GEOMETRY OF GRAPHS AND APPLICATIONS IN ARITHMETIC AND ALGEBRAIC GEOMETRY

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We survey recent results concerning the algebraic-geometric aspects of graphs and metric graphs, and discuss some applications in arithmetic and algebraic geometry.

All graphs considered here are supposed to be connected.

1. Algebraic geometry of metric graphs

In this section, we provide some background on algebraic geometry of metric graphs, and explain the link from algebraic geometry of curves to that of metric graphs. The presentation follows those in [5, 6, 9]; more details can be found in [4, 11, 12, 50, 11].

1.1. **Metric graphs.** Given $n \in \mathbb{Z}_{\geq 1}$, we define $S_n \subset \mathbb{C}$ to be a "star with n branches", i.e., a topological space homeomorphic to the union of the convex hull in $\mathbb{R}^2$ of $(0,0)$ and any point among a set of n points no two of them lie on a common line through the origin. We also define $S_0 = \{0\}$. A finite topological graph Γ is the topological realization of a finite graph: Γ is a compact (zero or) one dimensional topological space such that for any point $p \in \Gamma$, there exists a neighborhood U_p of p in Γ homeomorphic to some S_n; moreover there are only finitely many points p with U_p homeomorphic to S_n with $n \neq 2$.

The unique integer n such that U_p is homeomorphic to S_n is called the valence of p and denoted $\mathrm{val}(p)$. A point of valence different from 2 is called an essential vertex of Γ: they are of two types, v with $\mathrm{val}(v) \geq 3$ which are called branching points, and v for which $\mathrm{val}(v) = 1$ which are called ends of Γ. The set of tangent directions at p is $T_p(\Gamma) = \varinjlim_{U_p} \pi_0(U_p \setminus \{p\})$, where the limit is taken over all neighborhoods of p isomorphic to a star with n branches. The set $T_p(\Gamma)$ has precisely $\mathrm{val}(p)$ elements.

A metric graph (Γ, ℓ) is a compact connected metric space, such that for every $p \in \Gamma$ there is a radius $r_p \in \mathbb{R}_{>0}$ such that there is a neighborhood U_p around p which is isometric to the star shaped domain $S(\mathrm{val}(p), r_p) := \{re^{2\pi i m/\mathrm{val}(p)} : 0 < r < r_p, 1 \leq m \leq \mathrm{val}(p)\} \subset \mathbb{C}$ equipped with the path-metric. We will usually drop the metric ℓ from the notation and simply refer to Γ as the metric, and the corresponding topological, graph. We use the notation $T_p(\Gamma)$ to denote the set of all unit tangent vectors emanating from p in Γ (which gets identified with the unit vectors $e^{2\pi i m/\mathrm{val}(p)}$ in $\mathbb{C}$ under the isometry of U_p with $S(\mathrm{val}(p), r_p)$).

For a function $f : \Gamma \to \mathbb{C}$, a point $p \in \Gamma$ and a unit tangent vector $w \in T_p(\Gamma)$, the directional derivative $d_w f(x)$ of f at p in the direction of w, which we simply call the outgoing slope of f at p along w, is defined by:

$$d_w f(x) = \lim_{t \downarrow 0} \frac{f(x+tw) - f(x)}{t},$$

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if the limit exists. Note that the above expression makes sense by (isometrically) identifying a small enough neighborhood U_p of p with a star shaped domain $S(\mathrm{val}(p), r_p)$ in $\mathbb{C}$, and by restricting f to $S(\mathrm{val}(p), r_p)$.

Let Γ be a metric graph. A vertex set $V(\Gamma)$ is a finite subset of the points of Γ which contains all the essential points of Γ. An element of a fixed vertex set $V(\Gamma)$ is called a vertex of Γ, and the closure of a connected component of $\Gamma \setminus V(\Gamma)$ is called an edge of Γ. We denote by $E(\Gamma)$ the set of all edges of Γ with respect to the vertex set $V(\Gamma)$. The (combinatorial) graph $G = (V(\Gamma), E(\Gamma))$ is called a model of Γ. A model G of Γ is simple if there is no loop edge or double edge in E. Since Γ is a metric graph, we can associate to each edge e of a model $G = (V, E)$ its length $\ell(e) \in \mathbb{R}_{>0}$.

The genus $g(\Gamma)$ of a metric graph Γ is by definition equal to its first Betti number. If $G = (V, E)$ is a model of Γ, then $g = |E| - |V| + 1$.

The model $G = (V, E)$ of a metric graph Γ with V the set of all essential points of Γ is called the minimal model of Γ. We denote by $\ell_{\min}$ the minimum length of the edges in the minimal model of Γ. The volume $\mu(\Gamma)$ of Γ is the sum of the edge lengths in any model G of Γ. We denote by $d_{\max}$ the maximum valence of points of Γ.

1.2. **Divisor theory on metric graphs.** We recall some basic definitions concerning the divisor theory of metric graphs. See [12, 50] for more details.

For a metric graph Γ, let $\mathrm{Div}(\Gamma)$ be the free abelian group on points of Γ. An element D of $\mathrm{Div}(\Gamma)$ is called a divisor on Γ and can uniquely be written as

$$D = \sum_{v \in \Gamma} a_v(v), \text{ with } a_v \in \mathbb{Z},$$

where all but finitely many a_v are zero. The degree of D is $\deg(D) = \sum_{v\in\Gamma} a_v$. A divisor D is effective if $a_v \geq 0$ for all $v \in \Gamma$. The coefficient of D at v is also denoted by $D(v)$.

The set of points v for which a_v is nonzero is called the support of D and is denoted by $\mathrm{supp}(D)$.

