{\mathbf{u},\mathbf{v}} = \prod_{i=1}^{d}\{u_i, v_i\}$.

For any integer $k \in \mathbb{N}$, denote by $\mathscr{G}_k^{nd}$ the set of all non-degenerate k-simplices of $\mathscr{G}$. Each element σ of $\mathscr{G}_k^{nd}$ is a sequence $\mathbf{u}_0 < \mathbf{u}_1 < \cdots < \mathbf{u}_k$ of vertices $\mathbf{u}_i \in \mathscr{G}_0$ such that $\{\mathbf{u}_j, \mathbf{u}_{j+1}\}$ is a 1-simplex of $\mathscr{G}$, for any $0 \leq j \leq k-1$.

Let σ be a non-degenerate k-simplex of $\mathscr{G}$. For two indices $1 \leq i, j \leq d$ lying both in $I(\mathbf{u}_t, \mathbf{u}_{t+1})$ for some $0 \leq t \leq k-1$, define

$$\widetilde{R}_{\sigma,i,j} := \sum_{\substack{\mathbf{w}\in\mathbf{e}_{\mathbf{u}_t,\mathbf{u}_{t+1}}\\ w_i = u_{t,i}}} C_{\mathbf{w}} - \sum_{\substack{\mathbf{w}\in\mathbf{e}_{\mathbf{u}_t,\mathbf{u}_{t+1}}\\ w_j = u_{t,j}}} C_{\mathbf{w}}.$$

For any k-simplex $\sigma \in \mathscr{G}_k$, denote by C_σ the product (with multiplicity) of $C_{\mathbf{v}}$ over all vertices of σ, i.e., $C_\sigma := \prod_{\mathbf{v}\in\sigma} C_{\mathbf{v}}$. Using ($\mathscr{R}1$) and ($\mathscr{R}3$), one verifies that $C_\sigma \widetilde{R}_{\sigma,i,j} \in \mathscr{I}_{\mathrm{rat}}$.

For any $k \in \mathbb{N}$, denote by $\mathbb{Z}\langle\mathscr{G}_k^{nd}\rangle$ the $\mathbb{Z}$-submodule of $Z(\mathscr{G})$ generated by all elements C_σ for $\sigma \in \mathscr{G}_k^{nd}$. By definition of the simplicial structure, one sees that for an element $\sigma \in \mathscr{G}_k^{nd}$ consisting of vertices $\mathbf{u}_0 < \cdots < \mathbf{u}_k$, and for $i, j \in I(\mathbf{u}_t, \mathbf{u}_{t+1})$ as above, the product $C_\sigma \widetilde{R}_{\sigma,i,j}$ lies in $\mathbb{Z}\langle\mathscr{G}_k^{nd}\rangle$. Denote by $\mathscr{I}_k^{nd}$ the $\mathbb{Z}$-submodule of $\mathbb{Z}\langle\mathscr{G}_k^{nd}\rangle$ generated by all the elements $C_\sigma \widetilde{R}_{\sigma,i,j}$, for $\sigma \in \mathscr{G}_k^{nd}$ and $i, j \in I(\mathbf{u}_t, \mathbf{u}_{t+1})$ as above. We have the following theorem.

Theorem 1.5. *For any non-negative integer* k*, we have*

$$\mathrm{Chow}_{\mathrm{GS}}^{k+1}(\mathscr{G}) \simeq \mathbb{Z}\langle\mathscr{G}_k^{nd}\rangle / \mathscr{I}_k^{nd}.$$

The existence of a surjection from $\mathbb{Z}\langle\mathscr{G}_k^{\mathrm{nd}}\rangle / \mathscr{I}_k^{nd}$ to $\mathrm{Chow}_{\mathrm{GS}}^{k+1}(\mathscr{G})$ is a consequence of the moving lemma, c.f. Theorem 2.7. The proof of the injectivity of this map, on the other hand, turns out to be particularly tricky and technical. This is given in Section 3. For this we define an interesting filtration on $Z^{k+1}(\mathscr{G})$, and then we show by induction that it induces a trivial filtration on the kernel of the above map. We ignore if this filtration has any geometric origin.

1.2.3. *Combinatorics of the degree map.* Let $\square^d = \{0,1\}^d$ be the d-dimensional hypercube, which is the d-fold product of the complete graph K_2 on two vertices $0 < 1$ with its standard simplicial structure. It follows from the structure theorem that the Chow ring $\mathrm{Chow}_{\mathrm{GS}}(\square^d)$ is of rank one in graded degree $d+1$, i.e., $\mathrm{Chow}^{d+1}_{\mathrm{GS}}(\square^d) \simeq \mathbb{Z}$, generated by C_σ for any non-degenerate d-simplex σ of $\square^d$. This leads to a well-defined degree map

$$\deg : \mathrm{Chow}^{d+1}_{\mathrm{GS}}(\square^d) \to \mathbb{Z}.$$

Combining this with the localization theorem, and the vanishing of $\mathrm{Chow}^i_{\mathrm{GS}}(\square^d)$ in degree $i \geq d+2$, which follows for example from the structure theorem, or the moving lemma, we infer that for any collection of simple connected graphs $G_1 = (V_1, E_1), \dots, G_d = (V_d, E_d)$, we have $\mathrm{Chow}^{d+1}_{\mathrm{GS}}(\mathscr{G}) \simeq \mathbb{Z}^{|\mathscr{E}|}$. Therefore we get a degree map $\deg : \mathrm{Chow}^{d+1}_{\mathrm{GS}}(\mathscr{G}) \simeq \mathbb{Z}^{|\mathscr{E}|} \to \mathbb{Z}$, by aditionning the coordinates in $\mathbb{Z}^{|\mathscr{E}|}$.

Our next result gives a combinatorial formula for the value of the degree map. Combined with the map $\alpha_{\mathfrak{X}}$ in (3), this results in a concrete effective description of the local degrees in the Chow ring $\mathrm{Chow}^c_{\mathfrak{X}_s}(\mathfrak{X})$ for the Gross-Schoen desingularization $\mathfrak{X}$ of a product of semi-stable R-curves $X_1, \dots, X_d$, generalizing (1) to higher dimension.

Since $\mathrm{Chow}^{d+1}_{\mathrm{GS}}(\mathscr{G})$ is generated by monomials, we can restrict to the case of a monomial, and by localization theorem, and the definition of the degree map, it will be enough to treat the case of the hypercube $\square^d$.

By definition of the simplicial structure, each (possibly degenerate) d-simplex σ of $\square^d$ is of the form $\mathbf{v}_1^{n_1}\mathbf{v}_2^{n_2}\dots\mathbf{v}_k^{n_k}$ with $\mathbf{v}_1 < \mathbf{v}_2 < \dots < \mathbf{v}_k$, and $n_i \geq 1$ with $\sum_i n_i = d+1$. We have $0 \leq |\mathbf{v}_1| < \dots < |\mathbf{v}_k| \leq d$, where for any $\mathbf{v} \in \square^d$, we denote by $|\mathbf{v}|$ the *length* of $\mathbf{v}$ defined as the number of coordinates of $\mathbf{v}$ equal to one. Consider the set $[d] := \{0, 1, \dots, d\}$. Let us say a point $|\mathbf{v}_i|$ is a *neighbor* of a point $x \in [d] \setminus \big\{\, |\mathbf{v}_1|, \dots, |\mathbf{v}_k| \,\big\}$ if the interval formed by x and $|\mathbf{v}_i|$ does not contain any other point of $\big\{\, |\mathbf{v}_1|, \dots, |\mathbf{v}_k| \,\big\}$ beside $|\mathbf{v}_i|$. In this way each point $x \in [d] \setminus \big\{\, |\mathbf{v}_1|, \dots, |\mathbf{v}_k| \,\big\}$ has either one or two neighbors among the points $|\mathbf{v}_1|, \dots, |\mathbf{v}_k|$.

Assume now that n_i chips are placed on the point $|\mathbf{v}_i|$ in $[d]$. The total number of chips is thus $\sum_i n_i = d+1$. We assume further that the chips are labelled, and each point $|\mathbf{v}_i| \in [d]$ chooses, once for all, one of its n_i chips that she wants to keep, and decides to give all the extra remaining chips to $n_i - 1$ of its neighbors in $[d] \setminus \big\{\, |\mathbf{v}_1|, \dots, |\mathbf{v}_k| \,\big\}$ in such a way that at the end, each point of $[d]$ holds precisely one chip. In how many ways this can be done? The following theorem states that, up to a $\pm$ sign, the degree of $\mathbf{v}_1^{n_1}\dots\mathbf{v}_k^{n_k}$ in $\mathrm{Chow}_{\mathrm{GS}}(\square^d)$ is given by this number.

Theorem 1.6. *Notations as above, let $C_\sigma = C_{\mathbf{v}_1}^{n_1}\dots C_{\mathbf{v}_k}^{n_k}$. One of the two following cases can happen.*

(1) If there exists an $1 \leq i < k$ such that $n_1 + \dots + n_i > |\mathbf{v}_{i+1}|$, then $C_\sigma = 0$. Similarly, if there exists an $k \geq i \geq 2$ such that $n_i + \dots + n_k > d - |\mathbf{v}_{i-1}|$, then $C_\sigma = 0$.

(2) Otherwise, there exists a sequence of integers $y_0, x_1, y_1, x_2, y_2, \dots, x_{k-1}, y_{k-1}, x_k$ verifying the following properties

- $|\mathbf{v}_1| = y_0$.

- *For all* $i = 2, \dots, k$, $|\mathbf{v}_i| = |\mathbf{v}_{i-1}| + x_i + y_i + 1$.
- $n_i = y_{i-1} + x_i + 1$ *for all* $i = 1, \dots, k$,

and, in this case, we have

$$\deg(C_\sigma) = (-1)^{d+1-k}\binom{y_0+x_1}{y_0}\binom{x_1+y_1}{x_1}\binom{y_1+x_2}{y_1}\dots\binom{x_{k-1}+y_{k-1}}{y_{k-1}}\binom{y_{k-1}+x_k}{x_k}.$$

Note that the product $\binom{y_0+x_1}{y_0}\binom{x_1+y_1}{x_1}\binom{y_1+x_2}{y_1}\dots\binom{x_{k-1}+y_{k-1}}{y_{k-1}}\binom{y_{k-1}+x_k}{x_k}$ in (2) is precisely the number of ways the extra (labelled) chips can be placed on the points of $[d] \setminus \{\ell_1, \dots, \ell_k\}$ so that each point receives precisely one chip from one of its neighbors; in case (1), this number is zero.

1.2.4. *Fourier transform and a vanishing theorem.* Identifying the points of $\square^d$ with the elements of the vector space $\mathbb{F}_2^d$, it is possible to give a dual description of the Chow ring of the hypercube using the Fourier duality. So let $\langle\,,\rangle$ be the scalar product on $\mathbb{F}_2^d$ defined by

$$\forall\, \mathbf{u}, \mathbf{v} \in \mathbb{F}_2^d, \qquad \langle \mathbf{v}, \mathbf{u}\rangle := \sum_{i=1}^d v_i.u_i \in \mathbb{F}_2.$$

For $\mathbf{w} \in \mathbb{F}_2^d$, define $F_\mathbf{w}$ by

$$F_\mathbf{w} := \sum_{\mathbf{v}\in\square^d} (-1)^{\langle \mathbf{v},\mathbf{w}\rangle} C_\mathbf{v}.$$

By Fourier duality, we have for any $\mathbf{v} \in \mathbb{F}_2^d$,

$$C_\mathbf{v} = \frac{1}{2^d}\sum_{\mathbf{v}\in\square^d} (-1)^{\langle \mathbf{v},\mathbf{w}\rangle} F_\mathbf{w}.$$

It follows that the set $\{F_\mathbf{w}\}_{\mathbf{w}\in\square^d}$ forms another system of generators for the Chow ring $\mathrm{Chow}_{\mathrm{GS}}(\square^d)[\frac{1}{2}]$ localized at 2, that we call the Fourier dual of the set $\{C_\mathbf{v}\}_{\mathbf{v}\in\square^d}$.

Denote by $\mathbf{1}$, and $\mathbf{0}$, the points of $\square^d$ whose coordinates are all equal to one, and zero, respectively. Let $\{\mathbf{e}_1, \dots, \mathbf{e}_d\}$ be the standard basis of $\mathbb{F}_2^d$, where $\mathbf{e}_i$ is the element of $\mathbb{F}_2^d$ which has i-th coordinate equal to 1, and all the other coordinates equal to 0. Kolb proved in [18] that the following set of relations are verified by $\{F_\mathbf{w}\}$ in $\mathrm{Chow}_{\mathrm{GS}}(\square^d)$:

($\mathscr{R}^*1$) For any $\mathbf{w} \in \mathbb{F}^\mathbf{d}$, we have $F_\mathbf{0} F_\mathbf{w} = 0$;
($\mathscr{R}^*2$) For any $i \in [d]$, and any $\mathbf{w}, \mathbf{z} \in \mathbb{F}^\mathbf{d}$, we have $F_{\mathbf{e}_i}(F_\mathbf{w} - F_{\mathbf{w}+\mathbf{e}_i})(F_\mathbf{z} + F_{\mathbf{z}+\mathbf{e_i}}) = 0$;
($\mathscr{R}^*3$) For any pair of indices $i, j \in [d]$, and any $\mathbf{w}, \mathbf{z}$, we have $(F_{\mathbf{w}+\mathbf{e}_i+\mathbf{e}_j} - F_\mathbf{w})(F_{\mathbf{z}+\mathbf{e}_i+\mathbf{e}_j} - F_\mathbf{z}) = (F_{\mathbf{w}+\mathbf{e}_i} - F_{\mathbf{w}+\mathbf{e}_j})(F_{\mathbf{z}+\mathbf{e}_i} - F_{\mathbf{z}+\mathbf{e}_j})$.

We have the following more precise statement. Let $\widetilde{\mathscr{I}_{\mathrm{rat}}}(\square^d)$ be the ideal of $\mathbb{Z}[F_\mathbf{w}]_{\mathbf{w}\in\square^d}$ generated by the relations given by ($\mathscr{R}^*1$), ($\mathscr{R}^*2$), and ($\mathscr{R}^*3$) above, and define

$$\widetilde{\mathrm{Chow}}_{\mathrm{GS}}(\square^d) := \mathbb{Z}[F_\mathbf{w}]_{\mathbf{w}\in\square^d} \,/\, \widetilde{\mathscr{I}_{\mathrm{rat}}}(\square^d).$$

Theorem 1.7. *The set of relations* ($\mathscr{R}^*1$), ($\mathscr{R}^*2$), *and* ($\mathscr{R}^*3$) *generate the ideal* $\mathscr{I}_{\mathrm{rat}}(\square^d)$ *in* $\mathbb{Z}[\frac{1}{2}][C_\mathbf{v}]_{\mathbf{v}\in\square^d} = \mathbb{Z}[\frac{1}{2}][F_\mathbf{w}]_{\mathbf{w}\in\square^\mathbf{d}}$. *In particular, we have* $\widetilde{\mathrm{Chow}}_{\mathrm{GS}}(\square^d)[\frac{1}{2}] = \mathrm{Chow}_{\mathrm{GS}}(\square^d)[\frac{1}{2}]$.

We now describe a criterion guaranteeing the vanishing of a monomial of the form $F_{\mathbf{w}_0}\dots F_{\mathbf{w}_d}$, for elements $\mathbf{w}_0, \dots, \mathbf{w}_d \in \mathbb{F}_2^d$.

Let $\mathcal{P} = \{P_1, \dots, P_k\}$ be a partition of $\{1, \dots, d\}$ into k disjoint non-empty sets. For each $\mathbf{w}_i$, denote by $\alpha(\mathbf{w}_i, \mathcal{P})$ the number of indices $1 \leq i \leq k$ such that there exists $j \in P_i$ with $\mathbf{w}_j = 1$.

Theorem 1.8. *If $\sum_{i=0}^{d} \alpha(\mathbf{w}_i, \mathcal{P}) < d + k$, then we have $F_{\mathbf{w}_0} \dots F_{\mathbf{w}_d} = 0$ in the Chow ring.*

This property was conjectured by Kolb and is required in [19] in order to get the analytic description of the local degree map, that we briefly describe in the next section.

1.3. **Analytic description of the local intersection numbers.** Let X be a smooth proper curve over a complete discretely valued field K with an algebraically closed residue field. The Berkovich analytification X^{an} of X is a compact path-wise connected Hausdorff topological space which deformation retracts to a compact metric graph Γ [6, 4, 10]. If X admits a regular semi-stable model $\mathfrak{X}$ over the valuation ring of K, the metric graph Γ has a model (G, ℓ) given by the dual graph $G = (V, E)$ of $\mathfrak{X}$ and the edge length $\ell : E(G) \to \mathbb{R}$ given by $\ell(e) = 1$ for all edges $e \in E$. (So Γ is the metric realization of (G, ℓ) in the sense that each edge in G is replaced with an interval of length one, see e.g. [3, 5].)

Any Cartier divisor D on $\mathfrak{X}$ with support in the special fiber $\mathfrak{X}$ gives a map $f : V \to \mathbb{Z}$, that we can extend to Γ by linear interpolation on interior points of the intervals in Γ corresponding to the edges of G. For two Cartier divisors $D_1, D_2 \in \mathrm{Chow}^1_{\mathfrak{X}_s}(\mathfrak{X})$ with functions $f_1, f_2 : \Gamma \to \mathbb{R}$, the degree map given by the pairing (1) gives a number $\deg(D_1 D_2)$, which can be described analytically as

$$\deg(D_1 D_2) = \langle f_1, f_2 \rangle_{\mathrm{Dir}} = -\int_{\Gamma} f_1' f_2'. \tag{4}$$

Here $\langle . , . \rangle_{\mathrm{Dir}}$ denotes the Dirichlet pairing on piecewise smooth functions on Γ [5, 21].

By an approximation argument involving semi-stable models of curves $X_{K'}$ for finite extensions K'/K, and viewing Cartier divisors with support in the special fibers of semi-stable models of $X_{K'}$ as piecewise linear functions on Γ, one can continuously extend the (degree) pairing between divisors to the full class of piecewise smooth functions on Γ such that the equation above remains valid for this more general class of functions [21].

Motivated by applications in arithmetic geometry, Zhang derived in [23] a generalization of the analytic formula (4) for the degree pairing in the case of a 2-fold product of a smooth proper curve X over K. Kolb [19] later generalized this to d-fold products of X assuming the validity of the vanishing Theorem 1.8. We state his result in the more general setting of a product of smooth proper curves $X_1, \dots, X_d$.

Let $X_1, \dots, X_d$ be smooth proper curves over K, that we suppose (up to passing to a finite extension of K), to have regular strict semi-stable models $\mathfrak{X}_1, \dots, \mathfrak{X}_d$ over the valuation ring R. Denote by $G_1, \dots, G_d$ the dual graphs of the special fibers of $\mathfrak{X}_1, \dots, \mathfrak{X}_d$, respectively, and let $\mathscr{G}$ be the product $G_1 \times \dots \times G_d$ with its simplicial structure. The Gross-Schoen desingularization gives a regular proper strict semi-stable model $\mathfrak{X}$ of the product $X = X_1 \times \dots \times X_d$ with special fiber $\mathfrak{X}_s$ having a dual complex isomorphic to $\mathscr{G}$. The geometric realization of $\mathscr{G}$ is a locally affine space which embeds in the Berkovich analytification X^{an} of X (by a general theorem of Berkovich [7]). Each Cartier divisor D with support in the special fiber $\mathfrak{X}_s$ induces a metric on the trivial line bundle corresponding to a piecewise affine function on the geometric realization of $\mathscr{G}$ (for metric on lines bundles see e.g. [22]). The intersection pairing given by the degree map induces a multi-linear pairing between piecewise affine functions (by passing to finite extensions of K if necessary), and we have

$$\langle f_{D_0}, \dots, f_{D_d}\rangle = \deg(D_0 \dots D_d).$$

This pairing can be viewed as the local contribution to the intersection product of metrized line bundles in non-Archimedean Arakelov theory. It is useful to extend this pairing to a larger class of metrized line bundles. By approximation, for each of the piecewise smooth functions f_i, one may take a sequence of piecewise linear functions converging to f_i, and extend the pairing as the limit of the pairing between piecewise linear functions. This has been carried out in great detail in [19]. The well-definedness of the extension as well as the analytic generalization of the Formula (4) is guaranteed if the vanishing condition in Theorem 1.8 holds. To state the theorem, we need to introduce some notations.

For each graph G_i, denote by Γ_i the metric graph associated to (G, ℓ) with length function $\ell \equiv 1$, the constant function. For each $n \in \mathbb{N}$, denote by $G_i^{(n)} = (V_i^{(n)}, E_i^{(n)})$ the n-th subdivision of G_i, where each edge e is subdivided into n edges. The pair $(G_i^{(n)}, \ell^{(n)})$ with length function $\ell^{(n)} \equiv 1/n$ is a model of the same metric graph Γ_i. A total order on the vertex set of G_i naturally extends to a total order on the vertex set of $G_i^{(n)}$ such that the vertices of $G_i^{(n)}$ on each edge e of G form a monotone sequence. Denote by $\mathscr{G}^{(n)}$ the simplicial set on the product $G_1^{(n)} \times \dots \times G_d^{(n)}$. This provides a triangulation of the topological space $\mathscr{T} = \Gamma_1 \times \dots \times \Gamma_d$. The space $\mathscr{T}$ has a natural affine structure induced by the cubes $\square_{\mathbf{e}} \simeq [0,1]^d$ for each $\mathbf{e} \in \mathscr{E}^{(n)} = E_1^{(n)} \times \dots \times E_d^{(n)}$. Define the space $\mathcal{C}^\infty_\Delta(\mathscr{T})$ as the space of functions $f : \mathscr{T} \to \mathbb{R}$ which are smooth on simplices of $\mathscr{G}$ [19]. This means, for any cube $\square_{\mathbf{e}} \simeq [0,1]^d$, the restriction of f to each triangle Δ of $[0,1]^d$ can be extended to a smooth function in a neighborhood of Δ.

For each $f \in \mathcal{C}^\infty_\Delta(\mathscr{T})$, denote by $f^{(n)}$ the piecewise affine function on $\mathscr{T}$ obtained by interpolating the values of f on the vertices to all the interior points of σ, on each simplex σ of $\mathscr{G}^{(n)}$.

The graphs $G_i^{(n)}$ are the dual graphs of a semi-stable model $\mathfrak{X}_i^{(n)}$ of $X_{K'}$ for an appropriate finite extension K' of K, and the simplicial set $\mathscr{G}^{(n)}$ corresponds to the dual complex of the Gross-Schoen desingularization of the product $\mathfrak{X}_1^{(n)} \times \dots \mathfrak{X}_d^{(n)}$. Looking at $\mathbb{R}$-Cartier divisors with support in the special fiber $\mathfrak{X}_s^{(n)}$ as real valued functions defined on the vertices of $\mathscr{G}^{(n)}$, the degree map in the ring $\mathrm{Chow}^{\mathfrak{X}^{(n)}}_{\mathfrak{X}_s}$ leads to a pairing $\langle f_0^{(n)}, \dots, f_d^{(n)}\rangle$ for any collection of functions $f_0, \dots, f_d \in \mathcal{C}^\infty_\Delta(\mathscr{T})$. With these preliminaries, combining our Theorem 1.8 with the results in [19], we get the following generalization of Equation (4).

Theorem 1.9 (Kolb [19])**.** *For any collection of functions $f_0, \dots, f_d \in \mathcal{C}^\infty_\Delta(\mathscr{T})$, the limit*

$$\langle f_0, \dots, f_d\rangle := \lim_{n\to\infty} \langle f_0^{(n)}, \dots, f_d^{(n)}\rangle$$

exists, and admits the following analytic development

$$\langle f_0, \dots, f_d\rangle = \sum_{\substack{\textit{partitions}\\ \mathcal{P} \textit{ of } [d]}} \frac{1}{2^{|\mathcal{P}|+d}} \sum_{\substack{\mathbf{w}_0,\dots,\mathbf{w}_d \in \mathbb{F}_2^d\\ \sum \alpha(\mathbf{w}_i,\mathcal{P}) = d+|\mathcal{P}|}} \deg\Big(\prod_{i=0}^d F_{\mathbf{w}_i}\Big) \int_{\mathrm{Diag}_{\mathcal{P}}} \prod_{i=0}^d D^{\mathbf{w}_i}_{\alpha(\mathbf{w}_i,\mathcal{P})}(f_i).$$

In the above formula the generalized diagonal $\mathrm{Diag}_{\mathcal{P}}$ is the union of the generalized diagonals $\mathrm{Diag}^{\mathbf{e}}_{\mathcal{P}}$ in the hypercubes $\square_{\mathbf{e}} \simeq \square^d = [0,1]^d$ consisting of all the points $(x_1, \dots, x_d) \in [0,1]^d$ which verify $x_i = x_j$ for all $i, j \in [d]$ belonging to the same element of the partition $\mathcal{P}$. The term $D^{\mathbf{w}_i}_{\alpha(\mathbf{w}_i, \mathcal{P})}(f_i)$ is a *partial derivative of* f_i *of order* $\alpha(\mathbf{w}_i, \mathcal{P})$ *in the direction of* $\mathbf{w}_i$ *and along the generalized diagonal* $\mathrm{Diag}_{\mathcal{P}}$. For example, for the partition $\mathcal{P}$ of $[d]$ into singletons, we have $\alpha(\mathbf{w}, \mathcal{P}) = |\mathbf{w}|$ for any $\mathbf{w} \in \mathbb{F}_2^d$, and on any cube $\square_{\mathbf{e}} \simeq [0,1]^d$, we have $D^{\mathbf{w}_i}_{\alpha(\mathbf{w}_i, \mathcal{P})} = (\frac{\partial}{\partial x_1})^{w_1} \dots (\frac{\partial}{\partial x_d})^{w_d}$. We omit the formal definition and refer to [19] for more details.

As previously mentioned, the case $d = 2$ in the above theorem was proved by Zhang in [23], and, was shown by him there to have interesting applications in arithmetic geometry (see [9]).

Finally, we refer to [8, 15] for a general approach to non-Archimedean Arakelov geometry using Berkovich theory and tropical geometry.

1.4. **Organization of the paper.** In Section 2, we give the formal definition of the simplicial structure of $\mathscr{G}$, and prove several basic properties of the Chow ring which will be used all through the paper. The structure theorem is proved in Section 3. The proof of the localization theorem is given in Section 3.1. In Section 4, we prove Theorem 1.6. Section 5 is devoted to the study of the structure of the Chow ring in the Fourier dual basis. In particular, the vanishing Theorem 1.8 is proved in that section.

2. Basic definitions and properties

In this section, we define the simplicial set structure on products of graphs, and prove basic results on the structure of the combinatorial Chow ring.

All through this section, by $G_1 = (V_1, E_1), \dots, G_d = (V_d, E_d)$ we denote d simple connected graphs. All graphs are finite.

2.1. **Simplicial set structure on the product of graphs.** We view G_i as a simplicial set of dimension one in a natural way. Suppose that for each $i = 1, \dots, d$, a total order $\leq_{G_i}$, or simply $\leq_i$ if there is no risk of confusion, on the vertices of G_i is fixed. We can endow the product $\mathscr{G} := G_1 \times \cdots \times G_d$ with a simplicial set structure of dimension d induced by orders $\leq_i$. This works as follows. The set of vertices (0-simplices) of $\mathscr{G}$ is $\mathscr{G}_0 = V_1 \times \cdots \times V_d$. For two vertices $\mathbf{v} = (v_1, \dots, v_d)$ and $\mathbf{u} = (u_1, \dots, u_d)$ in $\mathscr{G}_0$, we say $\mathbf{u} \leq \mathbf{v}$ if for any $i = 1, \dots, d$, we have $u_i \leq_i v_i$. A 1-simplex of $\mathscr{G}$ is a pair of vertices $\mathbf{u} = (u_1, \dots, u_d)$ and $\mathbf{v} = (v_1, \dots, v_d)$ in $\mathscr{G}_0$ such that $\mathbf{u} \leq \mathbf{v}$ and such that, in addition, for each $1 \leq i \leq d$, either $u_i = v_i$, or if $u_i <_i v_i$, then $\{u_i, v_i\}$ is an edge of G_i. The set of 1-simplices of $\mathscr{G}$ is denoted by $\mathscr{G}_1$. An element $\{\mathbf{u}, \mathbf{v}\} \in \mathscr{G}_1$ as above is *non-degenerate* if $\mathbf{u} \neq \mathbf{v}$. For any 1-simplex $\{\mathbf{u}, \mathbf{v}\}$ with $\mathbf{u} \leq \mathbf{v}$, denote by $I(\mathbf{u}, \mathbf{v})$ the set of all indices $i \in \{1, \dots, d\}$ with $u_i < v_i$. In particular, if $\mathbf{u} = \mathbf{v}$, we have $I(\mathbf{u}, \mathbf{v}) = \emptyset$.

More generally, for $k \in \mathbb{N}$, the set $\mathscr{G}_k$ of k-simplices of $\mathscr{G}$ is defined as follows. A k-simplex σ is a sequence $\mathbf{v}_0 \leq \cdots \leq \mathbf{v}_k$ of vertices in $\mathscr{G}_0$ such that for each $0 \leq j \leq k-1$, the pair $\{\mathbf{v}_j, \mathbf{v}_{j+1}\}$ belongs to $\mathscr{G}_1$, and in addition, the sets $I(\mathbf{v}_j, \mathbf{v}_{j+1})$ are all pairwise disjoint. We denote by $I(\sigma)$ the union of all the disjoint sets $I(\mathbf{v}_j, \mathbf{v}_{j+1})$, for $0 \leq j \leq k-1$.
We say σ is *non-degenerate* if we have $\mathbf{v}_0 < \mathbf{v}_1 < \cdots < \mathbf{v}_k$. The set of non-degenerate k-simplices of $\mathscr{G}$ is denoted by $\mathscr{G}_k^{nd}$.

Here is an alternative way to describe the simplicial structure of $\mathscr{G}$. First, for each $1 \le i \le d$, we orient the edges of G_i with respect to the total order $\le_i$ in such a way that any edge $\{u, v\} \in E_i$ gets orientation uv with $u <_i v$. By an abuse of the notation, we use as well E_i to denote the set of oriented edges of G_i given by the total order $\le_i$.

Let $\mathscr{E} = E_1 \times \dots \times E_d$, and for each $\mathbf{e} = (e_1, \dots, e_d) \in \mathscr{E}$, for oriented edges $e_1 \in E_1, \dots, e_d \in E_d$, denote by $\square_{\mathbf{e}}$ the product $e_1 \times \dots \times e_d$. We identify $\square_{\mathbf{e}}$ with the d-dimensional cube $\square^d$ with vertices $\{0, 1\}^d$ via the identification of each oriented edge $e_i = u_i v_i$ with $\{0, 1\}$, identifying thus u_i with 0 and v_i with 1. We endow the hypercube $\square^d$ with its standard simplicial structure. Namely, identify $\square^d$ with the vertex set of the hypercube $[0, 1]^d$, and for each element σ of the symmetric group $\mathfrak{S}_d$ of order d, define

$$\Delta_\sigma := \left\{ (x_1, \dots, x_d) \in [0, 1]^d \;\middle|\; 0 \le x_{\sigma(1)} \le \dots \le x_{\sigma(d)} \le 1 \right\}.$$

The non-degenerate d-simplices of $\square^d$ are the vertices of Δ_σ, for any element $\sigma \in \mathfrak{S}_d$.

Notation. All through the paper, we use bold letters $\mathbf{u}, \mathbf{v}, \mathbf{w},$ etc. to denote a 0-simplex in a product of graphs. For graphs $G_1, \dots, G_d$ with product $\mathscr{G}$, if $\mathbf{u}, \mathbf{v}, \mathbf{w}$, etc. is a vertex in $\mathscr{G}_0$, we use u_i, v_i, w_i, etc., respectively, to denote the corresponding vertex of the graph G_i, so we have $\mathbf{v} = (v_1, \dots, v_d), \mathbf{u} = (u_1, \dots, u_d), \mathbf{w} = (w_1, \dots, w_d),$ etc.

2.2. **Definition of the combinatorial Chow ring.** We recall the definition of the Chow ring given in the introduction, and use the opportunity to introduce a few useful notations. Denote by $Z(\mathscr{G})$ the free polynomial ring with coefficients in $\mathbb{Z}$ generated by the vertices of $\mathscr{G}$, namely,

$$Z(\mathscr{G}) := \mathbb{Z}[C_{\mathbf{v}} \,|\, \mathbf{v} \in \mathscr{G}_0],$$

where the variables $C_{\mathbf{v}}$ are associated to the vertices in $\mathscr{G}_0$. We view $Z(\mathscr{G})$ as a graded ring where each variable $C_{\mathbf{v}}$ is of degree one. For $k \in \mathbb{N}$, denote by $Z^k(\mathscr{G})$ the graded piece consisting of homogeneous polynomials of degree k.

Let $\mathscr{I}_{\mathrm{rat}}(\mathscr{G})$, or simply $\mathscr{I}_{\mathrm{rat}}$ if there is no risk of confusion, be the graded ideal of all the elements of $Z(\mathscr{G})$ which are rationally equivalent to zero: this is the (homogenous) ideal generated by the following generators

($\mathscr{R}$1) $C_{\mathbf{v}_1} C_{\mathbf{v}_2} \dots C_{\mathbf{v}_k}$ for $k \in \mathbb{N}$ and elements $\mathbf{v}_j \in \mathscr{G}_0$ such that $\mathbf{v}_1, \dots, \mathbf{v}_k$ do not form a simplex in $\mathscr{G}$;

($\mathscr{R}$2) $C_{\mathbf{u}} \Big(\sum_{\mathbf{v} \in \mathscr{G}_0} C_{\mathbf{v}} \Big)$ for any vertex $\mathbf{u} \in \mathscr{G}_0$; and

($\mathscr{R}$3) $C_{\mathbf{u}} C_{\mathbf{w}} \Big(\sum_{\mathbf{v} \in \mathscr{G}_0 : v_i = u_i} C_{\mathbf{v}} \Big)$ for any pair of vertices $\mathbf{u}, \mathbf{w} \in \mathscr{G}_0$ and any index $1 \le i \le d$ with $u_i \ne w_i$.

For two elements $\alpha, \beta \in Z(\mathscr{G})$, we write $\alpha \sim_{\mathrm{rat}} \beta$ iff $\alpha - \beta \in \mathscr{I}_{\mathrm{rat}}$.

The combinatorial Chow ring of $\mathscr{G}$ is the ring $\mathrm{Chow}_{\mathrm{GS}}(\mathscr{G}) := Z(\mathscr{G}) / \mathscr{I}_{\mathrm{rat}}$. It has a natural grading, and for $k \in \mathbb{N}$, we denote by $\mathrm{Chow}^k_{\mathrm{GS}}(\mathscr{G})$ the graded piece of degree k.

Remark 2.1. We mention here that there are other types of cohomological rings one can associate to a product of graphs, e.g., the Stanley ring of the product of graphs (with its natural cubical structure) [16], the tropical Chow ring of products of (metric) graphs [2, 20], the Chow ring of matroids [1, 11], and the tropical homology groups associated to tropical varieties [17]. It is not clear how these different groups are related. A complete description of the above Chow rings are not available beside the results proved in [1].

Remark 2.2. As it was mentioned before, the Chow ring $\mathrm{Chow}_{\mathrm{GS}}(\mathscr{G})$ comes with a map $\alpha_{\mathfrak{X}} : \mathrm{Chow}_{\mathrm{GS}}(\mathscr{G}) \to \mathrm{Chow}_{\mathfrak{X}_s}(\mathfrak{X})$ for the Gross-Schoen desingularization $\mathfrak{X}$ of a product of regular proper semi-stable curves $\mathfrak{X}_1, \dots, \mathfrak{X}_d$ over discrete valuation ring R where the dual graph of the special fiber of each $\mathfrak{X}_{i,s}$ is G_i. It seems natural to expect that under some genericity condition on the semi-stable curves $\mathfrak{X}_{i,s}$, and the regular smoothings $\mathfrak{X}_i$ of $\mathfrak{X}_{i,s}$, the ring $\mathrm{Chow}_{\mathrm{GS}}(\mathscr{G})$ becomes isomorphic to the subring $\mathrm{Chow}^c_{\mathfrak{X}_s}(\mathfrak{X})$ of the Chow ring $\mathrm{Chow}_{\mathfrak{X}_s}(\mathfrak{X})$ generated by the irreducible components of the special fiber of $\mathfrak{X}$.

It will be useful to introduce the following.

Definition 2.3 (The ideal $\mathscr{I}_1$)**.** Denote by $\mathscr{I}_1$ the ideal of the polynomial ring $Z(\mathscr{G}) = \mathbb{Z}[C_{\mathbf{v}} | \mathbf{v} \in \mathscr{G}_0]$ generated by the relations ($\mathscr{R}1$), i.e., by the products $C_{\mathbf{v}_1} \dots C_{\mathbf{v}_k}$ for any $k \in \mathbb{N}$ and $\mathbf{v}_1, \dots, \mathbf{v}_k \in \mathscr{G}_0$ which do not form a simplex.

2.3. **Functoriality.** Let $H_1, \dots, H_d$ be d simple connected graphs with orders $\leq_{H_i}$ on the vertices of each H_i. Define $\mathscr{H} = H_1 \times \dots \times H_d$ with the induced simplicial structure as described above. Suppose for each $i = 1, \dots, d$, a homomorphism of graphs $f_i : H_i \to G_i$ is given such that f_i respects also the two orderings $\leq_{H_i}$ and $\leq_{G_i}$, namely, for two vertices $u \leq_{H_i} v$ of H_i, we have $f(u) \leq_{G_i} f(v)$ in G_i. By the definition of the simplicial structure, the product of f_i leads to a morphism of simplicial sets $f : \mathscr{H} \to \mathscr{G}$. Moreover, f induces the map of graded rings $f^* : Z(\mathscr{G}) \to Z(\mathscr{H})$, which is defined on the level of generators by sending $C_{\mathbf{v}}$ for $\mathbf{v} \in \mathscr{G}_0$ to

$$f^*(C_{\mathbf{v}}) := \sum_{\substack{\mathbf{u} \in \mathscr{H}_0 \\ f(\mathbf{u}) = \mathbf{v}}} C_{\mathbf{u}}.$$

Proposition 2.4. *Notations as above, the map f^* sends $\mathscr{I}_{\mathrm{rat}}(\mathscr{G})$ to $\mathscr{I}_{\mathrm{rat}}(\mathscr{H})$ and induces a map of Chow rings $f^* : \mathrm{Chow}_{\mathrm{GS}}(\mathscr{G}) \to \mathrm{Chow}_{\mathrm{GS}}(\mathscr{H})$.*

Proof. We need to prove that generators of $\mathscr{I}_{\mathrm{rat}}(\mathscr{G})$ given by ($\mathscr{R}1$), ($\mathscr{R}2$) and ($\mathscr{R}3$) are sent to $\mathscr{I}_{\mathrm{rat}}(\mathscr{H})$.

Let $k \in \mathbb{N}$ and $\mathbf{v}_1, \dots, \mathbf{v}_k \in \mathscr{G}_0$ such that $\mathbf{v}_1, \dots, \mathbf{v}_k$ do not form a simplex in $\mathscr{G}$. Since $f : \mathscr{H} \to \mathscr{G}$ is a map of simplicial sets, it follows for any set of vertices $\mathbf{u}_1, \dots, \mathbf{u}_k \in \mathscr{H}_0$ with $f(\mathbf{u}_j) = \mathbf{v}_j$ for $j = 1, \dots, k$, the vertices $\mathbf{u}_1, \dots, \mathbf{u}_k$ do not form a simplex in $\mathscr{H}$. It follows that $f^*(C_{\mathbf{v}_1}) f^*(C_{\mathbf{v}_2}) \dots f^*(C_{\mathbf{v}_k}) \in \mathscr{I}_{\mathrm{rat}}(\mathscr{H})$.

Let now $\mathbf{v} \in \mathscr{G}_0$. We have

$$f^*\Big(C_{\mathbf{v}}\big(\sum_{\mathbf{w} \in \mathscr{G}_0} C_{\mathbf{w}}\big)\Big) = f^*(C_{\mathbf{v}})\Big(\sum_{\mathbf{w} \in \mathscr{G}_0} f^*(C_{\mathbf{w}})\Big) = f^*(C_{\mathbf{v}})\Big(\sum_{\mathbf{x} \in \mathscr{H}_0} C_{\mathbf{x}}\Big) \in \mathscr{I}_{\mathrm{rat}}(\mathscr{H}).$$

Finally, let $\mathbf{v}, \mathbf{w} \in \mathscr{G}_0$ and $i \in \{1, \dots, d\}$ such that $v_i \neq w_i$. We have

$$\begin{aligned} f^*\Big(C_{\mathbf{v}} C_{\mathbf{w}} \big(\sum_{\mathbf{z} \in \mathscr{G}_0 : z_i = v_i} C_{\mathbf{z}} \big)\Big) &= f^*(C_{\mathbf{v}}) f^*(C_{\mathbf{w}}) \big(\sum_{\mathbf{z} \in \mathscr{G}_0 : z_i = v_i} f^*(C_{\mathbf{z}}) \big) \\ &= \sum_{\substack{\mathbf{u} \in \mathscr{H}_0 \\ f(\mathbf{u}) = \mathbf{v}}} \sum_{\substack{\mathbf{x} \in \mathscr{H}_0 \\ f(\mathbf{x}) = \mathbf{w}}} C_{\mathbf{u}} C_{\mathbf{x}} \sum_{\substack{\mathbf{y} \in \mathscr{H}_0 \\ f_i(y_i) = v_i}} C_{\mathbf{y}} \\ &= \Big(\sum_{\substack{\mathbf{u} \in \mathscr{H}_0 \\ f(\mathbf{u}) = \mathbf{v}}} \sum_{\substack{\mathbf{x} \in \mathscr{H}_0 \\ f(\mathbf{x}) = \mathbf{w}}} C_{\mathbf{u}} C_{\mathbf{x}} \sum_{\substack{\mathbf{y} \in \mathscr{H}_0 \\ y_i = u_i}} C_{\mathbf{y}} \Big) \\ &\qquad + \Big(\sum_{\substack{\mathbf{u} \in \mathscr{H}_0 \\ f(\mathbf{u}) = \mathbf{v}}} \sum_{\substack{\mathbf{x} \in \mathscr{H}_0 \\ f(\mathbf{x}) = \mathbf{w}}} C_{\mathbf{u}} C_{\mathbf{x}} \sum_{\substack{\mathbf{y} \in \mathscr{H}_0 \\ f_i(y_i) = v_i \\ y_i \neq u_i}} C_{\mathbf{y}} \Big). \end{aligned}$$

For $\mathbf{u}, \mathbf{x} \in \mathscr{H}_0$ with $f(\mathbf{u}) = \mathbf{v}$ and $f(\mathbf{x}) = \mathbf{w}$, we have $u_i \neq x_i$. Therefore, we have $C_{\mathbf{u}} C_{\mathbf{x}} \sum_{\substack{\mathbf{y} \in \mathscr{H}_0 \\ y_i = u_i}} C_{\mathbf{y}} \in \mathscr{I}_{\mathrm{rat}}(\mathscr{H})$.