A rational function on Γ is a continuous piecewise linear function on Γ whose outgoing slopes are all integers. The set of all rational functions on Γ is denoted by $R(\Gamma)$. The order of a rational function f at a point p of Γ, denoted by $\mathrm{ord}_p(f)$, is the sum of the outgoing slopes of f along the unique tangent directions in Γ emanating from p. As f is piecewise linear, and Γ is compact, the order of f is zero on all but finitely many points of Γ, and one gets a map

$$\mathrm{div} : R(\Gamma) \to \mathrm{Div}(\Gamma), f \mapsto \sum_p \mathrm{ord}_p(f)(p).$$

A divisor in the image of div is called a principal divisor. Two divisors, D and D' are called linearly equivalent, written $D \sim D'$, if they differ by a principal divisor, i.e., there is a rational function such that $D = \mathrm{div}(f) + D'$. The (complete) linear system $|D|$ of a divisor D is defined to be the set of all effective divisors which are linearly equivalent to D:

$$|D| := \{E \in \mathrm{Div}(\Gamma) : E \geq 0, E \sim D\}.$$

We denote by $R(D) := \{f \in \mathrm{Rat}(\Gamma) : D + \mathrm{div}(f) \geq 0\}$ the "set of all global sections of D". Note that $R(D)$ is closed under addition by constants and under taking maximum, i.e., for

$f, g \in R(D)$ and $c \in \mathbb{R}$, one has $c + f \in R(D)$ and $\max(f, g) \in R(D)$, in other words, $R(D)$ is a so called tropical semi-module.

The rank of a divisor D, denoted by $r(D)$ is defined by

$$r(D) := \min_{\{E: E \geq 0, |D-E| = \emptyset\}} \deg(D) - 1.$$

The canonical divisor K of Γ is by definition

$$K := \sum_{x \in \Gamma} (\mathrm{val}(x) - 2)(x).$$

Note that the above sum is actually only over essential vertices of Γ, and so is finite.

For the proof of the following Riemann-Roch theorem, we refer to [12, 50]. A generalization to the mixed (algebraic curve-metric graph) setting can be found in [4].

Theorem 1.1 (Riemann-Roch)**.** *Let Γ be a metric graph of genus g. For any divisor D of degree d, $r(D) - r(K - D) = \deg(D) - g + 1$.*

The divisorial gonality $\gamma_{div}(\Gamma)$ of a metric graph Γ is defined by

$$\gamma_{div}(\Gamma) := \min\{d : \text{ there exists a } D \in \mathrm{Div}(\Gamma), \text{ with } \deg(D) = d \text{ and } r(D) = 1\}.$$

1.3. **Harmonic morphisms, tropical modifications, and geometric gonality.** We recall some standard definitions regarding the morphisms between metric graphs, and the corresponding tropical curves, see [5] and the references there for a more detailed discussion of the following definitions with several examples.

Let Γ and Γ' be two metric graphs, and fix vertex sets $V = V(\Gamma)$ and $V' = V(\Gamma')$ for Γ and Γ', respectively. Denote by E and E' the edge sets $E(\Gamma)$ and $E(\Gamma')$, respectively. Let $\phi : \Gamma \to \Gamma'$ be a continuous map.

- The map ϕ is called a (V, V')-morphism of metric graphs if we have $\phi(V) \subset V'$, $\phi^{-1}(E') \subset E$, and the restriction of ϕ to any edge e in E is a dilation by some factor $d_e(\phi) \in \mathbb{Z}_{\geq 0}$.
- The map ϕ is called a morphism of metric graphs if there exists a vertex set $V = V(\Gamma)$ of Γ and a vertex set $V' = V(\Gamma')$ of Γ' such that ϕ is a (V, V')-morphism of metric graphs.
- The map ϕ is said to be finite if $d_e(\phi) > 0$ for any edge $e \in E(\Gamma)$.

The integer $d_e(\phi) \in \mathbb{Z}_{\geq 0}$ in the definition above is called the degree of ϕ along e. Let $p \in V(\Gamma)$, let $w \in T_p(\Gamma)$, and let $e \in E(\Gamma)$ be the edge of Γ in the direction of w. The directional derivative of ϕ in the direction w is by definition the quantity $d_w(\phi) := d_e(\phi)$. If we set $p' = \phi(p)$, then ϕ induces a map

$$d\phi(p) \ : \ \big\{w \in T_p(\Gamma) \ : \ d_w(\phi) \neq 0\big\} \to T_p(\Gamma')$$

in the obvious way.

Let $\phi : \Gamma \to \Gamma'$ be a morphism of metric graphs, let $p \in \Gamma$, and let $p' = \phi(p)$. The morphism ϕ is harmonic at p provided that, for every tangent direction $w' \in T_{p'}(\Gamma')$, the number

$$d_p(\phi) := \sum_{\substack{w \in T_p(\Gamma) \\ w \mapsto w'}} d_w(\phi)$$

is independent of w'. The number $d_p(\phi)$ is called the degree of ϕ at p.

We say that ϕ is harmonic if it is surjective and harmonic at all $p \in \Gamma$; in this case the number $\deg(\phi) = \sum_{p \mapsto p'} d_p(\phi)$ is independent of $p' \in \Gamma'$, and is called the degree of ϕ.

There is an equivalence relation between metric graphs, that we recall now; an equivalence class for this relation is called a tropical curve.

An elementary tropical modification of a metric graph Γ_0 is a metric graph $\Gamma = [0, \ell] \cup \Gamma_0$ obtained from Γ_0 by attaching a segment $[0, \ell]$ of (an arbitrary) length $\ell > 0$ to Γ_0 in such a way that $0 \in [0, \ell]$ gets identified with a point $p \in \Gamma_0$.

A metric graph Γ obtained from a metric graph Γ_0 by a finite sequence of elementary tropical modifications is called a tropical modification of Γ_0.