In addition, for such $\mathbf{u}, \mathbf{x} \in \mathscr{H}_0$, and for any $\mathbf{y} \in \mathscr{H}_0$ with $f_i(y_i) = v_i$ and $y_i \neq u_i$, since $v_i \neq w_i$, we have $y_i \neq x_i$. Thus, $C_{\mathbf{u}} C_{\mathbf{x}} C_{\mathbf{y}}$ is not a simplex in $\mathscr{H}$, and we have $C_{\mathbf{u}} C_{\mathbf{x}} C_{\mathbf{y}} \in \mathscr{I}_{\mathrm{rat}}(\mathscr{H})$. This shows that $f^*\Big(C_{\mathbf{v}} C_{\mathbf{w}} \big(\sum_{\mathbf{z} \in \mathscr{G}_0 : z_i = v_i} C_{\mathbf{z}} \big)\Big) \in \mathscr{I}_{\mathrm{rat}}(\mathscr{H})$, and the proposition follows. □

2.3.1. *Permutation of factors.* Let $\sigma \in \mathfrak{S}_d$ be an element of the permutation group of order d. Given simple graphs $G_1, G_2, \dots, G_d$, define $\mathscr{G}_\sigma := G_{\sigma(1)} \times \cdots \times G_{\sigma(d)}$, and denote by $\mathscr{V}_\sigma$ its vertex set. For any vertex $\mathbf{v} = (v_1, \dots, v_d) \in \mathscr{V}$, let $\mathbf{v}_\sigma := (v_{\sigma(1)}, \dots, v_{\sigma(d)})$. We have an isomorphic of polynomial rings $\eta_\sigma : Z(\mathscr{G}) \to Z(\mathscr{G}_\sigma)$ which sends the generator $C_{\mathbf{v}}$ to $C_{\mathbf{v}_\sigma}$. The following proposition is immediate.

Proposition 2.5. *Notations as above, the map η_σ induces an isomorphism of Chow rings* $\mathrm{Chow}_{\mathrm{GS}}(\mathscr{G}) \to \mathrm{Chow}_{\mathrm{GS}}(\mathscr{G}_\sigma))$.

2.4. **Intersection maps on the level of Chow groups.** For a graph $G = (V, E)$ with a total order $\leq$ on its vertex set V, and for any vertex v, we denote by $G[\leq v]$ (resp. $G[< v]$) the induced graph on the set of vertices $\{u \,|\, u \leq v\}$ (resp. $\{u \,|\, u < v\}$). Let $G_1, \dots, G_d$ be simples graphs, and let $\mathbf{v} = (v_1, \dots, v_d) \in \mathscr{V}$ with $v_i \in V(G_i)$, for $i = 1, \dots, d$. For each $1 \leq i \leq d$, define

$$H_i := \begin{cases} G_i[\leq_i v_i] & \text{if } i \neq k \\ G_k[< v_k] & \text{if } i = k, \end{cases}$$

and set $\mathscr{G}_{\mathbf{v},k} := H_1 \times \cdots \times H_d$. Denote by $\mathscr{V}_{\mathbf{v},k}$ the set of vertices of $\mathscr{G}_{\mathbf{v},k}$.

For each i, we have an inclusion $V(H_i) \subseteq V(G_i)$, which induces an inclusion $\mathscr{V}_{\mathbf{v},k} \subseteq \mathscr{V}$. Total orders $\leq_i$ induce total orders on the vertex set of each H_i, from which $\mathscr{G}_{\mathbf{v},k}$ inherits a simplicial structure, and the inclusion respects the simplicial structures. Thus, we can write $\mathscr{G}_{\mathbf{v},k} \subseteq \mathscr{G}$ as simplicial sets.

Consider the map of $\mathbb{Z}$-modules $\beta = \beta_{\mathbf{v},k} : \mathbb{Z}[\mathscr{V}_{\mathbf{v},k}] \to \mathbb{Z}[\mathscr{V}]$ defined by multiplication by $C_{\mathbf{v}}$

$$\forall\, i \in \mathbb{N} \;\; \forall\, \mathbf{w}_1, \dots, \mathbf{w}_i \in \mathscr{V}_{\mathbf{v},k}, \qquad \beta\big(C_{\mathbf{w}_1} C_{\mathbf{w}_2} \dots C_{\mathbf{w}_i}\big) := C_{\mathbf{w}_1} C_{\mathbf{w}_2} \dots C_{\mathbf{w}_i} C_{\mathbf{v}}.$$

We have

Proposition 2.6. *The map β induces a well-defined map of $\mathbb{Z}$-modules $\beta : \mathrm{Chow}_{\mathrm{GS}}(\mathscr{G}_{\mathbf{v},k}) \to \mathrm{Chow}_{\mathrm{GS}}(\mathscr{G})$.*

Proof. We will prove the three set of relations $(\mathscr{R}1), (\mathscr{R}2), (\mathscr{R}3)$ defining $\mathrm{Chow}_{\mathrm{GS}}(\mathscr{G}_{\mathbf{v},k})$ vanish by β in $\mathrm{Chow}_{\mathrm{GS}}(\mathscr{G})$, from which the result follows.

Using Proposition 2.5, and permuting factors if necessary, we can without loss of generality assume that $k = d$.

• $(\mathscr{R}1)$ If $\mathbf{w}_1, \dots, \mathbf{w}_i \in \mathscr{V}_{\mathbf{v},d}$ do not form a simplex in $\mathscr{G}_{\mathbf{v},d}$, then obviously, they do not form a simplex in $\mathscr{G}$, and we have

$$\beta(C_{\mathbf{w}_1} \dots C_{\mathbf{w}_i}) = C_{\mathbf{w}_1} \dots C_{\mathbf{w}_i} C_{\mathbf{v}} = 0 \text{ in } \mathrm{Chow}_{\mathrm{GS}}(\mathscr{G}).$$

• $(\mathscr{R}2)$ We show that for any $\mathbf{u} \in \mathscr{V}_{\mathbf{v},d}$, we have

$$\beta\big(C_{\mathbf{u}} \sum_{\mathbf{w} \in \mathscr{V}_{\mathbf{v},d}} C_{\mathbf{w}}\big) = 0 \qquad \text{in } \mathrm{Chow}_{\mathrm{GS}}(\mathscr{G}).$$

We have in the Chow ring $\mathrm{Chow}_{\mathrm{GS}}(\mathscr{G})$,

$$\begin{aligned}
\beta\big(C_{\mathbf{u}} \sum_{\mathbf{w} \in \mathscr{V}_{\mathbf{v},d}} C_{\mathbf{w}}\big) &= C_{\mathbf{u}}\big(\sum_{\mathbf{w} \in \mathscr{V}_{\mathbf{v},d}} C_{\mathbf{w}}\big) C_{\mathbf{v}} = C_{\mathbf{u}}\big(\sum_{\substack{\mathbf{z} \in \mathscr{V} \\ \mathbf{z} \leq \mathbf{v},\, z_d < v_d}} C_{\mathbf{z}}\big) C_{\mathbf{v}} \\
&= C_{\mathbf{u}}\big(\sum_{\substack{\mathbf{z} \in \mathscr{V} \\ \mathbf{z} \leq \mathbf{v},\, z_d = u_d}} C_{\mathbf{z}}\big) C_{\mathbf{v}} \text{ by } (\mathscr{R}1) \text{ since } u_d < v_d \text{ and } z_d < v_d \\
&= \big(\sum_{\substack{\mathbf{z} \in \mathscr{V} \\ z_d = u_d}} C_{\mathbf{z}}\big) C_{\mathbf{u}} C_{\mathbf{v}} \; - \; \big(\sum_{\substack{\mathbf{z} \in \mathscr{V} \\ \mathbf{z} \not\leq \mathbf{v},\, z_d = u_d < v_d}} C_{\mathbf{z}}\big) C_{\mathbf{u}} C_{\mathbf{v}} = 0.
\end{aligned}$$

In the last equation above, we have used the vanishing of the first term $\big(\sum_{\substack{\mathbf{z} \in \mathscr{V} \\ z_d = u_d}} C_{\mathbf{z}}\big) C_{\mathbf{u}} C_{\mathbf{v}} = 0$, implied by $(\mathscr{R}2)$ since $u_d \neq v_d$, and the vanishing for any $\mathbf{z} \not\leq \mathbf{v}$ with $z_d = u_d < v_d$ of the product $C_{\mathbf{z}} C_{\mathbf{u}} C_{\mathbf{v}}$, since in this case, $\mathbf{z}$ and $\mathbf{v}$ cannot form a simplex.

• $(\mathscr{R}3)$ We have to show that for all $j \in \{1, \dots, d\}$ and any $\mathbf{u}, \mathbf{w} \in \mathscr{V}_{\mathbf{v},d}$ with $u_j \neq w_j$, we have

$$\beta\big(C_{\mathbf{u}} C_{\mathbf{w}} \sum_{\substack{\mathbf{z} \in \mathscr{V}_{\mathbf{v},d} \\ z_j = u_j}} C_{\mathbf{z}}\,\big) = 0.$$

By $(\mathscr{R}1)$, we have $C_{\mathbf{u}} C_{\mathbf{w}} C_{\mathbf{z}} = 0$ for any $\mathbf{z}$ with $z_j \neq u_j, w_j$. It follows that

$$C_{\mathbf{u}} C_{\mathbf{w}} \sum_{\mathbf{z} \in \mathscr{V}_{\mathbf{v},d}} C_{\mathbf{z}} = C_{\mathbf{u}} C_{\mathbf{w}} \sum_{\substack{\mathbf{z} \in \mathscr{V}_{\mathbf{v},d} \\ z_j = u_j}} C_{\mathbf{z}} + C_{\mathbf{u}} C_{\mathbf{w}} \sum_{\substack{\mathbf{z} \in \mathscr{V}_{\mathbf{v},d} \\ z_j = w_j}} C_{\mathbf{z}}.$$

Since $\beta\big(C_{\mathbf{u}} C_{\mathbf{w}} \sum_{\mathbf{z} \in \mathscr{V}_{\mathbf{v},d}} C_{\mathbf{z}}\big) = 0$ in $\mathrm{Chow}_{\mathrm{GS}}(\mathscr{G})$, by $(\mathscr{R}2)$ that we just proved, we can assume further that $u_j < v_j$.

If $j = d$, then since $u_d, w_d < v_d$, and $u_d \neq w_d$, we have $\beta(C_{\mathbf{u}} C_{\mathbf{w}}) = C_{\mathbf{u}} C_{\mathbf{w}} C_{\mathbf{v}} = 0$ by $(\mathscr{R}1)$ in $\mathrm{Chow}_{\mathrm{GS}}(\mathscr{G})$, which directly gives the assertion.

So we can assume that $j \neq d$. We have

$$\begin{aligned}\beta\Big(C_{\mathbf{u}}C_{\mathbf{w}}\sum_{\substack{\mathbf{z}\in\mathscr{V}_{\mathbf{v},d}\\ z_j=u_j}}C_{\mathbf{z}}\Big) =& C_{\mathbf{u}}C_{\mathbf{w}}\Big(\sum_{\substack{\mathbf{z}\in\mathscr{V}_{\mathbf{v},d}\\ z_j=u_j}}C_{\mathbf{z}}\Big)C_{\mathbf{v}}\\ =& C_{\mathbf{u}}C_{\mathbf{w}}\Big(\sum_{\substack{\mathbf{z}\in\mathscr{V}\\ z_j=u_j}}C_{\mathbf{z}}\Big)C_{\mathbf{v}} - C_{\mathbf{u}}C_{\mathbf{w}}\Big(\sum_{\substack{\mathbf{z}\in\mathscr{V}\\ z_j=u_j\\ \mathbf{z}\not\leq\mathbf{v},\, z_d<v_d}}C_{\mathbf{z}}\Big)C_{\mathbf{v}}\\ &- C_{\mathbf{u}}C_{\mathbf{w}}\Big(\sum_{\substack{\mathbf{z}\in\mathscr{V}\\ z_j=u_j\\ \mathbf{z}\not\leq\mathbf{v},\, z_d=v_d}}C_{\mathbf{z}}\Big)C_{\mathbf{v}} - C_{\mathbf{u}}C_{\mathbf{w}}\Big(\sum_{\substack{\mathbf{z}\in\mathscr{V}\\ z_j=u_j\\ \mathbf{x}\not\leq\mathbf{v},\, z_d>v_d}}C_{\mathbf{z}}\Big)C_{\mathbf{v}}.\end{aligned}$$

Since $u_j < w_j$, by ($\mathscr{R}3$) in $\mathrm{Chow}_{\mathrm{GS}}(\mathscr{G})$, the first term in the above sum vanishes, i.e.,

$$C_{\mathbf{u}}C_{\mathbf{w}}\Big(\sum_{\substack{\mathbf{x}\in\mathscr{V}\\ x_j=u_j}}C_{\mathbf{x}}\Big)C_{\mathbf{v}} = 0.$$

For $\mathbf{x} \not\leq \mathbf{v}$ with $x_d < v_d$, $\mathbf{x}$ and $\mathbf{v}$ do not form a simplex, and so the second term in the sum is also zero, i.e.,

$$C_{\mathbf{u}}C_{\mathbf{w}}\Big(\sum_{\substack{\mathbf{z}\in\mathscr{V}\\ z_j=u_j\\ \mathbf{z}\not\leq\mathbf{v},\, z_d<v_d}}C_{\mathbf{z}}\Big)C_{\mathbf{v}} = 0.$$

Let now $\mathbf{z} \in \mathscr{V}$ with $\mathbf{z} \not\leq \mathbf{v}$ and $z_d = v_d$. Since $z_j = u_j < w_j$ and $z_d = v_d > w_d$, we infer that $\mathbf{z}$ and $\mathbf{w}$ do not form a simplex, and the third term in the sum vanishes as well, i.e.,

$$C_{\mathbf{u}}C_{\mathbf{w}}\Big(\sum_{\substack{\mathbf{z}\in\mathscr{V}\\ z_j=u_j\\ \mathbf{z}\not\leq\mathbf{v},\, z_d=v_d}}C_{\mathbf{z}}\Big)C_{\mathbf{v}} = 0.$$

As for the last term, we have $z_j = u_j < v_j$ and $z_d > v_d$, so $\mathbf{z}$ and $\mathbf{v}$ do not form a simplex, which gives

$$C_{\mathbf{u}}C_{\mathbf{w}}\Big(\sum_{\substack{\mathbf{z}\in\mathscr{V}\\ z_j=u_j\\ \mathbf{x}\not\leq\mathbf{v},\, z_d>v_d}}C_{\mathbf{z}}\Big)C_{\mathbf{v}} = 0.$$

Combining all this, we thus get $\beta\big(C_{\mathbf{u}}C_{\mathbf{w}}\sum_{\substack{\mathbf{z}\in\mathscr{V}_{\mathbf{v},d}\\ z_j=u_j}}C_{\mathbf{z}}\big) = 0$, and the proposition follows. $\square$

2.5. **Moving lemma.** The moving lemma for the Chow ring [18] is the statement that each graded piece $\mathrm{Chow}^{k+1}_{\mathrm{GS}}(\mathscr{G})$ is generated by monomials of the form $\prod_{\mathbf{v}\in\sigma}C_{\mathbf{v}}$, for $\sigma \in \mathscr{G}^{nd}_k$.

Theorem 2.7. *For any $k \in \mathbb{N}$, the Chow group $\mathrm{Chow}^{k+1}_{\mathrm{GS}}(\mathscr{G})$ is generated by monomials of the form $C_\sigma = C_{\mathbf{v}_0}\dots C_{\mathbf{v}_k}$, where σ is a non-degenerate simplex of dimension k in $\mathscr{G}$ with vertex set $\mathbf{v}_0,\dots,\mathbf{v}_k$.*

We give a proof of this theorem based on Proposition 2.9 below, which will be crucial later in the proof of the structure Theorem 1.5.

First we introduce some terminology. For any k-simplex τ of $\mathscr{G}$ with at least two different vertices, and for $i \in I(\tau)$ and $\epsilon \in \{0,1\}$, let $\{u_i, v_i\}$ be the corresponding edge of G_i with $u_i <_i v_i$, and define the element $R^\epsilon_{\tau,i}$ of $Z(\mathscr{G})$ by

$$R^0_{\tau,i} := \sum_{\substack{\mathbf{w}\in\mathscr{G}_0\\ w_i=u_i}} C_{\mathbf{w}}, \qquad \text{and} \qquad R^1_{\tau,i} := \sum_{\substack{\mathbf{w}\in\mathscr{G}_0\\ w_i=v_i}} C_{\mathbf{w}}.$$

Note that the product $C_\tau R^\epsilon_{\tau,i}$ is among the relations ($\mathscr{R}3$) and thus belongs to $\mathscr{I}_{\mathrm{rat}}$.

Definition 2.8. Let k, m be two natural numbers, and let $\tau \in \mathscr{G}^{nd}_k$ be a non-degenerate k-simplex. Define the set $\mathcal{A}_\tau(m)$ as the collection of all the multisets S of size m consisting of m (possibly equal) elements $(i_1,\epsilon_1),\dots,(i_m,\epsilon_m)$ with $i_1,\dots,i_m \in I(\tau)$ and $\epsilon_1,\dots,\epsilon_m \in \{0,1\}$.

With these notations, we have the following useful proposition.

Proposition 2.9. *Let $\sigma \in \mathscr{G}_k$ be a simplex with vertices $\mathbf{v}_0 \le \dots \le \mathbf{v}_k$ and with at least two distinct vertices. There exist an element $\beta \in \mathscr{I}_1$, and a collection of integers $a_{\tau,S} \in \mathbb{Z}$ for any $1 \le l \le k$, any $\tau \in \mathscr{G}^{nd}_l$ such that $\sigma \subseteq \tau$, and any $S \in \mathcal{A}_\tau(k-l)$, such that we have*

$$C_\sigma = \beta + \sum_{l=1}^{k} \sum_{\substack{\tau\in\mathscr{G}^{nd}_l\\ s.t.\, \sigma\subseteq\tau,\\ S\in\mathcal{A}_\tau(k-l)}} a_{\tau,S}\, C_\tau \prod_{(i,\epsilon)\in S} R^\epsilon_{\tau,i}.$$

Proof of Theorem 2.7. By Proposition 2.9, for any simplex $\sigma \in \mathscr{G}_k$ with at least two different vertices, we can write

$$C_\sigma = \beta + \sum_{l=1}^{k} \sum_{\substack{\tau\in\mathscr{G}^{nd}_l\\ S\in\mathcal{A}_\tau(k-l)}} a_{\tau,S}\, C_\tau \prod_{(i,\epsilon)\in X} R^\epsilon_{\tau,i},$$

for integers $a_{\tau,S}$ and $\beta \in \mathscr{I}_1 \subset \mathscr{I}_{\mathrm{rat}}$. For $l \neq k$, and any $S \in \mathcal{A}_\tau(k-l)$, the term $C_\tau \prod_{(i,\epsilon)\in S} R^\epsilon_{\tau,i}$ belongs to $\mathscr{I}_{\mathrm{rat}}$. It follows that

$$C_\sigma \sim_{\mathrm{rat}} \sum_{\tau\in G^{nd}_k} a_{\tau,\emptyset} C_\tau, .$$

Also note that for any integer $k \ge 2$ and any $\mathbf{u} \in \mathscr{G}_0$, we have by ($\mathscr{R}2$)

$$C^k_{\mathbf{u}} \sim_{\mathrm{rat}} - \sum_{\substack{\mathbf{v}\in\mathscr{G}_0\\ \{\mathbf{u},\mathbf{v}\}\in\mathscr{G}^{nd}_1}} C_{\mathbf{v}} C^{k-1}_{\mathbf{u}},$$

and so applying the previous case, it follows that all the monomials of degree k in $Z(\mathscr{G})$ are rationally equivalent to an integral linear combination of the monomials C_τ for $\tau \in \mathscr{G}^{nd}_k$, from which the theorem follows. □

Proof of Proposition 2.9. The proof goes by induction on k. For the base case $k = 1$, note that any 1-simplex with at least two distinct vertices is necessarily non-degenerate, and so the result trivially holds in this case. Let $k \ge 2$ be an integer, and assume the result holds for all k'-simplices with at least two distinct vertices for any $k' < k$. We prove it holds as well for any simplex $\sigma \in \mathscr{G}_k$ with at least two distinct vertices.

We proceed by a reverse induction on the number of different vertices of σ. If σ is non-degenerate, i.e., if it has $k+1$ distinct vertices, the result is obvious. Suppose that $2 \le l < k$, and the result holds for all simplices in $\mathscr{G}_k$ with at least $l+1$ distinct vertices. Let $\sigma \in \mathscr{G}_k$ with vertex set $\mathbf{v}_0 \le \mathbf{v}_1 \le \cdots \le \mathbf{v}_k$ such that the set $\{\mathbf{v}_0, \dots, \mathbf{v}_k\}$ is of size l. We prove the result for σ. Denote by $\mathbf{u}_1 < \mathbf{u}_2 < \cdots < \mathbf{u}_l$ all the different vertices of σ, and by $n_1, \dots, n_l$ the multiplicity of $\mathbf{u}_1, \dots, \mathbf{u}_l$ in σ, respectively. (I.e., the number of times each $\mathbf{u}_j$ appears among the vertices $\mathbf{v}_0, \dots, \mathbf{v}_k$ of σ.) We proceed by (a third) induction on the lexicographical order on ordered sequences $(n_1, \dots, n_l)$. Recall that for two ordered sequences $\mathbf{m} = (m_1, \dots, m_l)$ and $\mathbf{n} = (n_1, \dots, n_l)$, we have $\mathbf{m} \ge_{lex} \mathbf{n}$ if there exists $0 \le s \le l$ such that $m_{s+1} > n_{s+1}$, and $m_t \ge n_t$ for all $t \le s$.

Consider first the smallest ordered sequence $(n_1, \dots, n_l)$ in the lexicographical order, so that we have $n_1 = \cdots = n_{l-1} = 1$, and $n_l = k - l + 1 > 1$ (since $l < k$). Let $i \in I(\mathbf{u}_{l-1}, \mathbf{u}_l)$. There exists an element $\beta_0 \in \mathscr{I}_1$ such that we have

$$\begin{aligned} C_\sigma =& \beta_0 + C_{\mathbf{u}_1} \dots C_{\mathbf{u}_{l-1}} C_{\mathbf{u}_l}^{k-l+1} \\ =& \beta_0 + C_{\mathbf{u}_1} \dots C_{\mathbf{u}_{l-1}} C_{\mathbf{u}_l}^{n_l - 1} R^1_{\sigma,i} - \sum_{\substack{\mathbf{w} \in \mathscr{G}_0 \\ \mathbf{w} > u_l}} C_{\mathbf{u}_1} \dots C_{\mathbf{u}_{l-1}} C_{\mathbf{u}_l}^{n_l-1} C_{\mathbf{w}} \\ & - \sum_{\substack{\mathbf{w} \in \mathscr{G}_0 \\ \mathbf{u}_{l-1} < \mathbf{w} < \mathbf{u}_l \\ w_i = u_{l,i}}} C_{\mathbf{u}_1} \dots C_{\mathbf{u}_{l-1}} C_{\mathbf{w}} C_{\mathbf{u}_l}^{n_l - 1}. \end{aligned}$$

Each term $C_{\mathbf{u}_1} \dots C_{\mathbf{u}_{l-1}} C_{\mathbf{u}_l}^{n_l-1} C_{\mathbf{w}}$ in the above sum either belongs to $\mathscr{I}_1$ or is of the form C_τ for a k-simplex τ which has $l+1$ different vertices. Also the term $C_{\mathbf{u}_1} C_{\mathbf{u}_2} \dots C_{\mathbf{u}_l}^{n_l-1}$ is C_τ for a $(k-1)$-simplex τ with at least two distinct vertices. Thus the result follows by applying the induction hypothesis to each term appearing in the right hand side of the above equation.

By symmetry the same reasoning applies to the maximum ordered sequence $(n_1, \dots, n_l)$ in the lexicographical order which has $n_2 = \cdots = n_l = 1$.

Let now $\mathbf{n} = (n_1, \dots, n_l)$ be an arbitrary ordered sequence. We can assume that $\mathbf{n}$ is neither maximum nor minimum in the lexicographical order. Thus, there exists $1 < h \le l$ such that $n_h \ge 2$. Let $i \in I(\mathbf{u}_{h-1}, \mathbf{u}_h)$. Quite similarly as above, there exists $\beta_1 \in \mathscr{I}_1$ such that we have

$$\begin{aligned} C_\sigma = C_{\mathbf{u}_1}^{n_1} \dots C_{\mathbf{u}_h}^{n_h} \dots C_{\mathbf{u}_l}^{n_l} =& \beta_1 + C_{\mathbf{u}_1}^{n_1} \dots C_{\mathbf{u}_{h-1}}^{n_{h-1}} C_{\mathbf{u}_h}^{n_h - 1} \dots C_{\mathbf{u}_l}^{n_l} R^1_{\sigma,i} \\ & - \sum_{\substack{\mathbf{w} \in \mathscr{G}_0 \\ \mathbf{w} > u_h}} C_{\mathbf{u}_1}^{n_1} \dots C_{\mathbf{u}_{h-1}}^{n_{h-1}} C_{\mathbf{u}_h}^{n_h - 1} \dots C_{\mathbf{u}_l}^{n_l} C_{\mathbf{w}} \\ & - \sum_{\substack{\mathbf{w} \in \mathscr{G}_0 \\ \mathbf{u}_{h-1} < \mathbf{w} \le u_h \\ w_i = u_{h,i}}} C_{\mathbf{u}_1}^{n_1} \dots C_{\mathbf{u}_{h-1}}^{n_{h-1}} C_{\mathbf{u}_h}^{n_h - 1} \dots C_{\mathbf{u}_l}^{n_l} C_{\mathbf{w}}. \end{aligned}$$

The hypothesis of the induction applies to the first term as $C_{\mathbf{u}_1}^{n_1} \dots C_{\mathbf{u}_{h-1}}^{n_{h-1}} C_{\mathbf{u}_h}^{n_h-1} \dots C_{\mathbf{u}_l}^{n_l}$ has degree $k-1$. In the second term of the equation above, the induction hypothesis applies to each term $C_{\mathbf{u}_1}^{n_1} \dots C_{\mathbf{u}_{h-1}}^{n_{h-1}} C_{\mathbf{u}_h}^{n_h-1} \dots C_{\mathbf{u}_l}^{n_l} C_{\mathbf{w}}$ in the sum: if $\mathbf{w} = \mathbf{u}_j$ for some $j \ge h+1$ since the ordered sequence $(n_1, \dots, n_{h-1}, n_h - 1, \dots, n_j + 1, \dots, n_l)$ is smaller than $\mathbf{n}$ in the lexicographical order. If $\mathbf{w} \neq \mathbf{u}_j$ for all j, then the term is either in $\mathscr{I}_1$ or is of the form $C_{\sigma'}$ with $\sigma' \in \mathscr{G}_k$ with more than l distinct vertices.

Similarly, the hypothesis of the induction applies to each term $C_{\mathbf{u}_1}^{n_1} \dots C_{\mathbf{u}_{h-1}}^{n_{h-1}} C_{\mathbf{u}_h}^{n_h-1} \dots C_{\mathbf{u}_l}^{n_l} C_{\mathbf{w}}$ in the last sum, as each of those terms has $l+1$ distinct vertices. □

3. Proofs of Theorem 1.5 and Theorem 1.4

By Theorem 2.7, we have a surjective map $\mathbb{Z}\langle \mathscr{G}_k^{nd} \rangle \longrightarrow \mathrm{Chow}_{\mathrm{GS}}^{k+1}$. The structure Theorem 1.5 describes the kernel of this surjection.

First recall that for two elements $\mathbf{u} < \mathbf{v}$ of $\mathscr{G}_0$ such that $\{\mathbf{u}, \mathbf{v}\} \in \mathscr{G}_1$, we denote by $\mathbf{e}_{\mathbf{u},\mathbf{v}}$ the cube of dimension $|I(\mathbf{u}, \mathbf{v})|$ formed by all the vertices $\mathbf{z} = (z_1, \dots, z_d)$ in $\mathscr{G}_0$ with $z_i \in \{u_i, v_i\}$, i.e., $\mathbf{e}_{\mathbf{u},\mathbf{v}} = \prod_{i=1}^d \{u_i, v_i\}$.

Let σ be a k-simplex in $\mathscr{G}_k^{nd}$ with vertices $\mathbf{u}_0 < \mathbf{u}_1 < \dots < \mathbf{u}_k$. For two indices $1 \leq i, j \leq d$ lying both in $I(\mathbf{u}_t, \mathbf{u}_{t+1})$ for some $0 \leq t \leq k-1$, we defined

$$\widetilde{R}_{\sigma,i,j} := \sum_{\substack{\mathbf{w} \in \mathbf{e}_{\mathbf{u}_t, \mathbf{u}_{t+1}} \\ w_i = u_{t,i}}} C_{\mathbf{w}} - \sum_{\substack{\mathbf{w} \in \mathbf{e}_{\mathbf{u}_t, \mathbf{u}_{t+1}} \\ w_j = u_{t,j}}} C_{\mathbf{w}}.$$

Remark that we have

$$\begin{aligned} C_\sigma \widetilde{R}_{\sigma,i,j} = & \sum_{\substack{\mathbf{u}_t < \mathbf{w} < \mathbf{u}_{t+1} \\ \mathbf{w} \in \mathbf{e}_{\mathbf{u}_t, \mathbf{u}_{t+1}}, w_i = u_{t,i}}} C_{\mathbf{u}_0} \dots C_{\mathbf{u}_t} C_{\mathbf{w}} C_{\mathbf{u}_{t+1}} \dots C_{\mathbf{u}_k} \\ & - \sum_{\substack{\mathbf{u}_t < \mathbf{w} < \mathbf{u}_{t+1} \\ \mathbf{w} \in \mathbf{e}_{\mathbf{u}_t, \mathbf{u}_{t+1}}, w_j = u_{t,j}}} C_{\mathbf{u}_0} \dots C_{\mathbf{u}_t} C_{\mathbf{w}} C_{\mathbf{u}_{t+1}} \dots C_{\mathbf{u}_k}, \end{aligned}$$

and so $C_\sigma \widetilde{R}_{\sigma,i,j} \in \mathbb{Z}\langle \mathscr{G}_k^{nd} \rangle$. Define $\mathscr{I}_k^{nd}$ as the submodule of $\mathbb{Z}\langle \mathscr{G}_k^{nd} \rangle$ generated by all the elements $C_\sigma \widetilde{R}_{\sigma,i,j}$, for any σ, t, i, j as above.

For σ, t, i, j as above, we define

$$R_{\sigma,i,j} := R^0_{\sigma,i} - R^0_{\sigma,j} = \sum_{\substack{\mathbf{w} \in \mathscr{G}_0 \\ w_i = u_{t,i}}} C_{\mathbf{w}} - \sum_{\substack{\mathbf{w} \in \mathscr{G}_0 \\ w_i = u_{t,j}}} C_{\mathbf{w}}.$$

Note that we have $C_\sigma R_{\sigma,i,j} \sim_{\mathrm{rat}} 0$ by ($\mathscr{R}3$).

The following proposition is straightforward.

Proposition 3.1. *Notations as above, there exists $\beta \in \mathscr{I}_1$ such that we have $C_\sigma R_{\sigma,i,j} = \beta + C_\sigma \widetilde{R}_{\sigma,i,j}$.*

This shows that $\mathscr{I}_k^{nd} \subset \mathscr{I}_{\mathrm{rat}}$, and therefore, passing to the quotient, we get a surjection

$$\mathbb{Z}\langle \mathscr{G}_k^{nd} \rangle / \mathscr{I}_k^{nd} \longrightarrow \mathrm{Chow}_{\mathrm{GS}}^{k+1}.$$

In this section, we prove this map is injective, which implies Theorem 1.5.

Let $\alpha = \sum_{\tau \in \mathscr{G}_k^{nd}} a_\tau C_\tau \in \mathbb{Z}\langle \mathscr{G}_k^{nd} \rangle$, with $a_\tau \in \mathbb{Z}$ for all $\tau \in \mathscr{G}_k^{nd}$, be an element in the kernel, so we have $\alpha \simeq_{\mathrm{rat}} 0$. We shall prove that $\alpha \in \mathscr{I}_k^{nd}$.

Consider the graded piece $Z^{k+1}(\mathscr{G})$ consisting of all polynomials of homogenous degree $k+1$ with integral coefficients in variables $C_{\mathbf{v}}$ for $\mathbf{v} \in \mathscr{G}_0$. We define on $Z^{k+1}(\mathscr{G})$ a decreasing filtration $\mathcal{F}^\bullet: \mathcal{F}^{-1} = Z^{k+1}(\mathscr{G}) \supset \mathcal{F}^0 \supset \dots \supset \mathcal{F}^{k-1} \supset \mathcal{F}^k$ as follows.

Definition 3.2. Define $\mathcal{F}^{-1} := Z^{k+1}(\mathscr{G})$, $\mathcal{F}^0 := \mathscr{I}^{k+1}_{\mathrm{rat}}(\mathscr{G})$, and for each $1 \leq l \leq k-1$, define $\mathcal{F}^l$ as the set of all elements α which verify the following property: there exists an element $\beta \in \mathscr{I}_1$, and for each $l \leq t \leq k-1$, there are integers $a_{\tau,S}$ associated to any $\tau \in \mathscr{G}^{nd}_t$ and $S \in \mathcal{A}_\tau(k-t)$ such that we have

$$\alpha = \beta + \sum_{t=l}^{k-1} \sum_{\substack{\tau \in \mathscr{G}^{nd}_t \\ S \in \mathcal{A}_\tau(k-t)}} a_{\tau,S}\, C_\tau \prod_{(i,\epsilon)\in S} R^{\epsilon}_{\tau,i}.$$

Finally, define $\mathcal{F}^k := \mathscr{I}^{nd}_k$.

(Note that the inclusion $\mathcal{F}^0 \subset \mathcal{F}^1$ is implied from Proposition 3.1.)

Theorem 1.5 is a consequence of the following two lemmas.

Lemma 3.3. *Let $\alpha \in \mathbb{Z}\langle \mathscr{G}^{nd}_k \rangle$ so that $\alpha \sim_{\mathrm{rat}} 0$. Then we have $\alpha \in \mathcal{F}^{k-1}$.*

Lemma 3.4. *We have $\mathbb{Z}\langle \mathscr{G}^{nd}_k \rangle \cap \mathcal{F}^{k-1} = \mathcal{F}^k = \mathscr{I}^{nd}_k$.*

The rest of this section is devoted to the proof of these two lemmas.

Before giving the proof of Lemma 3.3, we introduce few extra notations, and state a useful proposition.

Let $1 \leq l \leq k-1$ be an integer. For any l-simplex $\tau \in \mathscr{G}^{nd}_l$ with vertices $\mathbf{u}_0 < \cdots < \mathbf{u}_l$, fix a subset J_τ of $I(\tau)$ of size l with the property that $\big|J_\tau \cap I(\mathbf{u}_j, \mathbf{u}_{j+1})\big| = 1$ for all $0 \leq j \leq l-1$.
Define the projection $\pi_\tau : I(\tau) \to J_\tau$ which projects the subset $I(\mathbf{u}_j, \mathbf{u}_{j+1}) \subset I(\sigma)$ to the unique element of the intersection $I(\mathbf{u}_j, \mathbf{u}_{j+1}) \cap J_\tau$.

For any set $S \in \mathcal{A}_\tau(k-l)$, define $\pi_\tau(S)$ as the multiset consisting of all the pairs $(\pi(i), \epsilon)$ for any pair $(i,\epsilon) \in S$. We have

Proposition 3.5. *Let $1 \leq l \leq k-1$ be an integer. For any $\tau \in \mathscr{G}^{nd}_l$ and any $S \in \mathcal{A}_\tau(k-l)$, we have*

$$C_\tau \prod_{(i,\epsilon)\in S} R^{\epsilon}_i - C_\tau \prod_{(i,\epsilon)\in \pi(S)} R^{\epsilon}_i \in \mathcal{F}^{l+1}.$$

Proof. Let $m = k-l$, and denote by $(i_1,\epsilon_1), \dots, (i_m,\epsilon_m)$ all the elements of S. Let j be the index with $i_1 \in I(\mathbf{u}_j, \mathbf{u}_j + 1)$. We can write

$$R^{\epsilon_1}_{\tau,i_1} = R^{\epsilon_1}_{\tau,i_1} - R^{\epsilon_1}_{\tau,\pi(i_1)} + R^{\epsilon_1}_{\tau,\pi(i_1)} = (-1)^{\epsilon_1} R_{\tau,i_1,\pi(i_1)} + R^{\epsilon_1}_{\tau,\pi(i_1)}.$$

By Proposition 3.1, we have

$$C_\tau R_{\tau,i_1,\pi(i_1)} = C_\tau \widetilde{R}_{\tau,i_1,\pi(i_1)} + \beta,$$

for some $\beta \in \mathscr{I}_1$. Since $C_\tau \widetilde{R}^{\epsilon_1}_{i_1,\pi(i_1)} \in \mathbb{Z}\langle \mathscr{G}^{nd}_{l+1} \rangle$, setting $S' := S \setminus \{(i_1,\epsilon_1)\}$, we infer that $C_\tau \widetilde{R}^{\epsilon_1}_{i_1,\pi(i_1)} \prod_{(i,\epsilon)\in S'} R^{\epsilon}_{\tau,i} \in \mathcal{F}^{l+1}$. Let $S_1 = S \cup \{(\pi(i_1), \epsilon)\} \setminus \{(i_1,\epsilon_1)\}$. Thus, since $\pi(\pi(i_1)) = \pi(i_1)$, we get

$$C_\tau \prod_{(i,\epsilon)\in S} R^{\epsilon}_{\tau,i} - C_\tau \prod_{(i,\epsilon)\in S_1} R^{\epsilon}_{\tau,i} =\in \mathcal{F}^{l+1}.$$

Proceeding by induction and applying the above reasoning to the set $S_t = S_{t-1} \cup \{(\pi(i_t), \epsilon_t)\} \setminus \{(i_t, \epsilon_t)\}$ and $(i_t, \epsilon_t) \in S_{t-1}$, for $t \geq 2$, we infer that for each t,

$$C_\tau \prod_{(i,\epsilon)\in S} R^\epsilon_{\tau,i} - C_\tau \prod_{(i,\epsilon)\in S_t} R^\epsilon_{\tau,i} \in \mathcal{F}^{l+1}.$$

For $t = k-l$, we have $S_{k-l} = \pi(S)$, and the proposition follows. □

The following proposition is a direct consequence of the definition of the simplicial structure on $\mathscr{G}$.

Proposition 3.6. *Let $\tau \in \mathscr{G}^{nd}_l$. For any $i \in I(\tau)$, we have*

$$C_\tau(\sum_{\mathbf{v}\in\mathscr{G}_0} C_{\mathbf{v}}) - C_\tau(R^0_{\tau,i} + R^1_{\tau,i}) \in \mathscr{I}_1.$$

With these preliminaries, we are ready to prove Lemma 3.3.

Proof of Lemma 3.3. Let $\alpha \in \mathbb{Z}\langle\mathscr{G}^{nd}_k\rangle$ be an element with $\alpha \in \mathcal{F}^0 = \mathscr{I}_{\mathrm{rat}}(\mathscr{G})$. Proceeding by induction, we will show that for any $1 \leq l \leq k-1$, we have $\alpha \in \mathcal{F}^l$; for $l = k-1$ we get the lemma.

For the base of our induction, we need to show that $\alpha \in \mathcal{F}^1$. Since $\alpha \sim_{\mathrm{rat}} 0$, by definition of the rational equivalence, there are elements $\beta \in \mathscr{I}_1$ and $\alpha_0 \in \mathbb{Z}(\mathscr{G})$, and for any $\sigma \in \mathscr{G}^{nd}_1$, and $i \in I(\sigma)$ and $\epsilon \in \{0,1\}$, there is an element $\alpha_{\sigma,i,\epsilon} \in \mathbb{Z}^{k-1}(\mathscr{G})$ such that we have

$$\alpha = \beta + \alpha_0(\sum_{\mathbf{w}\in\mathscr{G}_0} C_{\mathbf{w}}) + \sum_{\substack{\sigma\in\mathscr{G}^{nd}_1\\(i,\epsilon)\in\mathcal{A}_\sigma(1)}} \alpha_{\sigma,i,\epsilon} C_\sigma R^\epsilon_{\sigma,i}. \tag{5}$$

For any $\mathbf{u} \in \mathscr{G}_0$, we compare the coefficient of $C^{k+1}_{\mathbf{u}}$ on both sides of Equation (5). On the left hand side, the coefficient is zero since α has support in the non-degenerate simplices. Thus, the coefficient of $C^{k+1}_{\mathbf{u}}$ in the right hand side of the equality must be zero, and since all the monomials in the last sum have at least two distinct variables among $C_{\mathbf{v}}$, we infer that the coefficient of $C^k_{\mathbf{u}}$ in α_0 must be zero. Therefore, we can write $\alpha_0 = \beta_0 + \sum_{\sigma\in\mathscr{G}^{nd}_1} \gamma_\sigma C_\sigma$ for $\beta_0 \in \mathscr{I}_1$, and $\gamma_\sigma \in Z^{k-1}(\mathscr{G})$. For any $\sigma \in \mathscr{G}^{nd}_1$, picking an arbitrary $i_\sigma \in I(\sigma)$, and using Proposition 3.6, we decompose

$$C_\sigma \sum_{\mathbf{w}\in\mathscr{G}_0} C_{\mathbf{w}} = \beta_\sigma + C_\sigma R^0_{\sigma,i_\sigma} + C_\sigma R^1_{\sigma,i_\sigma},$$

for an element $\beta_\sigma \in \mathscr{I}_1$. Therefore, we have

$$\alpha_0(\sum_{\mathbf{w}\in\mathscr{G}_0} C_{\mathbf{w}}) = \beta'_0 + \sum_{\substack{\sigma\in\mathscr{G}^{nd}_1\\ \epsilon\in\{0,1\}}} \gamma_\sigma C_\sigma R^\epsilon_{\sigma,i_\sigma}.$$

for some $\beta'_0 \in \mathscr{I}_1$.

Thus, replacing β of Equation (5) with $\beta + \beta'_0$, and replacing $\alpha_{\sigma,i_\sigma,\epsilon}$ for each $\sigma \in \mathscr{G}^{nd}_1$ and $\epsilon \in \{0,1\}$ with $\alpha_{\sigma,i_\sigma,\epsilon} + \gamma_{\sigma,i_\sigma,\epsilon}$, we can ensure to have $\alpha_0 = 0$, and get

$$\alpha = \beta + \sum_{\substack{\sigma\in\mathscr{G}^{nd}_1\\(i,\epsilon)\in\mathcal{A}_\sigma(1)}} \alpha_{\sigma,i,\epsilon} C_\sigma R^\epsilon_{\sigma,i}. \tag{6}$$

Let $\sigma \in \mathscr{G}_1^{nd}$ and $(i,\epsilon) \in \mathcal{A}_\sigma(1)$. Each monomial in $\alpha_{\sigma,i,\epsilon}C_\sigma$ is either in $\mathscr{I}_1$ or has at least two distinct vertices. Thus, applying Proposition 2.9 to each of the monomial terms in $\alpha_{\sigma,i,\epsilon}C_\sigma$, for any $\sigma \in \mathscr{G}_1^{nd}$, and $(i,\epsilon) \in \mathcal{A}_\sigma(1)$, we finally infer the existence of $\beta_2 \in \mathscr{I}_1$, and for each $1 \le t \le k-1$, the existence of integers $a_{\tau,S}$ associated to $\tau \in \mathscr{G}_t^{nd}$ and $S \in \mathcal{A}_\tau(k-t)$, such that we can write

$$\alpha = \beta_2 + \sum_{t=1}^{k-1} \sum_{\substack{\tau\in\mathscr{G}_t^{nd}\\ S\in\mathcal{A}_\tau(k-t)}} a_{\tau,S}\, C_\tau \prod_{(i,\epsilon)\in S} R^\epsilon_{\tau,i}.$$

This shows that $\alpha \in \mathcal{F}^1$.