Tropical modifications generate an equivalence relation $\sim$ on the set of metric graphs. A tropical curve is an equivalence class of metric graphs with respect to $\sim$.

There exists a unique rational tropical curve, which is denoted by $\mathbb{TP}^1$: it is the class of all finite metric trees (which are all equivalent under tropical modifications).

A tropical morphism of tropical curves $\phi : C \to C'$ is a harmonic morphism of metric graphs between some metric graph representatives of C and C', considered up to tropical equivalence.

A tropical curve C is said to have a (non-metric) graph G as its combinatorial type if C admits a representative whose underlying graph is G.

A tropical curve C is called d-gonal if there exists a tropical morphism $C \to \mathbb{TP}^1$ of degree d. A metric graph Γ has geometric gonality d, if the tropical curve associated to Γ is d-gonal, and d is the smallest integer satisfying this condition. The geometric gonality of a metric graph is denoted by $\gamma_{gm}(\Gamma)$.

It is easy to see that the fibers of any finite harmonic morphisms from a metric graph Γ to a finite tree are linearly equivalent, and define a linear equivalence class of divisors on Γ of rank at least one. It thus follows that

$$\gamma_{gm}(\Gamma) \geq \gamma_{\mathrm{div}}(\Gamma)$$

for any metric graph Γ.

1.4. **Berkovich analytic curves.** We provide a brief discussion of the structure of Berkovich analytic curves; This will allow to explain in paragraphs 1.5 and 1.6, the link between algebraic geometry of curves and that of metric graphs, presented in the previous paragraph. For further details, we refer to [15, 16, 31, 60].

Let $X/\mathbb{K}$ be an algebraic variety. The topological space underlying the Berkovich analytification X^{an} of X is described as follows. Each point x of X^{an} corresponds to a scheme-theoretic point X, with residue field $\mathbb{K}(x)$, and an extension $|\,|_x$ of the absolute value on $\mathbb{K}$ to $\mathbb{K}(x)$. The topology on X^{an} is the weakest one for which $U^{\mathrm{an}} \subset X^{\mathrm{an}}$ is open for every open affine subset $U \subset X$ and the function $x \mapsto |f|_x$ is continuous for every $f \in \mathcal{O}_X(U)$. By definition, the set $X(\mathbb{K})$ of closed points of X is naturally included in X^{an}, and has a dense image. The space X^{an} is locally compact, Hausdorff, and locally path-connected. Furthermore, X^{an} is compact iff X is proper, and path-connected iff X is connected. Analytifications of algebraic varieties is a subcategory of a larger category of $\mathbb{K}$-analytic spaces, and e.g., open subsets of X^{an} come with a $\mathbb{K}$-analytic structure in a natural way [16].

For any point x of X^{an}, the completion of the residue field $\mathbb{K}(x)$ of X with respect to $|\,|_x$ is denote by $\mathcal{H}(x)$, and the residue field of the valuation field $(\mathcal{H}(x), |\,|_x)$ is denoted by $\widetilde{\mathcal{H}(x)}$.

1.4.1. *Structure of analytic curves.* For an analytic curve X^{an}, the points can be classified into four types. By Abhyankar's inequality, $\text{tr-deg}\Big(\widetilde{\mathcal{H}(x)}/\kappa\Big) + \text{rank}\Big(|\mathcal{H}(x)^\times|/|\mathbb{K}^\times|\Big) \leq 1$, where the rank is that of a finitely generated abelian group. The point x is then of type I if it belongs to $X(\mathbb{K})$ in which case, $\mathcal{H}(x) \cong \mathbb{K}$, of type II if the transcendence degree of $\widetilde{\mathcal{H}(x)}/\kappa$ is one, of type III if the rank of the valuations extension is one, and of type IV otherwise.

1.4.2. *Semistable vertex sets and skeleta.* A semistable vertex set for X^{an} is a finite set V of points of X^{an} of type II such that $X^{\mathrm{an}} \setminus V$ is isomorphic to a disjoint union of a finite number of open annuli and an infinite number of open balls. By semistable reduction theorem [28], semistable vertex sets always exist, and more generally, any finite set of points of type II in X^{an} is contained in a semistable vertex set. The skeleton $\Gamma = \Sigma(X, V)$ of X^{an} with respect to a semistable vertex set V is the subset of X^{an} defined as the union of V and the skeleton of each of the open annuli in the semistable decomposition associated to V. Using the canonical metric on the skeleton of the open annuli, Γ comes naturally equipped with the structure of a finite metric graph contained in X^{an}. In addition, Γ has a natural model $G = (V, E)$ where the edges are in correspondence with the annuli in the semistable decomposition. In this paper, we only consider semistable vertex sets whose associated model is a simple graph, i.e., without loops and multiple edges.

Semistable vertex sets for X^{an} correspond bijectively to semistable formal models $\mathfrak{X}$ for X over R [16, 30, 15].

1.4.3. *Retraction to the skeleton.* Let Γ be a skeleton of X^{an} defined by a semistable vertex set V. There is a canonical retraction map $\tau : X^{\mathrm{an}} \twoheadrightarrow \Gamma$ which is in fact a strong deformation retraction [16]. In terms of the semistable decomposition, τ is identity on Γ, sends the points of each open ball B to the unique point of Γ in the closure $\overline{B}$ of B, called the end of B, and is the retraction to the skeleton for the open annuli [16, 15].