Assume now that we have $\alpha \in \mathcal{F}^l$ for an integer $1 \le l < k-1$. We shall prove that $\alpha \in \mathcal{F}^{l+1}$. By definition of $\mathcal{F}^l$, there is an element $\beta_l \in \mathscr{I}_1$, and for all $l \le t \le k-1$, there are integers $a^l_{\tau,S} \in \mathbb{Z}$ associated to $\tau \in \mathscr{G}_t^{nd}$ and $S \in \mathcal{A}_\tau(k-t)$, so that we have

$$\alpha = \beta_l + \sum_{t=l}^{k-1} \sum_{\substack{\tau\in\mathscr{G}_t^{nd}\\ S\in\mathcal{A}_\tau(k-t)}} a^l_{\tau,S}\, C_\tau \prod_{(i,\epsilon)\in S} R^\epsilon_{\tau,i}. \tag{7}$$

In order to prove $\alpha \in \mathcal{F}^{l+1}$, it will be enough to show that

$$\sum_{\substack{\tau\in\mathscr{G}_l^{nd}\\ S\in\mathcal{A}_\tau(k-l)}} a^l_{\tau,S}\, C_\tau \prod_{(i,\epsilon)\in S} R^\epsilon_{\tau,i} \in \mathcal{F}^{l+1}.$$

We show the following stronger statement.

Claim 3.7. *We have for any $\tau \in \mathscr{G}_l^{nd}$*

$$\sum_{S\in\mathcal{A}_\tau(k-l)} a^l_{\tau,S}\, C_\tau \prod_{(i,\epsilon)\in S} R^\epsilon_{\tau,i} \in \mathcal{F}^{l+1}.$$

Fix a non-degenerate l-simplex $\tau \in \mathscr{G}_l^{nd}$ with vertices $\mathbf{u}_0 < \cdots < \mathbf{u}_l$. Let $J_\tau \subset I(\tau)$ be the set of size l which intersects each $I(\mathbf{u}_j, \mathbf{u}_{j+1})$ in a unique element. By Proposition 3.5, we have

$$\sum_{S\in\mathcal{A}_\tau(k-l)} a^l_{\tau,S} C_\tau \prod_{(i,\epsilon)\in S} R^\epsilon_{\tau,i} - \sum_{S\in\mathcal{A}_\tau(k-l)} a^l_{\tau,S} C_\tau \prod_{(j,\epsilon)\in\pi(S)} R^\epsilon_{\tau,j} \in \mathcal{F}^{l+1}. \tag{8}$$

Denote by $i_0, \dots, i_{l-1}$ all the elements of J_τ with $i_j \in I(\mathbf{u}_j, \mathbf{u}_{j+1})$ for $j = 0, \dots, l-1$.

Proposition 3.6 implies that for any $0 \le j < l$, we have

$$C_\tau R^0_{\tau,i_j} + C_\tau R^1_{\tau,i_j} - C_\tau R^0_{\tau,i_l} - C_\tau R^1_{\tau,i_l} \in \mathscr{I}_1. \tag{9}$$

Let $\mathcal{B}_\tau(k-l)$ be the set of all mutisets S of size $k-l$ such that each element $(i,\epsilon) \in S$ belongs to the set of pairs $\{(i_0,0), (i_1,0), \dots, (i_{l-1},0), (i_{l-1},1)\}$.

Combining (8) and (9), we infer the existence of integers $b_{\tau,S}$ for any $S \in \mathcal{B}_\tau(k-l)$ so that we have

$$\sum_{S\in\mathcal{A}_\tau(k-l)} a^l_{\tau,S}\, C_\tau \prod_{(i,\epsilon)\in S} R^\epsilon_{\tau,i} - \sum_{S\in\mathcal{B}_\tau(k-l)} b_{\tau,S}\, C_\tau \prod_{(i,\epsilon)\in S} R^\epsilon_{\tau,i} \in \mathcal{F}^{l+1}. \tag{10}$$

Thus, Claim 3.7 will be a consequence of the following statement.

Claim 3.8. *For any $\tau \in \mathscr{G}_l^{nd}$ and any $S \in \mathcal{B}_\tau(k-l)$, we have $b_{\tau,S} = 0$.*

Each element in $\mathcal{B}_\tau(k-l)$ is given by an ordered sequence $\mathbf{n} = (n_0, \dots, n_{l-1}, n_l)$ of multiplicities of $(i_0, 0), \dots, (i_{l-1}, 0), (i_{l-1}, 1)$, respectively, with the property that $n_s \geq 0$, and $n_0 + \dots + n_l = k - l$. For such an ordered sequence $\mathbf{n}$, denote by $S_{\mathbf{n}}$ the element of $\mathcal{A}_\tau(k-l)$ associated to $\mathbf{n}$.

For $\tau \in \mathscr{G}_l^{nd}$ with vertices $\mathbf{u}_0 < \dots < \mathbf{u}_l$, consider in the sum on the right hand side of Equation (7), the sum of the monomials which are in the polynomial ring $\mathbb{Z}[C_{\mathbf{u}_0}, \dots, C_{\mathbf{u}_l}]$. This polynomial is precisely

$$C_\tau \sum_{\substack{\mathbf{n}=(n_0,\dots,n_l)\in \mathbb{Z}_{\geq 0}^{l+1}\\ n_0+\dots+n_l = k-l}} b_{\tau,S_{\mathbf{n}}} \left(C_{\mathbf{u}_0}\right)^{n_0}\left(C_{\mathbf{u}_0}+C_{\mathbf{u}_1}\right)^{n_1}\dots\left(C_{\mathbf{u}_0}+\dots+C_{\mathbf{u}_{l-1}}\right)^{n_{l-1}}\left(C_{\mathbf{u}_l}\right)^{n_l},$$

which must be thus vanishing since α is supported on non-degenerated simplices, and $k-l \geq 1$. Since we have an isomorphism of polynomials rings

$$\mathbb{Z}[C_{\mathbf{u}_0}, \dots, C_{\mathbf{u}_l}] \simeq \mathbb{Z}[C_{\mathbf{u}_0}, C_{\mathbf{u}_0}+C_{\mathbf{u}_1}, \dots, C_{\mathbf{u}_0}+\dots+C_{\mathbf{u}_{l-1}}, C_{\mathbf{u}_l}],$$

it follows that the coefficients $b_{\tau,S_{\mathbf{n}}}$ are all zero, which proves Claim 3.8, and finishes the proof of Lemma 3.3. □

We now prove Lemma 3.4, finishing the proof of Theorem 1.5

Proof of Lemma 3.4. Let $\alpha \in \mathbb{Z}\langle \mathscr{G}_k^{nd}\rangle \cap \mathcal{F}^{k-1}$. By definition of the filtration, we can write α in the form

$$\alpha = \beta + \sum_{\substack{\tau\in\mathscr{G}_{k-1}\\ (i,\epsilon)\in\mathcal{A}_\tau(1)}} a_{\tau,(i,\epsilon)} C_\tau R_{\tau,i}^{\epsilon}. \tag{11}$$

for some $\beta \in \mathscr{I}_1$, and integers $a_{\tau,(i,\epsilon)}$ associated to $\tau \in \mathscr{G}_{k-1}^{nd}$ and $(i,\epsilon) \in \mathcal{A}_\tau(1)$.

Let $\tau \in \mathscr{G}_{k-1}^{nd}$ with vertex set $\mathbf{u}_0 < \dots < \mathbf{u}_{k-1}$. For each $0 \leq j \leq k-2$, define

$$\rho_{\tau,j} = \sum_{i\in I(\mathbf{u}_j,\mathbf{u}_{j+1})} a_{\tau,(i,1)},$$

and for each $0 \leq j \leq k-2$, define

$$\ell_{\tau,j} = \sum_{i\in I(\mathbf{u}_j,\mathbf{u}_{j+1})} a_{\tau,(i,0)}.$$

For an integer $0 \leq s \leq k-1$, the coefficient of $C_\tau C_{\mathbf{u}_s}$ on the right hand side of Equation (11) is $\sum_{j=0}^{s-1}\rho_{\tau,j} + \sum_{j=s}^{k-2}\ell_{\tau,j}$, which, since $\alpha \in \mathbb{Z}\langle\mathscr{G}_{k-1}^{nd}\rangle$, must be zero. We infer that

$$0 \leq s \leq k-1, \qquad \sum_{j=0}^{s-1}\rho_{\tau,j} + \sum_{j=s}^{k-2}\ell_{\tau,j} = 0. \tag{12}$$

Subtracting these equations for the values of $s = j$ and $s = j+1$, we get in particular,

$$\forall\, 0 \leq j \leq k-2, \quad \rho_{\tau,j} = \ell_{\tau,j}.$$

Applying Proposition 3.6 to any $i \in I(\tau)$, we infer the existence of $\beta' \in \mathscr{I}_1$ such that we have

$$\begin{aligned}
\alpha &= \beta' + \sum_{i \in I(\tau)} \big(a_{\tau,(i,0)} - a_{\tau,(i,1)}\big) C_\tau R^0_{\tau,i} + \Big(\sum_{i \in I(\tau)} a_{\tau,(i,1)}\Big)\big(\sum_{\mathbf{w} \in \mathscr{G}_0} C_{\mathbf{w}}\big) \\
&= \beta' + \sum_{i \in I(\tau)} \big(a_{\tau,(i,0)} - a_{\tau,(i,1)}\big) C_\tau R^0_{\tau,i} + \big(\sum_{j=0}^{k-2} \rho_{\tau,j}\big)\big(\sum_{\mathbf{w} \in \mathscr{G}_0} C_{\mathbf{w}}\big) \\
&= \beta' + \sum_{i \in I(\tau)} \big(a_{\tau,(i,0)} - a_{\tau,(i,1)}\big) C_\tau R^0_{\tau,i} \quad \text{(by vanishing Equation (12) for } s = k-1\text{)}.
\end{aligned}$$

Let J_τ be the subset of $I(\tau)$ of size $k-1$ which intersects each interval $I(\mathbf{u}_t, \mathbf{u}_{t+1})$ in a single element. By Proposition 3.5, we can further write

$$\begin{aligned}
\alpha &= \beta' + \sum_{i \in I(\tau)} \big(a_{\tau,(i,0)} - a_{\tau,(i,1)}\big) C_\tau R^0_{\tau,i} \\
&= \beta'' + \sum_{i \in I(\tau)} \big(a_{\tau,(i,0)} - a_{\tau,(i,1)}\big) C_\tau \widetilde{R}_{\tau,i,\pi(i)} + \sum_{i \in I(\tau)} \big(a_{\tau,(i,0)} - a_{\tau,(i,1)}\big) R^0_{\tau,\pi(i)},
\end{aligned}$$

for an element $\beta'' \in \mathscr{I}_1$. For each $0 \leq j \leq k-2$, let i_j to be the unique element in the intersection $I(\tau) \cap J_\tau$, so that we have $\pi(I(\mathbf{u}_j, \mathbf{u}_{j+1})) = \{i_j\}$. We further get

$$\begin{aligned}
\alpha &= \beta'' + \sum_{i \in I(\tau)} \big(a_{\tau,(i,0)} - a_{\tau,(i,1)}\big) C_\tau \widetilde{R}_{\tau,i,\pi(i)} + \sum_{j=0}^{k-2} \sum_{i \in I(\mathbf{u}_j, \mathbf{u}_{j+1})} \big(a_{\tau,(i,0)} - a_{\tau,(i,1)}\big) R^0_{\tau,i_j} \\
&= \beta'' + \sum_{i \in I(\tau)} \big(a_{\tau,(i,0)} - a_{\tau,(i,1)}\big) C_\tau \widetilde{R}_{\tau,i,\pi(i)} + \sum_{j=0}^{k-2} \big(\ell_{\tau,j} - \rho_{\tau,j}\big) R^0_{\tau,i_j} \\
&= \beta'' + \sum_{i \in I(\tau)} \big(a_{\tau,(i,0)} - a_{\tau,(i,1)}\big) C_\tau \widetilde{R}_{\tau,i,\pi(i)} \qquad \text{(by the equality } \rho_{\tau,j} = \ell_{\tau,j}\text{)}.
\end{aligned}$$

We finally conclude that $\beta'' = 0$ and $\alpha \in \mathscr{I}^{nd}_k$, and the lemma follows. □

3.1. **Proof of the localization theorem.** We now prove the localization theorem 1.4. We retain the terminology from the previous sections. For $\mathbf{e} \in \mathscr{E} = E_1 \times \cdots \times E_d$, let $\square_{\mathbf{e}} = e_1 \times \cdots \times e_d$. Regarding each edge e_i as a subgraph of G with the induced total order from G_i on its vertices, and applying the functoriality to the inclusions $e_i \hookrightarrow G_i$, we get a map $\iota^*_{\mathbf{e}} : \mathrm{Chow_{GS}}(\mathscr{G}) \to \mathrm{Chow_{GS}}(\square_{\mathbf{e}}) \simeq \mathrm{Chow_{GS}}(\square^d)$ associated to the inclusion map of simplicial sets

$$\iota_{\mathbf{e}} : \square_{\mathbf{e}} \hookrightarrow \mathscr{G}.$$

By definition, the map $\iota^*_{\mathbf{e}}$ is identity on the generators associated to the vertices of $\square_{\mathbf{e}}$, and is zero otherwise. For an element $\alpha \in Z(\mathscr{G})$, and for $\mathbf{e} \in \mathscr{E}$, we denote by $\alpha_{|\mathbf{e}} \in Z(\square_{\mathbf{e}})$ the restriction of α to the hypercube $\square_{\mathbf{e}}$, i.e., $\alpha_{|\mathbf{e}} = \iota^*_{\mathbf{e}}(\alpha)$.

We need the following proposition which follows directly from the definition.

Proposition 3.9. *For any collection of connected subgraphs $H_1, \ldots, H_d$ of $G_1, \ldots, G_d$, respectively, let $\mathscr{H} = H_1 \times \cdots \times H_d$ with its induced simplicial structure. We have for any $1 \leq k \leq d$, $\mathscr{I}^{nd}_k(\mathscr{H}) \subseteq \mathscr{I}^{nd}_k(\mathscr{G})$.*

With these notations, we first prove the injectivity part.

Theorem 3.10. *The map of graded rings* $\mathrm{Chow}_{\mathrm{GS}}(\mathscr{G}) \to \prod_{\mathbf{e}\in\mathscr{E}} \mathrm{Chow}_{\mathrm{GS}}(\square_{\mathbf{e}})$ *is injective.*

Proof. Let $\beta \in \mathrm{Chow}^k_{\mathrm{GS}}(\mathscr{G})$ be an element such that $\beta_{|\mathbf{e}} = 0$ in $\mathrm{Chow}^k_{\mathrm{GS}}(\square_{\mathbf{e}})$ for all $\mathbf{e} \in \mathscr{E}$. We show that $\beta = 0$ in $\mathrm{Chow}_{\mathrm{GS}}(\mathscr{G})$. By Theorem 2.7, β is represented in $\mathrm{Chow}^k_{\mathrm{GS}}(\mathscr{G})$ by an element $\alpha \in \mathbb{Z}\langle \mathscr{G}^{nd}_k\rangle$. Consider the element α such that the number of hypercubes $\mathbf{e} \in \mathscr{E}$ with $\alpha_{|\mathbf{e}} = 0$ is maximized. We claim that $\alpha = 0$ which proves the theorem.

Suppose this is not the case, and consider a hypercube $\square_{\mathbf{e}}$ with $\alpha_{|\mathbf{e}} \neq 0$. Since $\alpha_{|\mathbf{e}}$ is zero in $\mathrm{Chow}_{\mathrm{GS}}(\square_{\mathbf{e}})$, by Theorem 1.5, we get $\alpha_{|\mathbf{e}} \in \mathscr{I}^{nd}_{k+1}(\square_{\mathbf{e}})$. Using the inclusion $\mathscr{I}^{nd}_{k+1}(\square_{\mathbf{e}}) \subseteq \mathscr{I}^{nd}_{k+1}(\mathscr{G})$, which follows from Proposition 3.9, we find that $\alpha_{|\mathbf{e}} \sim_{\mathrm{rat}} 0$ in $\mathscr{G}$. Setting $\alpha' = \alpha - \alpha_{|\mathbf{e}}$, it follows that $\alpha \sim_{\mathrm{rat}} \alpha'$ in $\mathscr{G}$. On the other hand, we have $\alpha'_{|\mathbf{e}} = 0$, and for any other $\mathbf{e}'$ with $\alpha_{|\mathbf{e}'} = 0$, we also have $\alpha'_{|\mathbf{e}'} = 0$. This contradicts the choice of α, and finishes the proof of the theorem. $\square$

We now move to the proof of the second part of the theorem.

Let $(\alpha_{\mathbf{e}})$ be a collection of elements in $Z^{k+1}(\square_{\mathbf{e}})$, for $\mathbf{e} \in \mathscr{E}$, such that $(\alpha_{\mathbf{e}})$ is in the kernel of the map j. It follows that for two hypercubes $\mathbf{e}$ and $\mathbf{e}'$ sharing a facet, we have $\alpha_{\mathbf{e}|\mathbf{e}'\cap\mathbf{e}} \sim_{\mathrm{rat}} \alpha_{\mathbf{e}'|\mathbf{e}'\cap\mathbf{e}}$ in $Z(\square_{\mathbf{e}\cap\mathbf{e}'})$. Using Theorem 2.7, and Proposition 2.4, we can assume that $\alpha_{\mathbf{e}} \in \mathbb{Z}\langle \square^{nd}_{\mathbf{e},k}\rangle$ for all $\mathbf{e} \in \mathscr{E}$. We show the existence of $\gamma_e \in \mathbb{Z}\langle \square^{nd}_{\mathbf{e},k}\rangle$ for any $\mathbf{e} \in \mathscr{E}$ such that

(*i*) for any $\mathbf{e} \in \mathscr{E}$, we have $\gamma_{\mathbf{e}} \sim_{\mathrm{rat}} \alpha_{\mathbf{e}}$ in $\square_{\mathbf{e}}$; and
(*ii*) for any two hypercubes $\mathbf{e}$ and $\mathbf{e}'$ sharing a facet, we have $\gamma_{\mathbf{e}|\mathbf{e}\cap\mathbf{e}'} = \gamma_{\mathbf{e}'|\mathbf{e}\cap\mathbf{e}'}$.

Assuming this, we get an element $\gamma \in \mathbb{Z}\langle \mathscr{G}^{nd}_k\rangle$ such that the class of $(\alpha_{\mathbf{e}})_{\mathbf{e}\in\mathscr{E}}$ in $\prod_{\mathbf{e}} \mathrm{Chow}_{\mathrm{GS}}(\square_{\mathbf{e}})$ is in the image of the restriction map $\mathrm{Chow}_{\mathrm{GS}}(\mathscr{G}) \to \prod_{\mathbf{e}\in\mathscr{E}} \mathrm{Chow}_{\mathrm{GS}}(\square_{\mathbf{e}})$ and the theorem follows.

Let $N = |\mathscr{E}|$, and enumerate all the elements of $\mathscr{E}$ as $\mathbf{e}_1, \dots, \mathbf{e}_N$. Define $\gamma_{\mathbf{e}_1} = \alpha_{\mathbf{e}_1}$. Proceeding inductively, for each $1 \leq l \leq N-1$, suppose that $\gamma_{\mathbf{e}_1}, \dots, \gamma_{\mathbf{e}_l}$ have been defined, and (*i*) and (*ii*) are verified for $\gamma_{\mathbf{e}_1}, \dots, \gamma_{\mathbf{e}_l}$. Consider the hypercube $\mathbf{e}_{l+1}$. Denote by δ_1 the restriction of $\alpha_{\mathbf{e}_{l+1}} - \gamma_{\mathbf{e}_1}$ to the intersection $\mathbf{e}_{l+1} \cap \mathbf{e}_1$. Then δ_1 is rationally equivalent to zero, and so by Theorem 1.5 and Proposition 3.9, belongs to $\mathscr{I}^{nd}_k(\square_{\mathbf{e}_{l+1}\cap\mathbf{e}_1}) \subset \mathscr{I}^{nd}_k(\mathscr{G})$. Set $\lambda_1 = \alpha_{\mathbf{e}_{l+1}} - \delta_1 \sim \alpha_{\mathbf{e}_{l+1}}$, and note that λ_1 and $\alpha_{\mathbf{e}_1}$ coincide on the intersection $\mathbf{e}_1 \cap \mathbf{e}_{l+1}$. Proceeding inductively, for $t = 1, \dots, l$, define δ_t as the restriction of $\lambda_{t-1} - \gamma_t$ to the intersection $\mathbf{e}_{l+1} \cap \mathbf{e}_t$, and note that $\delta_t \in \mathscr{I}^{nd}_k(\square_{\mathbf{e}_{l+1}\cap\mathbf{e}_t}) \subset \mathscr{I}^{nd}_k(\mathscr{G})$. Define $\lambda_{t+1} := \lambda_t - \delta_t$.

Defining $\gamma_{\mathbf{e}_{l+1}} := \lambda_{l+1}$, we get a collection of elements $\gamma_{\mathbf{e}_1}, \dots, \gamma_{\mathbf{e}_{l+1}}$ which verify (*i*) and (*ii*) above, and this completes the proof of the second part of Theorem 1.4.

4. Combinatorics of the degree map: Proof of Theorem 1.6

Let $\square^d = \{0,1\}^d$ be the d-dimensional hypercube with its standard simplicial structure, which is the d-fold product of the complete graph K_2 on two vertices $0 < 1$. After stating some results concerning the structure of the Chow ring $\mathrm{Chow}_{\mathrm{GS}}(\square^d)$, we prove Theorem 1.6.

First recall from Section 2 that the elements of $\square^d = \{0,1\}^d$ are the vertices of the hypercube $[0,1]^d$ in $\mathbb{R}^d$, and the non-degenerate d-simplices σ of $\square^d$ are in bijection with the elements ρ of the permutation group $\mathfrak{S}_d$, as follows: denoting by $\mathbf{e}_1, \dots, \mathbf{e}_d$ the standard basis of $\mathbb{R}^d$, the d-simplex σ_ρ associated to $\rho \in \mathfrak{S}_d$ has vertices $\mathbf{0}, \mathbf{e}_{\rho(1)}, \mathbf{e}_{\rho(1)} + \mathbf{e}_{\rho(2)}, \dots, \mathbf{e}_{\rho(1)} + \dots + \mathbf{e}_{\rho(d)}$. We have the following corollary of the structure Theorem 1.5.

Proposition 4.1. *For any two non-degenerate d-simplices $\sigma_1, \sigma_2 \in \square^d$, we have $C_{\sigma_1} = C_{\sigma_2}$ in $\mathrm{Chow}_{\mathrm{GS}}(\square^d)$, and the Chow group $\mathrm{Chow}_{\mathrm{GS}}^{d+1}(\square^d)$ is canonically isomorphic to $\mathbb{Z}$.*

Proof. Since by Theorem 2.7, the non-degenerate simplices generate the Chow ring, the second part of the proposition follows from the first part.