1.4.4. *Residue curves and the genus formula.* A point $x \in X^{\mathrm{an}}$ of type II has a (double) residue field $\widetilde{\mathcal{H}(x)}$ which is of transcendence degree one over κ. We denote by $\mathfrak{C}_x$ the unique smooth proper curve over κ with function field $\widetilde{\mathcal{H}(x)}$, and denote by g_x the genus of $\mathfrak{C}_x$. If V is any semistable vertex set for X^{an}, then for any point of type II in $X^{\mathrm{an}} \setminus V$, $g_x = 0$, and by semistable reduction theorem, we have the following genus formula:

$$g = g(X) = g(\Gamma) + \sum_{x \in V} g_x,$$

where $g(\Gamma) = |E| - |V| + 1$, for $G = (V, E)$ the model of the skeleton $\Gamma = \Sigma(X, V)$, is the first Betti number of Γ. We extend the definition of $g(\cdot)$ to all points of Γ by declaring $g(x) = 0$ if x is not a point of type II in X^{an}, obtaining in this way an augmented metric graph in the terminology of [5].

1.4.5. *Tangent vectors.* Denote by $\mathbf{H}(X^{\mathrm{an}})$ the set of points of type II and III in X^{an}. There is a canonical metric on $\mathbf{H}(X^{\mathrm{an}})$ which restricts to the metric on $\Gamma = \Sigma(X^{\mathrm{an}}, V)$ for any semistable vertex set V for X^{an}.

A geodesic segment starting at $x \in X^{\mathrm{an}} \setminus X(\mathbb{K})$ is an isometric embedding $\alpha : [0,\theta] \to X^{\mathrm{an}} \setminus X(\mathbb{K})$ for some $\theta > 0$ such that $\alpha(0) = x$. Two geodesic segments starting at x are called equivalent if they agree on a neighborhood of 0. As usual, a tangent direction at a point x is an equivalence class of geodesic segments starting at x. We denote by $\mathrm{T}_x = \mathrm{T}_x(X^{\mathrm{an}})$ the set of all tangent directions at x.

For any simply connected neighborhood U of $x \in X^{\mathrm{an}}$, there is a natural bijection between T_x and the connected components of $U \setminus \{x\}$. There is only one tangent direction at x when x is of type I; for x of type III we have $|\mathrm{T}_x| = 2$. (For x of type IV we have $|T_x| = 1$.) For a point x of type II, there is a canonical bijection between T_x and $\mathfrak{C}_x(\kappa)$, the set of closed points of the smooth proper curve $\mathfrak{C}_x$ associated to x. Points of $\mathfrak{C}_x(\kappa)$ correspond to discrete valuations on $\widetilde{\mathcal{H}(x)}$ which are trivial on κ, and the resulting bijection with T_x associates to a vector $\nu \in \mathrm{T}_x$, a discrete valuation $\mathrm{ord}_\nu : \kappa(\mathfrak{C}_x)^\times \to \mathbb{Z}$: If x^ν denotes the corresponding point of $\mathfrak{C}_x(\kappa)$ then, for every nonzero rational function $\widetilde{f} \in \kappa(\mathfrak{C}_x)$, we have $\mathrm{ord}_\nu(\widetilde{f}) = \mathrm{ord}_{x^\nu}(\widetilde{f})$.

1.5. **Specialization of divisors from curves to metric graphs.**

1.5.1. *Reduction of rational functions and the slope formula.* Let $x \in X^{\mathrm{an}}$ be a point of type 2. For a nonzero rational function f on X, there is an element $c \in \mathbb{K}^\times$ such that $|f|_x = |c|$. Define $\widetilde{f} \in \kappa(\mathfrak{C}_x)^\times$ to be the image of $c^{-1}f$ in $\widetilde{\mathcal{H}(x)} \cong \kappa(\mathfrak{C}_x)$. Note that if the valuation of $\mathbb{K}$ has a section (which is the case for algebraically closed fields [49, Lemma 2.1.15]), this can be made well-defined; otherwise, it is well-defined up to a multiplicative scalar.

If H is a $\mathbb{K}$-linear subspace of $\mathbb{K}(X)$, the collection of all possible reductions of nonzero elements of H, together with $\{0\}$, forms a κ-vector space $\widetilde{H}$. In addition, we have $\dim \widetilde{H} = \dim H$ (c.f. [4]).

A function $F : X^{\mathrm{an}} \to \mathbb{R}$ is piecewise linear if for any geodesic segment $\alpha : [a,b] \hookrightarrow X^{\mathrm{an}} \setminus V$, the pullback map $F \circ \alpha : [a,b] \to \mathbb{R}$ is piecewise linear. The outgoing slope of a piecewise linear function F at a point $x \in X^{\mathrm{an}}$ along a tangent direction $\nu \in \mathrm{T}_x$ is defined by

$$d_\nu F(x) = \lim_{t \to 0} (F \circ \alpha)'(t),$$

where $\alpha : [0,\theta] \hookrightarrow X^{\mathrm{an}}$ is a geodesic segment starting at x which represents ν. A piecewise linear function F is called harmonic at a point $x \in X^{\mathrm{an}} \setminus V$ if the outgoing slope $d_\nu F(x)$ is zero for all but finitely many $\nu \in \mathrm{T}_x$, and in addition $\sum_{\nu \in \mathrm{T}_x} d_\nu F(x) = 0$.

The following theorem will be essential [15, 17, 61]. It is called the slope formula in [15] and is also a consequence of the non-Archimedean Poincaré-Lelong formula [61].

Theorem 1.2 (Slope formula)**.** *Let X be a smooth proper curve over $\mathbb{K}$, and f be a nonzero rational function in $\mathbb{K}(X)$. Let $F = -\log|f| : X^{\mathrm{an}} \to \mathbb{R} \cup \{\pm\infty\}$. Let V be a semistable vertex set of X such that zeros and poles of f are mapped to vertices in V under the retraction map τ from X^{an} to the skeleton $\Gamma = \Sigma(X,V)$. We have*

1. *F is piecewise linear with integer slopes, and F is linear on each edge of $\Gamma \hookrightarrow X^{\mathrm{an}}$.*
2. *If x is a type-2 point of X^{an} and $\nu \in \mathrm{T}_x$, then $d_\nu F(x) = \mathrm{ord}_\nu(\widetilde{f}_x)$.*
3. *F is harmonic at all $x \in \mathbf{H}(X^{\mathrm{an}})$.*

(4) *Let x be a point in the support of* $\mathrm{div}(f)$*, let e be the ray in X^{an} with one endpoint x and another endpoint $y \in V$, and let $\nu \in \mathrm{T}_y$ be the tangent direction represented by e. Then $d_\nu F(y) = \mathrm{ord}_x(f)$.*

1.5.2. *Baker's Specialization Lemma.* Let X be a smooth proper curve over an algebraically closed complete non-Archimedean field K with a non-trivial valuation.

Consider the deformation retraction $\tau : X^{\mathrm{an}} \to \Sigma(X, V)$. Identifying $X(K)$ with points of type 1 on X^{an}, this induces a morphism $\tau_* : \mathrm{Div}(X) \to \mathrm{Div}(\Sigma(X, V))$ which is called the specialization map.

Remark 1.3. For curves defined over an arbitrary non-trivially valued non-Archimedean field, one can find an equivalent (more classical) definition of the specialization map without reference to the analytification in [21, 11, 63]. The advantage of the above presentation is that the analytification of the curve over the algebraic closure of the completion of the base field, takes care of the renormalization by ramification indices of (the choice of) the finite base field extension over which the original curve admits semistable reduction.

To each nonzero rational function f on X and each semistable vertex set V for X, one associates a corresponding rational function $F = -\log|f|$ on the skeleton Γ.

As an application of Theorem 1.2, we obtain the following [4, 11]: For every nonzero rational function f on X,

$$\tau_*(\mathrm{div}(f)) = \mathrm{div}(F).$$

Let X be a smooth proper curve over K and let Γ be a metric graph associated to X. Baker's specialization lemma [11] states that for any divisor D on X one has $r(D) \le r(\tau_*(D))$. Formulated in terms of the analytification of the curve, the statement is a direct consequence of the slope formula [15, 61] for Berkovich curves, stated above, see [4]. A more refined version of the specialization lemma, taking into account the genus or the geometry of points of $\Gamma \hookrightarrow X^{\mathrm{an}}$ can be found in [7, 4].

1.6. **Morphisms of curves induce morphisms of tropical curves.** Let X and X' be two smooth proper curves over an algebraically closed complete non-Archimedean field K. Consider a morphism $\phi` : X \to X'$, and let $\phi : X^{\mathrm{an}} \to X'^{\mathrm{an}}$ be the induced morphism between the Berkovich analytifications of X and X'^{an}.

The proof of the following theorem, as well as more precise statements concerning stronger skeletonized versions of some foundational results of Liu-Lorenzini [44], Coleman [23], and Liu [43] on simultaneous semistable reduction of curves, can be found in [5].

Theorem 1.4. *Let $\phi : X \to X'$ be a fintie morphism of smooth proper curves over K of degree d. Let C and C' be the tropical curves associated to X and X'. Then ϕ induces a tropical morphism $\phi : C \to C'$ of degree d.*

Note that, in particular, the (algebraic) gonality of X over K is bounded below by the geometric gonality of C. In general the inequality $\gamma(X) \ge \gamma(C)$ can be strict (see [6] for an example of a genus 27 tropical curve C of gonality 4 such that any X over K of genus 27 with associated tropical curve C has gonality at least 5).

In general if the base non-Archimedean field K is not algebraically closed, and $\phi : X \to Y$ is a finite morphism between two smooth proper geometrically connected curves X and Y

over K, then one gets a morphism between two tropical curves C and C' by looking at ϕ over the completion of an algebraic closure of K.

2. Geometry of graphs

In this section, we discuss some results which concern the geometry of graphs and metric graphs.

2.1. Eigenvalue estimates in graphs.

For any graph G on n vertices, denote by $\lambda_0(G) = 0 < \lambda_1(G) \leq \lambda_2(G) \leq \cdots \leq \lambda_{n-1}(G)$ all the eigenvalues of the graph Laplacian $\Delta = \Delta_G$. Recall that Δ is the positive semidefinite operator defined on the space of real valued functions on the vertices of G by

$$\Delta(f)(v) = \sum_{u:uv\in E} f(v) - f(u),$$

for any function $f : V \to \mathbb{R}$.

We first recall basic results concerning the eigenvalues of general graphs, and then restrict to some special families of graphs. For more details concerning these materials, see e.g., [27, 39]. Let S be a subset of V. The (edge) boundary of S, denoted by $B(S)$, is the set of edges $E(S, S^c)$ between a vertex in S and a vertex in its complementary $S^c = V \setminus S$. Its size is denoted by $b(S)$. The expansion of a subset S of vertices is by definition $b(S)/|S|$. The (edge) expansion of G is defined as follows:

$$exp(G) = \min_{S\subset V, |S|\leq \frac{|V|}{2}} \frac{b(S)}{|S|}.$$

By definition, the expansion is bounded by minimum degree of G.

The following theorem of Alon-Milman shows that the spectral gap of G, which is by definition the first non-trivial eigenvalue λ_1 of the Laplacian Δ, controls the expansion factor of G if G is regular.

Theorem 2.1 (Alon-Milman [2]). *Let G be a d-regular graph. Then*

$$\frac{\lambda_1}{2} \leq \exp(G) \leq \sqrt{2d\lambda_1}\,.$$

The following classical theorem of Alon-Boppana provides a lower bound on the spectral gap for a regular graph.

Theorem 2.2 (Alon-Boppana [51]). *Let G be a d-regular graph. We have*

$$\lambda_1 \leq d - 2\sqrt{d-1} + o(1)\,.$$

Friedman [34] has proved that for any ϵ, random d-regular graphs have asymptotically almost surely $|\lambda_1 - d + 2\sqrt{d-1}| \leq \epsilon$.