So let σ_1 and σ_2 be two non-degenerate d simplices of $\square^d$, and denote by ρ_1, ρ_2 the corresponding elements of $\mathfrak{S}_d$, respectively.

Writing $\rho_1^{-1}\rho_2$ as the product of the transpositions of the form $(i\,,\,i+1)$, for $1 \le i \le d-1$, it will be enough to prove the equality of C_{σ_1} and C_{σ_2} for $\rho_2 = \rho_1(i, i+1)$. Furthermore, using the action of $\mathfrak{S}_d$ on $\square^d$ via permutation of the factors and Proposition 2.5, we can further reduce to prove the equality of C_{σ_1} and C_{σ_2} for $\rho_1 = \mathrm{id}$ and $\rho_2 = (i, i+1)$.

In this case, the vertices of σ_1 are $\mathbf{v}_0 = \mathbf{0}$, and $\mathbf{v}_j = \mathbf{e}_1 + \dots + \mathbf{e}_j$, for $j \in [d]$, and the vertices of σ_2 are $\mathbf{u}_j$, $0 \le j \le d$ with $\mathbf{u}_j = \mathbf{v}_j$ for $j \neq i$, and $\mathbf{u}_i = \mathbf{e}_1 + \dots + \mathbf{e}_{i-1} + \mathbf{e}_{i+1}$. Let τ be the $d-2$-simplex of $\square^d$ with vertices $\mathbf{v}_j = \mathbf{u}_j$ for $0 \le j \le d$ and $j \neq i, i+1$. The vanishing of $C_{\sigma_1} - C_{\sigma_2}$ in $\mathrm{Chow}_{\mathrm{GS}}(\square^d)$ now follows by observing that $C_{\sigma_1} - C_{\sigma_2} = C_\tau \widetilde{R}_{\tau,i,i+1} \in \mathscr{I}_{\mathrm{rat}}$. ☐

We define the degree map

$$\deg : \mathrm{Chow}_{\mathrm{GS}}^{d+1}(\square^d) \to \mathbb{Z}$$

to be the canonical isomorphism of the above proposition.

Corollary 4.2. *For any collection $G_1 = (V_1, E_1), \dots, (G_d, E_d)$ of d simple connected graphs, we have $\mathrm{Chow}_{\mathrm{GS}}^{d+1}(\mathscr{G}) \simeq \mathbb{Z}^{|\mathscr{E}|}$.*

Proof. This follows from the localization Theorem 1.5 for $\mathrm{Chow}_{\mathrm{GS}}^{d+1}(\mathscr{G})$, the previous proposition, and the vanishing of the Chow group $\mathrm{Chow}_{\mathrm{GS}}^{d+1}(\square^{d-1})$. ☐

Definition 4.3 (Degree map)**.** For $\mathscr{G} = G_1 \times \dots G_d$ a d-fold product of simple connected graphs $G_1, \dots, G_d$, we define the *degree map* $\deg : \mathrm{Chow}_{\mathrm{GS}}^{d+1}(\mathscr{G}) \to \mathbb{Z}$ by

$$\deg(x) := \sum_{\mathbf{e} \in \mathscr{E}} \deg_{\mathbf{e}}(\iota_{\mathbf{e}}^*(x)),$$

for any $x \in \mathrm{Chow}_{\mathrm{GS}}^{d+1}(\mathscr{G})$, where $\deg_{\mathbf{e}}$ is the degree map of $\mathrm{Chow}_{\mathrm{GS}}^{d+1}(\square_{\mathbf{e}}) \simeq \mathrm{Chow}_{\mathrm{GS}}^{d+1}(\square^d)$.

4.1. **Intersection maps for the inclusion of hypercubes.** Let $k < d$ be two natural numbers, and $\mathbf{v} \in \square^d$ be an element of length $|\mathbf{v}| = k + 1 \le d$. Let $I = \{i_1, \dots, i_{k+1}\}$ be the support of $\mathbf{v}$, i.e., the subset of $[d]$ consisting of all the indices i with $v_i = 1$. The first k indices $i_1, \dots, i_k$ define an inclusion $\eta : \square^k \hookrightarrow \square^d$ which is given by sending $\mathbf{w} = (w_1, \dots, w_k) \in \square^k$ to the point $\mathbf{u} = \eta(\mathbf{w}) \in \square^d$ with $u_{i_j} = w_j$ for all $j = 1, \dots, k$, and $u_i = 0$ for all $i \notin \{i_1, \dots, i_k\}$.

Denote as before by K_2 the complete graph on two vertices $0 < 1$, and let K_1 be the complete graph on a unique vertex 0. Note that $K_1 = K_2[\le 0] = K_2[< 1]$ in the terminology of Section 2. We can view the hypercube $\square^k$ as the product of graphs $H_1, \dots, H_d$, with $H_i = K_2$ for $i = i_1, \dots, i_k$, and $H_i = K_1$ for all the other values of i. The cube $\square^d$ corresponds to the d-fold product of the complete graph K_2. In this way the map $\eta : \square^k \to \square^d$ corresponds to the inclusion map $H_1 \times \cdots \times H_d \hookrightarrow K_2 \times \cdots \times K_2$. Consider now the map of $\mathbb{Z}$-modules $\beta = \beta_{\mathbf{v}} : \mathbb{Z}[\square^k] \to \mathbb{Z}[\square^d]$ defined by multiplication by $C_{\mathbf{v}}$

$$\forall\, i \in \mathbb{N}\ \forall\, \mathbf{w}_1, \dots, \mathbf{w}_i \in \square^k, \qquad \beta\big(C_{\mathbf{w}_1} C_{\mathbf{w}_2} \dots C_{\mathbf{w}_i}\big) := C_{\eta(\mathbf{w}_1)} C_{\eta(\mathbf{w}_2)} \dots C_{\eta(\mathbf{w}_i)} C_{\mathbf{v}}.$$

As a special case of Proposition 2.6, we get the following useful proposition.

Proposition 4.4. *The map β induces a well-defined map of $\mathbb{Z}$-modules $\beta : \mathrm{Chow}_{\mathrm{GS}}(\square^k) \to \mathrm{Chow}_{\mathrm{GS}}(\square^d)$.*

With these preliminaries, we are ready to present the proof of Theorem 1.6.

4.2. **Proof of Theorem 1.6.** Let σ be any d-simplex of $\square^d$ with vertices $\mathbf{v}_1 < \mathbf{v}_2 < \cdots < \mathbf{v}_k$ where each vertex $\mathbf{v}_i$ has has multiplicity n_i in σ for numbers $n_i \geq 1$ with $\sum_i n_i = d + 1$, as in the theorem. As before, we set $C_\sigma = C_{\mathbf{v}_1}^{n_1} \dots C_{\mathbf{v}_k}^{n_k}$.

We first prove the vanishing result, namely part (1) of the theorem.

Claim 4.5. *Assume there exists an $1 \leq i < k$ with $n_1 + \cdots + n_i > |\mathbf{v}_{i+1}|$. Then $C_\sigma = 0$.*

Claim 4.6. *Assume there exists an $1 < i \leq k$ with $n_i + \cdots + n_k > d - |\mathbf{v}_{i-1}|$. Then $C_\sigma = 0$.*

We only prove Claim 4.5, as the proof of Claim 4.6 is similar, and follows by symmetry.

Proof of Claim 4.5. We proceed by a decreasing induction on $n_1 + \cdots + n_i + |\mathbf{v}_{i+1}|$. The base of our induction is the case $i = 1$, $|\mathbf{v}_2| = 1$, and $n_1 = 2$. Since $\mathbf{v}_1 < \mathbf{v}_2$, this means $\mathbf{v}_1 = \mathbf{0}$. It will be enough to prove that $C_{\mathbf{v}_1}^2 C_{\mathbf{v}_2} = 0$. Without loss of generality, and using Proposition 2.5, we can suppose that $\mathbf{v}_2 = \mathbf{e}_1$. By relation ($\mathscr{R}3$) in the Chow ring, we have

$$C_{\mathbf{v}_1} C_{\mathbf{v}_2} \sum_{\substack{\mathbf{v} \in \square^d \\ v_1 = 0}} C_{\mathbf{v}} = 0.$$

Since for all $\mathbf{v} \neq \mathbf{0}$ with $v_1 = 0$, by ($\mathscr{R}1$), we have $C_{\mathbf{v}} C_{\mathbf{v}_1} C_{\mathbf{v}_2} = 0$, we infer that $C_{\mathbf{v}_1}^2 C_{\mathbf{v}_2} = 0$.

Let $N \geq 3$ be an integer, and suppose that the vanishing $C_\sigma = 0$ holds for any σ verifying condition of the claim for an $1 \leq i < k$ such that $n_1 + \cdots + n_i + |\mathbf{v}_{i+1}| < N$. We show the vanishing $C_\sigma = 0$ holds for any σ verifying hypothesis of the claim for an $1 \leq i < k$ with $n_1 + \cdots + n_i + |\mathbf{v}_{i+1}| = N$.

The proof is divided into the following three cases, depending on whether $i = 1$, or $i \geq 2$ and $n_i \geq 2$, or $i \geq 2$ and $n_i = 1$.

• *Consider first the case* $i = 1$. Thus, we have $n_1 > |\mathbf{v}_2|$. For any $j \in I(\mathbf{v}_1, \mathbf{v}_2)$, we have $v_{1,j} = 0, v_{2,j} = 1$, and by relation ($\mathscr{R}3$) in the Chow ring, we get

$$\sum_{\substack{\mathbf{v} \in \square^d \\ v_j = 0}} C_{\mathbf{v}_1}^{n_1 - 1} C_{\mathbf{v}} C_{\mathbf{v}_2}^{n_2} \dots C_{\mathbf{v}_k}^{n_k} = 0, \text{ which in turn implies} \tag{13}$$

$$C_{\mathbf{v}_1}^{n_1} \dots C_{\mathbf{v}_k}^{n_k} + \sum_{\substack{\mathbf{v} \in \square^d \\ \mathbf{v}_1 < \mathbf{v} < \mathbf{v}_2\,,\, v_j = 0}} C_{\mathbf{v}_1}^{n_1 - 1} C_{\mathbf{v}} C_{\mathbf{v}_2}^{n_2} \dots C_{\mathbf{v}_k}^{n_k} + \sum_{\substack{\mathbf{v} \in \square^d \\ \mathbf{v} < \mathbf{v}_1\,,\, v_j = 0}} C_{\mathbf{v}_1}^{n_1 - 1} C_{\mathbf{v}} C_{\mathbf{v}_2}^{n_2} \dots C_{\mathbf{v}_k}^{n_k} = 0. \tag{14}$$

− For $\mathbf{v} \in \square^d$ with $\mathbf{v}_1 < \mathbf{v} < \mathbf{v}_2$, we have the vanishing of the product $C_{\mathbf{v}_1}^{n_1 - 1} C_{\mathbf{v}} C_{\mathbf{v}_2}^{n_2} \dots C_{\mathbf{v}_k}^{n_k}$ by the hypothesis of our induction. Indeed, in this case, we have $n_1 - 1 > |\mathbf{v}_2| - 1 \geq |\mathbf{v}|$ and $n_1 - 1 + |\mathbf{v}_2| < N$.

– For $\mathbf{v} \in \square^d$ with $\mathbf{v} < \mathbf{v}_1$, we again have $C_{\mathbf{v}} C_{\mathbf{v}_1}^{n_1-1} C_{\mathbf{v}_2}^{n_2} \dots C_{\mathbf{v}_k}^{n_k} = 0$ by the hypothesis of our induction, since we have $1 + n_1 - 1 = n_1 > |\mathbf{v}_2| > |\mathbf{v}_1|$ and $n_1 - 1 + |\mathbf{v}_2| < N$.

The only remaining term in Equation (14) is $C_{\mathbf{v}_1}^{n_1} \dots C_{\mathbf{v}_k}^{n_k}$ which must be thus zero.

• *Consider now the case $i \geq 2$ and $n_i \geq 2$.* The proof in this case is similar to the above situation. Namely, take an index $j \in I(\mathbf{v}_i, \mathbf{v}_{i+1})$ so that $v_{i,j} = 0$ and $v_{i+1,j} = 1$. Using the equation of type ($\mathscr{R}3$),

$$C_{\mathbf{v}_1}^{n_1} \dots C_{\mathbf{v}_{i-1}}^{n_{i-1}} C_{\mathbf{v}_i}^{n_i-1} \Big(\sum_{\mathbf{v} \in \square^d : v_j = 0} C_{\mathbf{v}} \Big) C_{\mathbf{v}_{i+1}}^{n_{i+1}} \dots C_{\mathbf{v}_k}^{n_k} = 0,$$

we see as above that all the terms $\mathbf{v} \neq \mathbf{v}_i$ with $v_j = 0$ contribute zero to the above sum, i.e., $C_{\mathbf{v}_1}^{n_1} \dots C_{\mathbf{v}_{i-1}}^{n_{i-1}} C_{\mathbf{v}_i}^{n_i-1} C_{\mathbf{v}} C_{\mathbf{v}_{i+1}}^{n_{i+1}} \dots C_{\mathbf{v}_k}^{n_k} = 0$, either by ($\mathscr{R}1$) if the corresponding sequence does not form a simplex, or by the induction hypothesis as in the previous case. The only remaining term in the sum above is for $\mathbf{v} = \mathbf{v}_i$, and it follows that $C_{\mathbf{v}_1}^{n_1} \dots C_{\mathbf{v}_k}^{n_k} = 0$.

• *Finally, consider the case $i \geq 2$ and $n_i = 1$.* In this case, we have $n_1 + \dots + n_{i-1} = n_1 + \dots + n_i - 1 > |\mathbf{v}_{i+1}| - 1 \geq |\mathbf{v}_{i-1}|$ and by the hypothesis of our induction, we again have $C_{\mathbf{v}_1}^{n_1} \dots C_{\mathbf{v}_k}^{n_k} = 0$. $\square$

We now turn to the proof of the second part of the theorem. So suppose there is no $1 \leq i < k$ with $n_1 + \dots + n_i > |\mathbf{v}_{i+1}|$, and there is no $2 \leq i \leq k$ with $n_i + \dots + n_k > d - |\mathbf{v}_{i-1}|$. Since $\sum n_i = d + 1$, this means that for all $1 \leq i \leq k - 1$, we have

$$|\mathbf{v}_i| + 1 \leq n_1 + \dots + n_i \leq |\mathbf{v}_{i+1}|,$$

and, obviously, $|\mathbf{v}_k| + 1 \leq n_1 + \dots + n_k = d + 1$. We first show the existence of the sequence x_i, y_i verifying the properties stated in the theorem. Let $y_0 = |v_1|$, and define x_i, y_i, for $i = 1, \dots, k$, as follows:

$$x_i := n_1 + \dots + n_i - |v_i| - 1, \quad \text{and} \quad y_i := |v_{i+1}| - n_1 - \dots - n_i.$$

Note that $x_i, y_i \geq 0$ for all i, and $|\mathbf{v}_i| = |\mathbf{v}_{i-1}| + x_i + y_i + 1$ for $2 \leq i \leq k$, and $n_i = y_{i-1} + x_i + 1$ for $1 \leq i \leq k$. Thus x_i, y_i verify the three conditions stated in part (2) of the theorem.

Claim 4.7. *With the above notations, we have*

$$\deg(C_\sigma) = (-1)^{d+1-k} \binom{y_0 + x_1}{y_0} \binom{x_1 + y_1}{x_1} \binom{y_1 + x_2}{y_1} \dots \binom{x_{k-1} + y_{k-1}}{y_{k-1}} \binom{y_{k-1} + x_k}{x_k}.$$

We need to prove some preliminary results, which are all special cases of the above claim.

Proposition 4.8. *We have $C_{\mathbf{1}}^{d+1} = C_{\mathbf{0}}^{d+1} = (-1)^d$.*

Proof. By symmetry, we will only need to show $C_{\mathbf{1}}^{d+1} = (-1)^d$. Proceeding by induction, we show that for any $0 \leq i \leq d - 1$, we have

$$C_{\mathbf{1}}^{d+1} = (-1)^{i+1} C_{\mathbf{0}} C_{\mathbf{e}_1} C_{\mathbf{e}_1+\mathbf{e}_2} \dots C_{\mathbf{e}_1+\dots+\mathbf{e}_i} C_{\mathbf{1}}^{d-i},$$

which, for $i = d - 1$, gives the equality $C_{\mathbf{1}}^{d+1} = (-1)^d$, as required. (For $i = 0$, this means $C_{\mathbf{1}}^{d+1} = -C_{\mathbf{0}} C_{\mathbf{1}}^d$.)

First note that by ($\mathscr{R}2$), we have $\big(\sum_{\mathbf{v}\in\square^2} C_{\mathbf{v}}\big)C_{\mathbf{1}}^d = 0$, which implies that

$$C_{\mathbf{1}}^{d+1} = -C_{\mathbf{0}}C_{\mathbf{1}}^d - \sum_{\mathbf{v}\neq\mathbf{0},\mathbf{1}} C_{\mathbf{v}}C_{\mathbf{1}}^d.$$

By vanishing part of Theorem 1.6, that we established in Claims 4.5 and 4.6, we have $C_{\mathbf{v}}C_{\mathbf{1}}^d = 0$ for all $\mathbf{v}\neq\mathbf{0},\mathbf{1}$. This gives

$$C_{\mathbf{1}}^{d+1} = -C_{\mathbf{0}}C_{\mathbf{1}}^d.$$

Suppose that we have already proved for an $0\leq i<d-1$, that

$$C_{\mathbf{1}}^{d+1} = (-1)^{i+1}C_{\mathbf{0}}C_{\mathbf{e}_1}C_{\mathbf{e}_1+\mathbf{e}_2}\dots C_{\mathbf{e}_1+\dots+\mathbf{e}_i}C_{\mathbf{1}}^{d-i}.$$

By relation ($\mathscr{R}3$) in the Chow ring, we have

$$C_{\mathbf{0}}C_{\mathbf{e}_1}C_{\mathbf{e}_1+\mathbf{e}_2}\dots C_{\mathbf{e}_1+\dots+\mathbf{e}_i}\Big(\sum_{\mathbf{v}\in\square^d,v_{i+1}=1} C_{\mathbf{v}}\Big)C_{\mathbf{1}}^{d-i-1} = 0. \tag{15}$$

For any $\mathbf{v}\in\square^d$ with $v_{i+1}=1$, if $\mathbf{e}_1+\dots+\mathbf{e}_i\not<\mathbf{v}$, by the definition of the simplicial structure of $\square^d$ and the relation ($\mathscr{R}1$), we get $C_{\mathbf{0}}C_{\mathbf{e}_1}C_{\mathbf{e}_1+\mathbf{e}_2}\dots C_{\mathbf{e}_1+\dots+\mathbf{e}_i}C_{\mathbf{v}}C_{\mathbf{1}}^{d-i-1}=0$. On the other hand, for any $\mathbf{e}_1+\dots+\mathbf{e}_i+\mathbf{e}_{i+1}<\mathbf{v}<\mathbf{1}$ with $v_{i+1}=1$, by applying the vanishing criterium of Theorem 1.6, we get $C_{\mathbf{0}}C_{\mathbf{e}_1}C_{\mathbf{e}_1+\mathbf{e}_2}\dots C_{\mathbf{e}_1+\dots+\mathbf{e}_i}C_{\mathbf{v}}C_{\mathbf{1}}^{d-i-1}=0$. Indeed, in this situation, we have $|\mathbf{v}|\geq i+2$, which gives the inequality $d-i-1>d-|\mathbf{v}|$ as required in Claim 4.6. It follows from these observations and Equation (15) that

$$C_{\mathbf{0}}C_{\mathbf{e}_1}C_{\mathbf{e}_1+\mathbf{e}_2}\dots C_{\mathbf{e}_1+\dots+\mathbf{e}_i}C_{\mathbf{1}}^{d-i} = -C_{\mathbf{0}}C_{\mathbf{e}_1}C_{\mathbf{e}_1+\mathbf{e}_2}\dots C_{\mathbf{e}_1+\dots+\mathbf{e}_{i+1}}C_{\mathbf{1}}^{d-i-1},$$

and the lemma follows. $\square$

Let now $\mathbf{v}\in\square^d$ be any element of length $1<\ell=|\mathbf{v}|<d$ with support the subset $I\subset[d]$. Let $i_1\in I$ and $i_2\notin I$ be two elements of $[d]$, and define $I_1:=I\setminus\{i_1\}$ and $I_2:=I^c\setminus\{i_2\}$.

Let $m\in\mathbb{N}$ be an integer, and suppose $\mathbf{w}_1,\dots,\mathbf{w}_m$ are vertices of $\square^d$ with support in I_1. Consider an element $\alpha_1=C_{\mathbf{w}_1}^{a_1}\dots C_{\mathbf{w}_m}^{a_m}$ for $a_i\in\mathbb{N}$ with $a_1+\dots+a_m=\ell$.

Similarly, let $t\in\mathbb{N}$ be an integer, and suppose $\mathbf{z}_1,\dots,\mathbf{z}_t\in\square^d$ are such that for each $1\leq j\leq t$, $I\cup\{i_2\}\subseteq\mathrm{support}(\mathbf{z}_j)$. Consider an element of the form $\alpha_2=C_{\mathbf{z}_1}^{b_1}\dots C_{\mathbf{z}_t}^{b_t}$ for $b_j\in\mathbb{N}$ with $b_1+\dots+b_t=d-\ell$.

We write $\alpha_1|_{I_1}$ for the element of $\mathrm{Chow}_{\mathrm{GS}}(\square^{\ell-1})$ of degree ℓ obtained by viewing $\mathbf{w}_1,\dots,\mathbf{w}_m$ in $\square^{\ell-1}$ (keeping only the coordinates in I_1) and keeping the exponents $a_1,\dots,a_m$. Similarly, we write $\alpha_2|_{I_2}$ for the element of $\mathrm{Chow}_{\mathrm{GS}}(\square^{d-\ell-1})$ of degree $d-\ell$ obtained by restricting $\mathbf{z}_j$ to I_2, and keeping the exponents $b_1,\dots,b_t$.

Proposition 4.9. *Notations as above, we have*

$$\deg(\alpha_1C_{\mathbf{v}}\alpha_2) = \deg(\alpha_1|_{I_1})\deg(\alpha_2|_{I_2}).$$

Proof. Choose $\mathbf{v}_0=\mathbf{0}<\dots<\mathbf{v}_{\ell-1}<\mathbf{v}$ with support of $\mathbf{v}_i$ included in I_1 for each $1\leq i\leq\ell-1$. It follows from Proposition 4.4 and the fact that $\mathrm{Chow}_{\mathrm{GS}}(\square^{\ell-1})$ is one dimensional in degree ℓ generated by $C_{\mathbf{v}_0}\dots C_{\mathbf{v}_{\ell-1}}C_{\mathbf{v}}$ that

$$C_{\mathbf{w}_1}^{n_1}\dots C_{\mathbf{w}_m}^{n_m}C_{\mathbf{v}} = \deg(\alpha_1|_{I_1})C_{\mathbf{v}_0}C_{\mathbf{v}_1}\dots C_{\mathbf{v}_{\ell-1}}C_{\mathbf{v}}.$$

Similarly, chose $\mathbf{v}_{\ell+1} < \cdots < \mathbf{v}_d$ such that $I \cup \{i_2\}$ is included in the support of $\mathbf{v}_{\ell+1}$. We have

$$C_{\mathbf{v}} C_{\mathbf{z}_1}^{b_1} \dots C_{\mathbf{z}_t}^{b_t} = \deg(\alpha_2|_{I_2}) C_{\mathbf{v}} C_{\mathbf{v}_{\ell+1}} \dots C_{\mathbf{v}_d}.$$

We infer that

$$\begin{aligned}\alpha_1 C_{\mathbf{v}} \alpha_2 &= \deg(\alpha_1|_{I_1}) C_{\mathbf{v}_0} \dots C_{\mathbf{v}_{\ell-1}} C_{\mathbf{v}} \alpha_2 \\ &= \deg(\alpha_1|_{I_1}) \deg(\alpha_2|_{I_2}) C_{\mathbf{v}_0} C_{\mathbf{v}_1} \dots C_{\mathbf{v}_{\ell-1}} C_{\mathbf{v}} C_{\mathbf{v}_{\ell+1}} \dots C_{\mathbf{v}_d},\end{aligned}$$

from which the result follows. □

The previous proposition allows to prove the following generalization of Proposition 4.8.

Proposition 4.10. *For any $\mathbf{v} \in \square^d$, we have $C_{\mathbf{v}}^{d+1} = (-1)^d \binom{d}{|\mathbf{v}|}$.*

Proof. We proceed by induction on d. The case $d = 1$ trivially holds. Suppose the statement holds for $d-1$. Let now $\mathbf{v} \in \square^d$ be an element with $|\mathbf{v}| = \ell$. We can suppose that $\mathbf{v} \neq \mathbf{0}, \mathbf{1}$, since we already treated these cases. We have by $(\mathscr{R}2)$

$$\Big(\sum_{\mathbf{w} \in \square^d} C_{\mathbf{w}}\Big) C_{\mathbf{v}}^d = 0. \tag{16}$$

For all $\mathbf{w} \neq \mathbf{0}, \mathbf{1}, \mathbf{v}$, we have by applying either $(\mathscr{R}1)$ or by the vanishing criterium of Theorem 1.6, that $C_{\mathbf{w}} C_{\mathbf{v}}^d = 0$. Therefore, from Equation (16) we get

$$C_{\mathbf{v}}^{d+1} = -C_{\mathbf{0}} C_{\mathbf{v}}^d - C_{\mathbf{v}}^d C_{\mathbf{1}}.$$

Let $i \in \{1, \dots, d\}$ be an index with $v_i = 1$, and define $I := [d] \setminus \{i\}$. By Proposition 4.4, we have $\deg(C_{\mathbf{0}} C_{\mathbf{v}}^d) = \deg\big((C_{\mathbf{v}}|_I)^d\big)$, which applying the hypothesis of the induction, gives

$$\deg(C_{\mathbf{0}} C_{\mathbf{v}}^d) = \deg\big((C_{\mathbf{v}}|_I)^d\big) = (-1)^{d-1} \binom{d-1}{|\mathbf{v}|-1}.$$

Similarly, let j be an index with $v_j = 0$, and $J = [d] \setminus \{j\}$. We have

$$\deg(C_{\mathbf{v}}^d C_{\mathbf{1}}) = \deg(C_{\mathbf{v}}|_J^d) = (-1)^{d-1} \binom{d-1}{|\mathbf{v}|}.$$

The result now follows from the standard binomial identity $\binom{d}{|v|} = \binom{d-1}{|v|} + \binom{d-1}{|v|-1}$. □

We are ready to prove Claim 4.7, and complete the proof of Theorem 1.6.

Proof of Claim 4.7. The proof goes by induction on d. So suppose that the statement holds in all hypercubes $\square^{d'}$ for any positive integer $d' < d$. We show that it holds also in $\mathrm{Chow}_{\mathrm{GS}}(\square^d)$.

We already treated the case $k = 1$ in Proposition 4.10. So we can suppose $k \geq 2$.

Suppose first that $n_1 = 1$. In this case we must have $\mathbf{v}_1 = \mathbf{0}$, since otherwise, we would have $n_2 + \cdots + n_k = d > d - |\mathbf{v}_1|$, and by Claim 4.6, we would have $C_\sigma = 0$. Let now $i \in [d]$ be an index with $v_{2,i} = 1$, and let $I = [d] \setminus \{i\}$ and $\alpha_2 = C_{\mathbf{v}_2}^{n_2} \dots C_{\mathbf{v}_k}^{n_k}$. By Proposition 4.4, we get $\deg(C_{\mathbf{0}} \alpha_2) = \deg(\alpha_2|_I)$, and the statement then follows by the induction hypothesis in the hypercube $\square^{d-1}$.

So we can suppose that $n_1 \geq 2$. We divide the proof into two parts depending on whether $\mathbf{v}_1 = \mathbf{0}$ or not.

Suppose first that $\mathbf{v}_1 \neq 0$. Let $i \in I(\mathbf{v}_1, \mathbf{v}_2)$, so we have $v_{1,i} = 0$ and $v_{2,i} = 1$. By relation ($\mathscr{R}3$), we get

$$C_{\mathbf{v}_1}^{n_1-1}\Big[C_{\mathbf{v}_1}\Big(\sum_{\mathbf{w}\in\square^d, w_i=0} C_{\mathbf{w}}\Big)C_{\mathbf{v}_2}\Big]C_{\mathbf{v}_2}^{n_2-1}C_{\mathbf{v}_3}^{n_3}\dots C_{\mathbf{v}_k}^{n_k} = 0,$$

in the Chow ring, from which we deduce, by developing, and using ($\mathscr{R}1$) and the vanishing criterium in Theorem 1.6, that

$$C_{\mathbf{v}_1}^{n_1}\dots C_{\mathbf{v}_k}^{n_k} = -C_{\mathbf{0}}C_{\mathbf{v}_1}^{n_1-1}\dots C_{\mathbf{v}_k}^{n_k} - \sum_{\substack{\mathbf{v}_1<\mathbf{v}<\mathbf{v}_2\\ |\mathbf{v}|=|\mathbf{v}_1|+x_1}} C_{\mathbf{v}_1}^{n_1-1}C_{\mathbf{v}}C_{\mathbf{v}_2}^{n_2}\dots C_{\mathbf{v}_k}^{n_k}.$$

Therefore, we have

$$\deg(C_{\mathbf{v}_1}^{n_1}\dots C_{\mathbf{v}_k}^{n_k}) = -\deg(C_{\mathbf{0}}C_{\mathbf{v}_1}^{n_1-1}\dots C_{\mathbf{v}_k}^{n_k}) - \sum_{\substack{\mathbf{v}_1<\mathbf{v}<\mathbf{v}_2\\ |\mathbf{v}|=|\mathbf{v}_1|+x_1}} \deg(C_{\mathbf{v}_1}^{n_1-1}C_{\mathbf{v}}C_{\mathbf{v}_2}^{n_2}\dots C_{\mathbf{v}_k}^{n_k}). \tag{17}$$

Let j be an arbitrary element in the support of $\mathbf{v}_1$, and set $J := [d]\backslash\{j\}$. From Proposition 4.9, we get

$$\begin{aligned}\deg(C_{\mathbf{0}}C_{\mathbf{v}_1}^{n_1-1}\dots C_{\mathbf{v}_k}^{n_k}) &= \deg\Big((C_{\mathbf{v}_1}^{n_1-1}\dots C_{\mathbf{v}_k}^{n_k})|_J\Big)\\ &= (-1)^{d-k}\binom{y_0+x_1-1}{y_0-1}\binom{x_1+y_1}{x_1}\binom{y_1+x_2}{y_1}\dots\binom{x_{k-1}+y_{k-1}}{y_{k-1}}\binom{y_{k-1}+x_k}{x_k}.\end{aligned}$$

In the last inequality we used the hypothesis of our induction in $\square^{d-1}$.

Now for each $\mathbf{v}_1 < \mathbf{v} < \mathbf{v}_2$ with $|\mathbf{v}| = |\mathbf{v}_1| + x_1 = y_0 + x_1 = n_1 - 1$, let $i_{\mathbf{v},1} \in I(\mathbf{v}_1, \mathbf{v})$ be an arbitrary element, and define $I_{\mathbf{v},1} = \mathrm{support}(\mathbf{v}) \setminus \{i_{\mathbf{v},1}\}$. Similarly, let $i_{\mathbf{v},2} \in I(\mathbf{v}, v_2)$ be an arbitrary element, and define $I_{\mathbf{v},2} = [d] \setminus \Big(\mathrm{support}(\mathbf{v}) \cup \{i_{\mathbf{v},2}\}\Big)$. By Proposition 4.9, we have

$$\begin{aligned}\deg(C_{\mathbf{v}_1}^{n_1-1}C_{\mathbf{v}}C_{\mathbf{v}_2}^{n_2}\dots C_{\mathbf{v}_k}^{n_k}) &= \deg(C_{\mathbf{v}_1}^{n_1-1}|_{I_{\mathbf{v},1}})\deg\Big((C_{\mathbf{v}_2}^{n_2}\dots C_{\mathbf{v}_k}^{n_k})|_{I_{\mathbf{v},2}}\Big)\\ &= (-1)^{n_1-2}\binom{n_1-2}{|v_1|}\times(-1)^{d-|v|-k+1}\binom{y_1+x_2}{y_1}\dots\binom{x_{k-1}+y_{k-1}}{y_{k-1}}\binom{y_{k-1}+x_k}{x_k}\\ &= (-1)^{n_1-2+d-|v|-k+1}\binom{y_0+x_1-1}{y_0}\binom{y_1+x_2}{y_1}\dots\binom{x_{k-1}+y_{k-1}}{y_{k-1}}\binom{y_{k-1}+x_k}{x_k}\\ &= (-1)^{d-k}\binom{y_0+x_1-1}{y_0}\binom{y_1+x_2}{y_1}\dots\binom{x_{k-1}+y_{k-1}}{y_{k-1}}\binom{y_{k-1}+x_k}{x_k}\end{aligned}$$

Since $|\mathbf{v}_2| = x_1 + y_1$, there are in total $\binom{x_1+y_1}{x_1}$ choices for $\mathbf{v}_1 < \mathbf{v} < \mathbf{v}_2$ with $|\mathbf{v}| = |\mathbf{v}_1| + x_1$. It finally follows from Equation (17) and the calculation of degrees above that

$$\begin{aligned}\deg(C_\sigma) &= (-1)^{d+1-k}\Big[\binom{y_0+x_1-1}{y_0-1}+\binom{y_0+x_1-1}{y_0}\Big]\binom{x_1+y_1}{x_1}\dots\binom{y_{k-1}+x_k}{x_k}\\ &= (-1)^{d+1-k}\binom{y_0+x_1}{y_0}\binom{x_1+y_1}{x_1}\dots\binom{y_{k-1}+x_k}{x_k},\end{aligned}$$

and the theorem follows.

In the final case $\mathbf{v}_1 = \mathbf{0}$, using ($\mathscr{R}1$) and the vanishing criterium in Theorem 1.6, we have, similarly as in the previous case above, that

$$C_{\mathbf{v}_1}^{n_1} \dots C_{\mathbf{v}_k}^{n_k} = -\sum_{\substack{\mathbf{v}_1 < \mathbf{v} < \mathbf{v}_2 \\ |\mathbf{v}| = |\mathbf{v}_1| + x_1}} C_{\mathbf{v}_1}^{n_1 - 1} C_{\mathbf{v}} C_{\mathbf{v}_2}^{n_2} \dots C_{\mathbf{v}_k}^{n_k}.$$

from which the result again follows by the hypothesis of our induction using a similar argument as in the previous case $\mathbf{v}_1 \neq \mathbf{0}$. $\square$

5. Fourier transform and a dual description of $\mathrm{Chow}_{\mathrm{GS}}(\square^d)$

Identify the points of $\square^d$ with the elements of the vector space $\mathbb{F}_2^d$, and consider the scalar product $\langle\, , \rangle$ on $\mathbb{F}_2^d$ defined by $\langle \mathbf{v}, \mathbf{u} \rangle = \sum_{i=1}^d v_i.u_i \in \mathbb{F}_2$, for any $\mathbf{u}, \mathbf{v} \in \mathbb{F}_2^d$. Recall that for any $\mathbf{w} \in \mathbb{F}_2^d$, we defined $F_{\mathbf{w}}$ by

$$F_{\mathbf{w}} := \sum_{\mathbf{v} \in \square^d} (-1)^{\langle \mathbf{v}, \mathbf{w} \rangle} C_{\mathbf{v}},$$

and noticed that by Fourier duality, the set $\{F_{\mathbf{w}}\}_{\mathbf{w} \in \mathbb{F}_2^d}$ forms another system of generators for the localized Chow ring $\mathrm{Chow}_{\mathrm{GS}}(\square^d)[\frac{1}{2}]$. The following set of relations are verified by $F_{\mathbf{w}}$ in $\mathrm{Chow}_{\mathrm{GS}}(\square^d)$ [18].

($\mathscr{R}^*1$) For any $\mathbf{w} \in \mathbb{F}^{\mathbf{d}}$, we have $F_{\mathbf{0}} F_{\mathbf{w}} = 0$;
($\mathscr{R}^*2$) For any $i \in [d]$, and any $\mathbf{w}, \mathbf{z} \in \mathbb{F}^{\mathbf{d}}$, we have $F_{\mathbf{e}_i}(F_{\mathbf{w}} - F_{\mathbf{w}+\mathbf{e}_i})(F_{\mathbf{z}} + F_{\mathbf{z}+\mathbf{e_i}}) = 0$;
($\mathscr{R}^*3$) For any pair of indices $i, j \in [d]$, and any $\mathbf{w}, \mathbf{z}$, we have $(F_{\mathbf{w}+\mathbf{e}_i+\mathbf{e}_j} - F_{\mathbf{w}})(F_{\mathbf{z}+\mathbf{e}_i+\mathbf{e}_j} - F_{\mathbf{z}}) = (F_{\mathbf{w}+\mathbf{e}_i} - F_{\mathbf{w}+\mathbf{e}_j})(F_{\mathbf{z}+\mathbf{e}_i} - F_{\mathbf{z}+\mathbf{e}_j})$.

Consider the ideal $\widetilde{\mathscr{I}_{\mathrm{rat}}}(\square^d)$ of $\mathbb{Z}[F_{\mathbf{w}}]$ generated by ($\mathscr{R}^*1$), ($\mathscr{R}^*2$), and ($\mathscr{R}^*3$), and define $\widetilde{\mathrm{Chow}}_{\mathrm{GS}}(\square^d) := \mathbb{Z}[F_{\mathbf{w}}]/\widetilde{\mathscr{I}_{\mathrm{rat}}}(\square^d)$.

We now give a proof of these relations in the Chow ring, proving at the same time Theorem 1.7, which shows that $\widetilde{\mathrm{Chow}}_{\mathrm{GS}}(\square^d)[\frac{1}{2}] = \mathrm{Chow}_{\mathrm{GS}}(\square^d)[\frac{1}{2}]$.

Proof of Theorem 1.7. We shall show that inverting 2, we have $\widetilde{\mathscr{I}_{\mathrm{rat}}}(\square^d) = \mathscr{I}_{\mathrm{rat}}(\square^d)$, and $\widetilde{\mathrm{Chow}}_{\mathrm{GS}}(\square^d) = \mathrm{Chow}_{\mathrm{GS}}(\square^d)$.

First note that $F_{\mathbf{0}} = \sum_{\mathbf{v} \in \mathbb{F}_2^d} C_{\mathbf{v}}$, and so for any $\mathbf{w}$, we have

$$F_{\mathbf{0}} F_{\mathbf{w}} = \sum_{\mathbf{u} \in \mathbb{F}_2^d} (-1)^{\langle \mathbf{u}, \mathbf{w} \rangle} \Big(\sum_{\mathbf{v} \in \mathbb{F}_2^d} C_{\mathbf{v}} \Big) C_{\mathbf{u}}.$$

This shows that $F_{\mathbf{0}} F_{\mathbf{w}} \in \mathscr{I}_{\mathrm{rat}}$, and yields to the proof of ($\mathscr{R}^*1$). On the other hand, we see from the above description, and using the Fourier duality, that for any $\mathbf{u} \in \mathbb{F}_2^d$,

$$\Big(\sum_{\mathbf{v} \in \mathbb{F}_2^d} C_{\mathbf{v}} \Big) C_{\mathbf{u}} = \frac{1}{2^d} \sum_{\mathbf{w} \in \mathbb{F}_2^d} (-1)^{\langle \mathbf{u}, \mathbf{w} \rangle} F_{\mathbf{0}} F_{\mathbf{w}}.$$

This shows that any generator $\Big(\sum_{\mathbf{v}} C_{\mathbf{v}} \Big) C_{\mathbf{u}}$ of type ($\mathscr{R}2$) in $\mathscr{I}_{\mathrm{rat}}(\square^d)$ belongs to the ideal $\widetilde{\mathscr{I}_{\mathrm{rat}}}(\square^d)$ of $\mathbb{Z}[\frac{1}{2}][F_{\mathbf{w}}]$.

Let $\mathbf{e} = \mathbf{e}_i$ for some $i \in [d]$. Note that for any $\mathbf{w}, \mathbf{z} \in \mathbb{F}_2^d$, we have

$$F_{\mathbf{w}} - F_{\mathbf{w}+\mathbf{e}} = 2 \sum_{\mathbf{u} \in \mathbb{F}_2^d \colon u_i = 1} (-1)^{\langle \mathbf{u}, \mathbf{w} \rangle} C_{\mathbf{u}} \qquad , \qquad F_{\mathbf{z}} + F_{\mathbf{z}+\mathbf{e}} = 2 \sum_{\mathbf{v} \in \mathbb{F}_2^d \colon v_i = 0} (-1)^{\langle \mathbf{v}, \mathbf{z} \rangle} C_{\mathbf{v}}. \tag{18}$$

For any $\epsilon \in \{0, 1\}$, define

$$R_i^\epsilon := \sum_{\substack{\mathbf{y} \in \mathbb{F}_2^d \\ y_i = \epsilon}} C_{\mathbf{y}}.$$

From Equation (18), we get

$$F_{\mathbf{e}}(F_{\mathbf{w}} - F_{\mathbf{w}+\mathbf{e}})(F_{\mathbf{z}} + F_{\mathbf{z}+\mathbf{e}}) = 4\Big(\sum_{\mathbf{y} \in \mathbb{F}_2^d} (-1)^{y_i} C_{\mathbf{y}}\Big)\Big(\sum_{\substack{\mathbf{u} \in \mathbb{F}_2^d \\ u_i = 1}} (-1)^{\langle \mathbf{u}, \mathbf{w} \rangle} C_{\mathbf{u}}\Big)\Big(\sum_{\substack{\mathbf{v} \in \mathbb{F}_2^d \\ v_i = 0}} (-1)^{\langle \mathbf{v}, \mathbf{w} \rangle} C_{\mathbf{v}}\Big) \tag{19}$$

$$= 4 \sum_{\substack{\epsilon \in \{0,1\}; \mathbf{u}, \mathbf{v} \in \mathbb{F}_2^d \\ u_i = 1, v_i = 0}} (-1)^{\epsilon + \langle \mathbf{u}, \mathbf{w} \rangle + \langle \mathbf{v}, \mathbf{z} \rangle}) C_{\mathbf{u}} C_{\mathbf{v}} R_i^\epsilon, \tag{20}$$

Since $C_{\mathbf{u}} C_{\mathbf{v}} R_i^\epsilon \in \mathscr{I}_{\mathrm{rat}}$, this implies the relation $(\mathscr{R}^*2)$ among the $F_{\mathbf{w}}$ in the Chow ring $\mathrm{Chow}_{\mathrm{GS}}(\square^d)$.