We note by passing that a graph is called Ramanujan if for all the non-trivial eigenvalues, $|d - \lambda_i(G)| \leq 2\sqrt{d-1}$ [27]. Until recently it was unknown if an infinite family of Ramanujan existed in all degrees d; constructions were known for $d = q + 1$ for q a prime power. An elegant recent paper of Marcus, Speilman and Srivastava [47] solved this problem, by showing the existence of an infinite family of bipartite Ramanujan graphs of any given degree d.

2.2. Eigenvalue estimates in bounded tree-width and minor closed family of graphs.

2.2.1. *Tree-decomposition, minors, and graph minor theorem.* We first recall some basic terminology on tree-decompositions of finite graphs.

Let $G = (V, E)$ be a connected graph. A tree-decomposition of G is a pair $(T, \mathcal{X})$ where T is a finite tree on a set of vertices I, and $\mathcal{X} = \{X_i : i \in I\}$ is a collection of subsets of V, subject to the following three conditions:

(1) $V = \cup_{i \in I} X_i$,
(2) for any edge e in G, there is a set $X_i \in \mathcal{X}$ which contains both end-points of e,
(3) for any triple i_1, i_2, i_3 of vertices of T, if i_2 is on the path from i_1 to i_3 in T, then $X_{i_1} \cap X_{i_3} \subseteq X_{i_2}$.

Note that the point (3) in the above definition simply means that the subgraph of T induced by all the vertices i which contain a given vertex v of the graph G is connected.

The width of a tree-decomposition $(T, \mathcal{X})$ is defined as $w(T, \mathcal{X}) = \max_{i \in I} |X_i| - 1$. The tree-width of G, denoted by $tw(G)$, is the minimum width of any tree-decomposition of G.

There is a useful duality theorem concerning the tree-width wich allows in practice to bound the tree-width of graphs. The dual notion for tree-width is bramble (as named by B. Reed [52]): a bramble in a finite graph G is a collection of connected subsets of $V(G)$ such that the union of any two of these subsets forms again a connected subset of $V(G)$. (To be more precise, we should say the graph induced on these subsets is connected.) The order of a bramble is the minimum size of a subset of vertices which intersect any set in the bramble. The bramble number of G, denoted by $bn(G)$, is the maximum order of a bramble in G.

Theorem 2.3 (Seymour-Thomas [57])**.** *For any graph G, $tw(G) = bn(G) - 1$.*

More general forms of the duality theorem can be found in [10, 29].

Example 2.4. Let H be an $n \times n$ grid. It is easy to see that $bn(H) = n$ by taking brambles formed by crosses in the grid. This shows that grid graphs can have large tree-width. Thus, the tree-width can be unbounded on planar graphs.

The other important notion in graph theory is minors in graphs. A graph H is a minor of another graph G, and we write $H \preceq G$, if H can obtained from G by a sequence of operations consisting in

- contracting an edge of G, or
- removing an edge of G.

It is easy to see that tree-width is minor monotone, in the sense that if $H \preceq G$, then $tw(H) \leq tw(G)$. It follows that bounded tree-width graphs cannot have large grid minors.

The main theorem concerning the notion of graph minors is the Robertson-Seymour finiteness theorem which states:

Theorem 2.5 (Robertson-Seymour [54])**.** *Let $\mathcal{F}$ be a family of graphs which is stable under minors, i.e., if $G \in \mathcal{F}$ and H is a minor of G, then H belongs to $\mathcal{F}$. Then there is a finite number of graphs (possibly empty if $\mathcal{F}$ contains all finite graphs) $H_1, \dots, H_k$ such that G belongs to $\mathcal{F}$ if and only if G does not contain any of H_i as minor.*

In particular, the above theorem is a far reaching generalization of Kuratowski theorem which characterizes planar graphs as the family of graphs which do not contain the complete graph on five vertices K_5, and the complete bipartite graph $K_{3,3}$ on two parts of size three each.

Remark 2.6. Robertson and Seymour prove that tree-width is bounded on the class of graphs with forbidden H-minor if and only if H is planar.

2.2.2. *Eigenvalue estimates.* Let H be a given graph. Consider the family $\mathcal{F}_H$ of all connected graphs G which do not contain H as minor. Note that $\mathcal{F}_H$ is minor closed. The following theorem shows that graphs in $\mathcal{F}_H$ are far from being expanders.

Theorem 2.7 ([41])**.** *There is a constant $h = h(H)$ such that for any graph G in $\mathcal{F}_H$ and any $1 \leq k$, we have $\lambda_k(G) \leq \frac{hd_{\max}k}{|G|}$ where $d_{\max}$ is the maximum valence of vertices in G and $|G|$ is the number of vertices in G.*

For graphs which can be embedded in a surface of genus at most g, the following more precise statement holds

Theorem 2.8 ([8])**.** *There is a universal constant c such that for any graph G which can be embedded in a surface of genus at most g, we have*

$$\lambda_k^{nr}(G) \leq c\frac{d_{\max}(g+k)}{|G|},$$

where λ_k^{nr} are the eigenvalues of the normalized Laplacian of G, and $|G|$ is the number of vertices of G.

(Note that in any graph G, with min- and max-degrees $d_{\min}$ and $d_{\max}$, one has $d_{\min}\lambda_k^{nr}(G) \leq \lambda_k(G) \leq d_{\max}\lambda_k^{nr}(G)$, and similarly, $d_{\min}|G|/2 \leq m \leq d_{\max}|G|/2$.)

We end this subsection with a discussion of the above results in the case of bounded tree-width graphs. A graph of tree-width bounded by some constant N does not contain a grid of size $N \times N$ as minor. It follows that there is an increasing function $f : \mathbb{N} \to \mathbb{N}$ such that for a graph G of tree-width $tw(G)$, one has $\lambda_k(G) \leq f(tw(G))d_{\max}k/|G|$.

For λ_1, we have the following more precise result of Chandran-Subramanian [20].

Theorem 2.9 ([20])**.** *For any graph $G = (V, E)$, the following holds*

$$\lambda_1 \leq \frac{12\big(tw(G)+1\big)d_{\max}}{|G|}.$$

2.3. **Yang-Li-Yau inequality.**

2.3.1. *Classical Yang-Li-Yau inequality.* We first recall the Li-Yau inequality [42]. Let M be a compact surface with a Riemannian metric g. We denote by $d\mu$ the volume form corresponding to its metric, and by $\mu(M)$ the total volume of M. Consider the sphere $\mathbb{S}^2$ with its standard metric g_0, and let $\phi : M \to \mathbb{S}^2$ be a non-degenerate conformal map. The group of conformal diffeomorphisms of $\mathbb{S}^2$, denoted by $\mathrm{Diff}_c(\mathbb{S}^2)$ acts on the set of non-degenerate conformal maps from M to $\mathbb{S}^2$ in a natural way. Define $\mu_c(M, \phi)$ as the supremum volume of M with the respect to the volume forms induced on M from S^2 by the conformal maps in the orbit of ϕ, i.e.,

$$\mu_c(M,\phi) := \sup_{\psi \in \mathrm{Diff}_c(\mathbb{S}^2)} \int_M |\nabla(\psi \circ \phi)|^2 d\mu.$$

The conformal area $\mu_c(M)$ of M (with respect to the conformal structure on M induced by the metric g) is by definition the infimum of $\mu_c(M,\phi)$ over non-degenerate conformal maps $\phi : M \to \mathbb{S}^2$, i.e., $\mu_c(M) := \inf_\phi \mu_c(M,\phi)$.

Theorem 2.10 (Li-Yau [42])**.** *Denote by $\lambda_1 > 0$ the first non-zero eigenvalue of the Laplacian of (M, g). Then $\lambda_1\mu(M) \leq 2\mu_c(M)$.*

This refines earlier results of Hersch [38] and Szegö [59]. As a corollary we get the following previous result of Yang and Yau [62]. Let M be a Riemann surface, equipped with a metric of constant curvature in its conformal class, λ_1 and μ the first non-trivial eigenvalue of the Laplacian and the volume of M, respectively. Denote by $\gamma(M)$ the gonality of M, the minimum degree of a (branched) covering $M \to \mathbb{P}^1(\mathbb{C})$.

Theorem 2.11 (Yang-Yau [62])**.** *For any Riemann surface M,*

$$\lambda_1\,\mu(M) \leq 8\pi\gamma(M).$$

Proof. It is easy to see that for a conformal map of positive degree d from M to N, one has $\mu_c(M) \leq d\mu_c(N)$. It follows that

$$\lambda_1\,\mu(M) \leq 2\gamma(M)\,\mu_c(\mathbb{S}^2).$$

One conclude by observing that $\mu_c(\mathbb{S}^2) = 4\pi$. □

We quickly sketch the proof of Theorem 2.10, which, like the other above mentioned results, uses Hersch lemma.

Lemma 2.12 (Hersch lemma)**.** *Let $\phi : M \to \mathbb{S}^2$ a conformal map. Denote by x_1, x_2, x_3 the coordinate functions on $\mathbb{S}^2$ for the standard embedding $\mathbb{S}^2 \hookrightarrow \mathbb{R}^3$; $x_1^2 + x_2^2 + x_3^2 = 1$. There exists $\psi \in \mathrm{Diff}_c(\mathbb{S}^2)$ such that $\int_M x_i \circ \psi \circ \phi\, d\mu = 0$ for $i = 1, 2, 3$.*

Proof. Let p be a point of $\mathbb{S}^2$ and consider the stereographic projection π_p of $\mathbb{S}^2$ to the hyperplane H_p in $\mathbb{R}^3$ tangent to $\mathbb{S}^2$ at $-p$. For each $t \in (0, 1)$, let $\alpha_{t,p} : H_p \to H$ be the dilation by a factor $1/t$ in H_p, seen as an affine plane with origin at $-p$. Consider the family of conformal maps $\psi_{t,p} = \pi_p^{-1} \circ \alpha_{t,p} \circ \pi_p : \mathbb{S}^2 \to \mathbb{S}^2$. We claim the existence of a t such that for $\psi = \psi_{t,p}$ the conclusion of theorem holds. To see this, consider the map $T : (0, 1) \times \mathbb{S}^2 \to \mathbb{B}_3$, the closed unite ball in $\mathbb{R}^3$, which sends (t, p) to the point with coordinates $\int_M x_i \circ \psi_{t,p} \circ \phi\, d\mu$ for $i = 1, 2, 3$. The map T can be extended to a map $\overline{T} : [0, 1] \times \mathbb{S}^2/\{1\} \times \mathbb{S}^2 \sim \mathbb{B}^3$, so that on the boundary $\{0\} \times \mathbb{S}^2 = \partial\mathbb{B}_3$, $\overline{T}$ restricts to the identity map. Assuming 0 not being in the image of T, one gets a retraction of $\mathbb{B}_3$ to $\partial\mathbb{B}_3$, which leads to a contradiction. □

Proof of Theorem 2.10. Fix an $\epsilon > 0$ and let ϕ be a non-degenerate conformal map $M \to \mathbb{S}^2$ such that $\mu_c(M, \phi) \leq \mu(M) + \epsilon$. By Hersch Lemma, up to replacing ϕ by a conformal map in its orbit for the action of $\mathrm{Diff}_c(\mathbb{S}^2)$, we can assume that $\int_M x_i \circ \phi\, d\mu = 0$ for $i = 0, 1, 2$, and in addition $\int_M |\nabla\phi|^2 d\mu \leq \mu_c(M, \phi) \leq \mu_c(M) + \epsilon$.