Define now the two functions $f, g : \mathbb{F}_2 \times \mathbb{F}_2^d \times \mathbb{F}_2^d \to \mathbb{Z}[C_w]$, as follows. For any triple $(\epsilon, \mathbf{u}, \mathbf{v}) \in \mathbb{F}_2 \times \mathbb{F}_2^d \times \mathbb{F}_2^d$, set

$$f(\epsilon, \mathbf{u}, \mathbf{v}) := \begin{cases} C_{\mathbf{u}} C_{\mathbf{v}} R_i^\epsilon & \text{if } u_i = 1,\ v_i = 0 \\ 0 & \text{otherwise.} \end{cases}$$

For any triple $(\delta, \mathbf{w}, \mathbf{z}) \in \mathbb{F}_2 \times \mathbb{F}_2^d \times \mathbb{F}_2^d$, set

$$g(\delta, \mathbf{w}, \mathbf{z}) = \begin{cases} F_{\mathbf{e}}(F_{\mathbf{w}} - F_{\mathbf{w}+\mathbf{e}})(F_{\mathbf{z}} + F_{\mathbf{z}+\mathbf{e}}) & \text{if } \delta = 1 \\ \sum_{\substack{\mathbf{u}, \mathbf{v} \in \mathbb{F}_2^d \\ u_i = 1, v_i = 0}} (-1)^{\langle \mathbf{u}, \mathbf{w} \rangle + \langle \mathbf{v}, \mathbf{z} \rangle} C_{\mathbf{u}} C_{\mathbf{v}} F_{\mathbf{0}} & \text{if } \delta = 0. \end{cases}$$

Note that we have for any $(\delta, \mathbf{w}, \mathbf{z}) \in \mathbb{F}_2 \times \mathbb{F}_2^d \times \mathbb{F}_2^d$,

$$g(\delta, \mathbf{w}, \mathbf{z}) = 4 \sum_{\substack{\epsilon \in \{0,1\} \\ \mathbf{u}, \mathbf{v} \in \mathbb{F}_2^d}} (-1)^{\delta.\epsilon + \langle \mathbf{u}, \mathbf{w} \rangle + \langle \mathbf{v}, \mathbf{z} \rangle} f(\epsilon, \mathbf{u}, \mathbf{v}).$$

Indeed, for $\delta = 1$ this is identical to Equation (20), and for $\delta = 0$, both sides of the equations are equal by definition (the right hand side is a sum of the terms of the form $C_{\mathbf{u}} C_{\mathbf{v}} (R_i^1 + R_i^0) = C_{\mathbf{u}} C_{\mathbf{v}} (\sum_{\mathbf{x}} C_{\mathbf{x}}) = C_{\mathbf{u}} C_{\mathbf{v}} F_{\mathbf{0}}$. By Fourier duality in $\mathbb{F}_2^{2d+1}$, it follows that

$$f(\epsilon, \mathbf{u}, \mathbf{v}) = \frac{1}{2^{2d-1}} \sum_{(\delta, \mathbf{w}, \mathbf{z}) \in \mathbb{F}_2^{2d+1}} (-1)^{\delta.\epsilon + \langle \mathbf{u}, \mathbf{w} \rangle + \langle \mathbf{v}, \mathbf{z} \rangle} g(\delta, \mathbf{w}, \mathbf{z}).$$

Since g takes values in $\widetilde{\mathscr{I}_{\mathrm{rat}}}(\square^d)$, this shows that for any $\epsilon, \mathbf{u}, \mathbf{v}$, we have $f(\epsilon, \mathbf{u}, \mathbf{v}) \in \widetilde{\mathscr{I}_{\mathrm{rat}}}(\square^d)$. In particular, inverting 2, all the generators of type $(\mathscr{R}3)$ in $\mathscr{I}_{\mathrm{rat}}(\square^d)$ belong to $\widetilde{\mathscr{I}_{\mathrm{rat}}}(\square^d)$.

Finally, to prove ($\mathscr{R}^*3$), let $i, j \in [d]$, and $\mathbf{e} = \mathbf{e}_i$ and $\mathbf{e}' = \mathbf{e}_j$. By an easy computation, we have

$$(F_{\mathbf{w}+\mathbf{e}+\mathbf{e}'} - F_{\mathbf{w}})(F_{\mathbf{z}+\mathbf{e}+\mathbf{e}'} - F_{\mathbf{z}}) - (F_{\mathbf{w}+\mathbf{e}} - F_{\mathbf{w}+\mathbf{e}'})(F_{\mathbf{z}+\mathbf{e}} - F_{\mathbf{z}+\mathbf{e}'}) = \sum_{\mathbf{u},\mathbf{v}\in\mathbb{F}_2^d} (-1)^{\langle \mathbf{u},\mathbf{w}\rangle + \langle \mathbf{v},\mathbf{z}\rangle} h(u_i, u_j, v_i, v_j) C_{\mathbf{u}} C_{\mathbf{v}},$$

where $h : \mathbb{F}_2^4 \to \mathbb{Z}$ is the function given by

$$h(a,b,c,d) := \Big((-1)^{a+b} - 1\Big)\Big((-1)^{c+d} - 1\Big) - \Big((-1)^a - (-1)^b\Big)\Big((-1)^c - (-1)^d\Big),$$

for any $(a,b,c,d) \in \mathbb{F}_2^4$. In particular, we get $h(0,1,1,0) = h(1,0,0,1) = 8$, and h vanishes at all other points of $\mathbb{F}_2^4$. Now, note that for any pair of points $\mathbf{u}, \mathbf{v} \in \mathbb{F}_2^d$ such that $\{(u_i,u_j),(v_i,v_j)\} = \{(0,1),(1,0)\}$, since $\mathbf{u}$ and $\mathbf{v}$ do not form a simplex in $\square^d$, we have $C_{\mathbf{u}}C_{\mathbf{v}} \in \mathscr{I}_{\mathrm{rat}}(\square^d)$. This proves that the relation ($\mathscr{R}^*3$) holds among the $F_{\mathbf{w}}$.

To conclude, using Fourier duality in $\mathbb{F}_2^d \times \mathbb{F}_2^d$, we see that for any pair $i, j \in [d]$, $2^{2d} h(u_i, u_j, v-i, v_j) C_{\mathbf{u}} C_{\mathbf{v}}$ is a linear combination with $\pm$ coefficient of the terms $(F_{\mathbf{w}+\mathbf{e}+\mathbf{e}'} - F_{\mathbf{w}})(F_{\mathbf{z}+\mathbf{e}+\mathbf{e}'} - F_{\mathbf{z}}) - (F_{\mathbf{w}+\mathbf{e}} - F_{\mathbf{w}+\mathbf{e}'})(F_{\mathbf{z}+\mathbf{e}} - F_{\mathbf{z}+\mathbf{e}'})$, and thus belongs to $\widetilde{\mathscr{I}_{\mathrm{rat}}}(\square^d)$. For $\mathbf{u}$ and $\mathbf{v}$ which do not form a simplex in $\square^d$, there are indices $i, j \in [d]$, such that $\{(u_i,u_j),(v_i,v_j)\} = \{(0,1),(1,0)\}$. Since $h(0,1,0,1) = 8$, we infer that $C_{\mathbf{u}}C_{\mathbf{v}} \in \widetilde{\mathscr{I}_{\mathrm{rat}}}(\square^d)$.

This shows all the relations of type ($\mathscr{R}1$) in $\mathscr{I}_{\mathrm{rat}}(\square^d)$ belong to $\widetilde{\mathscr{I}_{\mathrm{rat}}}(\square^d)$, which finally shows that $\widetilde{\mathscr{I}_{\mathrm{rat}}}(\square^d) = \mathscr{I}_{\mathrm{rat}}(\square^d)$ in $\mathbb{Z}[\frac{1}{2}][C_{\mathbf{v}}] = \mathbb{Z}[\frac{1}{2}][F_{\mathbf{w}}]$, and the theorem follows. $\square$

5.1. **Functoriality of $\widetilde{\mathrm{Chow}}_{\mathrm{GS}}$ for the inclusion of hypercubes.** Let $r < d$ be two integers. Let $\widetilde{\mathrm{Chow}}_{\mathrm{GS}}(\square^d)$ be the Chow ring of $\square^d$, with generators $F_{\mathbf{u}}$ for $\mathbf{u} \in \mathbb{F}_2^d$ and with relations ($\mathscr{R}^*1$), ($\mathscr{R}^*2$), ($\mathscr{R}^*3$). Similarly, let $\widetilde{\mathrm{Chow}}_{\mathrm{GS}}(\square^r)$ be the Chow ring of $\square^r$, with generators $\widetilde{F}_{\mathbf{w}}$ for $\mathbf{w} \in \mathbb{F}_2^r$ and with relations ($\mathscr{R}^*1$), ($\mathscr{R}^*2$), ($\mathscr{R}^*3$).

Let $I \subseteq \{1,\dots,d\}$ be a subset of size r. Viewing $\mathbb{F}_2^r$ as the set of elements of $\mathbb{F}_2^d$ with support in I, we get an inclusion $\eta : \square^r \hookrightarrow \square^d$.

Proposition 5.1. *We have a morphism of graded rings $\eta_* : \widetilde{\mathrm{Chow}}_{\mathrm{GS}}(\square^r) \to \widetilde{\mathrm{Chow}}_{\mathrm{GS}}(\square^d)$ defined by sending $\widetilde{F}_{\mathbf{w}}$ to $F_{\eta(\mathbf{w})}$.*

Proof. Consider the map of polynomial rings $\mathbb{Z}[F_{\mathbf{w}}]_{\mathbf{w}\in\mathbb{F}_2^r} \to \mathbb{Z}[F_{\mathbf{u}}]_{\mathbf{u}\in\mathbb{F}_2^d}$ induced by η. The image by η of any relation of type ($\mathscr{R}^*1$), ($\mathscr{R}^*2$), ($\mathscr{R}^*3$) in $\widetilde{\mathrm{Chow}}_{\mathrm{GS}}(\square^r)$ is a relation of type ($\mathscr{R}^*1$), ($\mathscr{R}^*2$), ($\mathscr{R}^*3$) in $\widetilde{\mathrm{Chow}}_{\mathrm{GS}}(\square^d)$. It follows that passing to the quotient, we get a well-defined map $\eta_* : \widetilde{\mathrm{Chow}}_{\mathrm{GS}}(\square^r) \to \widetilde{\mathrm{Chow}}_{\mathrm{GS}}(\square^d)$ of Chow rings. $\square$

Corollary 5.2. *The inclusion $\eta : \square^I \to \square^d$ induces a map of localized Chow rings*

$$\eta_* : \mathrm{Chow}_{\mathrm{GS}}(\square^r)[\frac{1}{2}] \to \mathrm{Chow}_{\mathrm{GS}}(\square^d)[\frac{1}{2}].$$

Proof. This follows from the previous proposition and the isomorphism of Theorem 1.7. $\square$

The morphism $\eta : \square^r \to \square^d$ is induced from a morphism of graphs $\eta_i : H_i \to K_2$, for $i = 1,\dots,r$, with K_2 the complete graph on two vertices $0 < 1$, and $H_i = G_i$ for $i \in I$, and $H_i = K_2[< 1] \simeq K_1$ for $i \notin I$. It follows from Proposition 2.4 that we have a morphism of Chow rings $\eta^* : \mathrm{Chow}_{\mathrm{GS}}(\square^d)[\frac{1}{2}] \to \mathrm{Chow}_{\mathrm{GS}}(\square^d)[\frac{1}{2}]$.

Proposition 5.3. *The composition morphism $\eta^* \circ \eta_*$ of $\mathrm{Chow}_{\mathrm{GS}}(\square^r)[\frac{1}{2}]$ is the identity.*

Proof. It will be enough to prove this in degree one. Let $\mathbf{w} \in \square^r$. We have

$$\eta^* \circ \eta_*(\widetilde{F}_{\mathbf{w}}) = \eta^*(F_{\eta(\mathbf{w})}) = \eta_*\big(\sum_{\mathbf{v}\in\mathbb{F}_2^d} (-1)^{\langle \mathbf{v},\eta(\mathbf{w})\rangle} C_{\mathbf{v}}\big)$$
$$= \sum_{\mathbf{v}\in\mathbb{F}_2^d} (-1)^{\langle \mathbf{v},\eta(\mathbf{w})\rangle} \eta_*(C_{\mathbf{v}}) = \sum_{\mathbf{z}\in\mathbb{F}_2^r} (-1)^{\langle \mathbf{z},\mathbf{w}\rangle} C_{\mathbf{z}} = \widetilde{F}_{\mathbf{w}}.$$

□

5.2. **Vanishing theorem.** In this section, we prove the vanishing Theorem 1.8. So consider elements $\mathbf{w}_0, \dots, \mathbf{w}_d \in \mathbb{F}_2^d$. Let $\mathcal{P} = \{P_1, \dots, P_k\}$ be a partition of $[d]$ into k disjoint non-empty sets. Recall that for each $\mathbf{w}_i$, we denote by $\alpha(\mathbf{w}_i, \mathcal{P})$ the number of indices $1 \le i \le k$ such that there exists an index $j \in P_i$ with $\mathbf{w}_j = 1$. Suppose that the condition in Theorem 1.8 is verified, namely, $\sum_{i=0}^d \alpha(\mathbf{w}_i, \mathcal{P}) < d + k$. We have to prove that $F_{\mathbf{w}_0} \dots F_{\mathbf{w}_d} = 0$.

We first reformulate the condition in the theorem as follows. Let $\mathbf{w}_0, \dots, \mathbf{w}_d \in \mathbb{F}_2^d$, possibly with $\mathbf{w}_i = \mathbf{w}_j$ for $i \neq j$. We construct a bipartite graph $H = H(\mathcal{P}; \mathbf{w}_0, \dots, \mathbf{w}_d)$ as follows. The graph H has the vertex set partitioned into two separate parts W and V of size $d+1$ and d, respectively. Let $W = \{\omega_0, \dots, \omega_d\}$ be $d+1$ vertices on one side, and $V = \{1, \dots, d\}$ the d vertices on the other side. There is an edge between $\omega_i \in W$ and $j \in V$ if the j-coordinate of $\mathbf{w}_i$ is one. We have the following proposition.

Proposition 5.4. *The following three conditions are equivalent*

(i) *The graph H is disconnected;*
(ii) *There exists a partition $\mathcal{P} = \{P_1, P_2\}$ of $\{1, \dots, d\}$ such that $\sum_{i=0}^d \alpha(\mathbf{w}_i, \mathcal{P}) < d + 2$;*
(iii) *There exists an integer $k \in \mathbb{N}$ and a partition $\mathcal{P} = \{P_1, P_2, \dots, P_k\}$ such that*

$$\sum_{i=0}^d \alpha(\mathbf{w}_i, \mathcal{P}) < d + k.$$

Proof. To show that (i) implies (ii), note that if H is disconnected, then we can write $W = W_1 \sqcup W_2$ and $V = V_1 \sqcup V_2$ such that all the edges of H join a vertex of W_i to a vertex of V_i, for $i = 1, 2$. In this case, let $\mathcal{P} = \{V_1, V_2\}$. Then we have $\sum_{i=0}^d \alpha(\mathbf{w}_i, \mathcal{P}) \le |W_1| + |W_2| = d + 1 < d + 2$.

Obviously, (ii) implies (iii).

Finally, let $\mathcal{P} = \{P_1, P_2, \dots, P_k\}$ be a partition of V such that $\sum_{i=0}^d \alpha(\mathbf{w}_i, \mathcal{P}) < d + k$. Let $\widetilde{H}$ be the graph obtained from H by contracting each P_i into a single vertex. The number of edges of $\widetilde{H}$ is precisely $\sum_{i=0}^d \alpha(\mathbf{w}_i, \mathcal{P})$. On the other hand, the number of vertices of $\widetilde{H}$ is $d + 1 + k$. Since $|E(\widetilde{H})| < |V(\widetilde{H})| - 1$, the graph $\widetilde{H}$ is not connected, which shows that H cannot be connected neither. □

Let now $\mathbf{w}_0, \dots, \mathbf{w}_d$ be a sequence of elements of the hypercube such that the associated graph H is not connected. Thus, there exists a decomposition $W = W_1 \sqcup W_2$ and $V = V_1 \sqcup V_2$ such that there is no edge between V_i and W_j provided that $i \neq j$.

Since $|W_1| + |W_2| > |V_1| + |V_2|$, we can suppose without loss of generality that $|W_1| > |V_1|$. Let $r := |V_1|$, and note that $0 < r < d$. We have the following proposition, which clearly implies the vanishing theorem above. (Recall that for an element $\mathbf{w} \in \mathbb{F}_2^d$, the support of $\mathbf{w}$ is the set of all $j \in [d]$ with $w_j = 1$.)

Proposition 5.5. *Let r be a non-negative integer. Let $\mathbf{w}_0, \dots, \mathbf{w}_r$ be elements of $\mathbb{F}_2^d$ with support in a subset V_1 of $V = [d]$ of size r. Then for any $\mathbf{v} \in \mathbb{F}_2^d$, we have*

$$F_{\mathbf{w}_0} \dots F_{\mathbf{w}_r} F_{\mathbf{v}} = 0.$$

We need the following useful lemma, proved in [18, Proposition 4.29].

Lemma 5.6. *We have $F_{\mathbf{e}_1} F_{\mathbf{e}_2} \dots F_{\mathbf{e}_d} F_{\mathbf{e}_1+\dots+\mathbf{e}_d} = (-4)^d C_{\mathbf{0}} C_{\mathbf{e}_1} C_{\mathbf{e}_1+\mathbf{e}_2} \dots C_{\mathbf{e}_1+\dots+\mathbf{e}_d}$.*

Proof of Proposition 5.5. Consider the inclusion $\eta : \mathbb{F}_2^r \to \mathbb{F}_2^d$ induced by the subset V_1, and let $\eta_* : \widetilde{\mathrm{Chow}}_{\mathrm{GS}}(\square^r) \to \widetilde{\mathrm{Chow}}_{\mathrm{GS}}(\square^d)$ be the induced map on the level of Chow rings given by Proposition 5.1.

Since $\mathrm{Chow}_{\mathrm{GS}}(\square^r)$ is one dimensional in degree $r+1$, using Lemma 5.6, we infer the existence of a rational number $a \in \mathbb{Z}[\frac{1}{2}]$ such that we have $\widetilde{F}_{\mathbf{w}_0} \widetilde{F}_{\mathbf{w}_1} \dots \widetilde{F}_{\mathbf{w}_r} = a \widetilde{F}_{\mathbf{e}_1} \dots \widetilde{F}_{\mathbf{e}_r} \widetilde{F}_{\mathbf{e}_1+\dots+\mathbf{e}_r}$, where, by an abuse of the notation, $\mathbf{e}_1, \dots, \mathbf{e}_r$ denote the basis of $\mathbb{F}_2^r \hookrightarrow \mathbb{F}_2^d$ corresponding to the elements of V_1. It follows that we have

$$F_{\mathbf{w}_0} \dots F_{\mathbf{w}_r} = a F_{\mathbf{e}_1} \dots F_{\mathbf{e}_r} F_{\mathbf{e}_1+\dots+\mathbf{e}_r}$$

in the Chow ring $\mathrm{Chow}_{\mathrm{GS}}(\square^d)[\frac{1}{2}]$, and it will be enough to prove that for any $\mathbf{v} \in \square^d$, we have

$$F_{\mathbf{e}_1} \dots F_{\mathbf{e}_r} F_{\mathbf{e}_1+\dots+\mathbf{e}_r} F_{\mathbf{v}} = 0.$$

We prove this by induction on r.

- For the base of our induction $r = 1$, we need to prove that $F_{\mathbf{e}_1}^2 F_{\mathbf{v}} = 0$.

We have $2F_{\mathbf{e}_1}^2 F_{\mathbf{v}} = F_{\mathbf{e}_1}(F_{\mathbf{e}_1} + F_0)(F_{\mathbf{v}} - F_{\mathbf{v}+\mathbf{e}_1}) + F_{e_1}(F_{\mathbf{e}_1} - F_0)(F_{\mathbf{v}} + F_{\mathbf{v}+\mathbf{e}_1}) = 0$, which proves the claim.

- Suppose $r \geq 2$ and assume that the statement holds for $r-1$, we show it for r.

Write

$$\begin{aligned} 2F_{\mathbf{e}_1} \dots F_{\mathbf{e}_r} F_{\mathbf{e}_1+\dots+\mathbf{e}_r} F_{\mathbf{v}} =& F_{\mathbf{e}_1} \dots F_{\mathbf{e}_{r-1}} F_{\mathbf{e}_r} F_{\mathbf{e}_1+\dots+\mathbf{e}_r} (F_{\mathbf{v}} - F_{\mathbf{v}+\mathbf{e}_r}) \\ &+ F_{\mathbf{e}_1} \dots F_{\mathbf{e}_{r-1}} F_{\mathbf{e}_r} F_{\mathbf{e}_1+\dots+\mathbf{e}_r} (F_{\mathbf{v}} + F_{\mathbf{v}+\mathbf{e}_r}) \\ =& F_{\mathbf{e}_1} \dots F_{\mathbf{e}_{r-1}} F_{\mathbf{e}_r} \big(F_{\mathbf{e}_1+\dots+\mathbf{e}_r} + F_{\mathbf{e}_1+\dots+\mathbf{e}_{r-1}}\big)(F_{\mathbf{v}} - F_{\mathbf{v}+\mathbf{e}_r}) \\ &+ F_{\mathbf{e}_1} \dots F_{\mathbf{e}_{r-1}} F_{\mathbf{e}_r} \big(F_{\mathbf{e}_1+\dots+\mathbf{e}_r} - F_{\mathbf{e}_1+\dots+\mathbf{e}_{r-1}}\big)(F_{\mathbf{v}} + F_{\mathbf{v}+\mathbf{e}_r}), \end{aligned}$$

which is zero by the relation ($\mathscr{R}^*3$) satisfied by the generators $F_{\mathbf{w}}$ of $\widetilde{\mathrm{Chow}}_{\mathrm{GS}}(\square^d)$. In the last equalities, we used the induction assumption that $F_{\mathbf{e}_1} \dots F_{\mathbf{e}_{r-1}} F_{\mathbf{e}_1+\dots+\mathbf{e}_{r-1}} F_{\mathbf{e}_r} = 0$. $\square$

Remark 5.7. It would be interesting to find a combinatorial formula for the degree with respect to the dual basis $F_{\mathbf{w}}$. For any collection of $d+1$ elements $(\mathbf{w}_0, \dots, \mathbf{w}_d) \in (\mathbb{F}_2^d)^{d+1}$, we have

$$\deg(F_{\mathbf{w}_0} \dots F_{\mathbf{w}_d}) = \sum_{\mathbf{u}_0, \dots, \mathbf{u}_d} (-1)^{\sum_i \langle \mathbf{u}_i, \mathbf{w}_i \rangle} \deg(C_{\mathbf{u}_0} \dots C_{\mathbf{u}_d}).$$

So the question can be reformulated in asking for the Fourier transform in $(\mathbb{F}_2^d)^{d+1}$ of the degree map on $(\mathbb{F}_2^d)^{d+1}$ given by Theorem 1.6.

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