By variational characterization of λ_1, one has

$$\lambda_1 = \inf\,\frac{\int_M |\nabla f|^2\, d\mu}{\int_M f^2\, d\mu},$$

where the infimum is taken over all Lipschitz functions f on M with $\int_M f\, d\mu = 0$. In particular, one has

$$\lambda_1 \int_M (x_i \circ \phi)^2 d\mu \leq \int_M |\nabla x_i \circ \phi|^2\, d\mu.$$

Summing up over i, gives

$$\lambda_1\,\mu(M) \leq \int_M \sum_i |\nabla x_i \circ \phi|^2\,d\mu = \int_M \phi^*(\sum_i |\nabla x_i|^2 d\mu_{\mathbb{S}^2})$$
$$= 2\int_M \phi^*(d\mu_{\mathbb{S}^2}) = 2\int_M |\nabla\phi|^2\,d\mu \leq 2\mu_c(M) + 2\epsilon.$$

This holds for any $\epsilon > 0$, from which the theorem follows. □

In the next section, we provide a non-Archimedean version of the Yang-Li-Yau inequality.

2.3.2. *Yang-Li-Yau for metric graphs.* We state two types of Yang-Li-Yau inequalities, one concerning the geometric gonality and one concerning the divisorial gonality of metric graphs, as defined in the previous section. We only give the proof of the former, which is quite short.

Let C be a tropical curve with combinatorial type a graph G with set of vertices V and set of edges E. Let λ_1 be the first non-trivial eigenvalue of the Laplacian Δ of G. We have

Theorem 2.13 ([26])**.** *There is a constant A such that for any tropical curve C with combinatorial type G, we have*

$$\gamma_{gm}(C) \geq A\frac{\lambda_1}{d_{\max}}|G|,$$

where $d_{\max}$ denotes the maximum valence, and $|G|$ is the number of vertices in G.

The following is an alternative simpler proof. First, we have the following basic proposition relating the geometric gonality of a tropical curve with combinatorial type G to the tree-width of G.

Proposition 2.14. *For any tropical curve C with combinatorial type $G = (V, E)$, we have $2\gamma_{gm}(C) \geq tw(C)$.*

Proof. Let $\phi : C \to \mathbb{TP}^1$ be a morphism of degree $\gamma_{gm}(C)$. Consider the restriction of ϕ to a finite harmonic morphism from a metric graph representative Γ of C with a model graph G on vertex set V and edge set E, and denote by T the image of Λ in $\mathbb{TP}^1$, so T is a finite tree. Let I_1 be a vertex set for T which contains $\phi(V)$, and E_1 be the corresponding set of edges. For each edge e in T_1 take a point in the interior of e, and let I be the new vertex set for T obtained by adding to I_1 all these new vertices.

A tree decomposition $(T, \mathcal{X})$ of G can be defined as follows. For each vertex i in I, consider the preimage $\phi^{-1}(i)$ of i. This set consists of some (possibly empty) vertices $v_1, \dots, v_s$ of G and some (possibly empty) points $x_1, \dots, x_l$ in the interior of some edges $e_1 = u_1w_1, \dots, e_l = u_lw_l$ of G. Define $X_i = \{v_1, \dots, v_s, u_1, w_1, \dots, u_l, w_l\}$. Since ϕ is of degree $\gamma(C)$, $|\phi^{-1}(i)| \leq \gamma_{gm}(C)$ and thus, X_i has cardinality at most $2\gamma_{gm}(C)$. It is easy to check that $(T, \mathcal{X} = \{X_i\}_{i\in I})$ is a tree-decomposition of G. This proves the proposition. □

Proof of Theorem 2.13. This follows from the above proposition and the bound given in Theorem 2.9. □

As another corollary, note that if a graph G is a model of a tropical curve with bounded geometric gonality, then the tree-width of G is bounded, and thus, G cannot contain a large grid as minor. Combined with Proposition 2.14, one obtains the following corollary.

Corollary 2.15. *For any tropical curve C of combinatorial type G, one has $f(2\gamma(C)) \geq \lambda_k(G).|G|\frac{d_{\max}}{k}$. In particular, if in a family of tropical curves C_i of combinatorial type G_i, $d_{\max}$ is bounded and for some constant k, $\lambda_k(G_i).|G_i|$ tend to infinity, then one has $\gamma_{gm}(C_i) \to \infty$.*

Let now Γ be a metric graph, and denote by γ_{div} the divisorial gonality of Γ, which we recall, is the smallest integer d such that there exists a divisor of degree d and rank one on Γ. Since the fibers of any tropical morphism of degree d from the tropical C defined by Γ to $\mathbb{TP}^1$ is a divisor of degree d and rank at least one, it follows that $\gamma_{gm}(C) \geq \gamma_{\text{div}}(\Gamma)$.

Theorem 2.16 ([9])**.** *There exists a constant C such that for any compact metric graph Γ of total length $\mu(\Gamma)$ with first non-trivial eigenvalue $\lambda_1(\Gamma)$ of the Laplacian Δ, the following holds*

$$\gamma_{\text{div}}(\Gamma) \geq C\frac{\lambda_1(\Gamma)\ell_{\min}(\Gamma)\mu(\Gamma)}{d_{\max}}.$$

Here $\ell_{\min}$ is the minimum edge length in Γ.

In the above theorem, Δ is the Laplacian on a metric graph [13, 14, 63].

As a corollary of Theorem 2.16, and the specialization inequality, we get

Theorem 2.17. *Let X be a smooth proper curve over a non-Archimedean field K, and let Γ be a metric graph associated to X. We have*

$$\gamma(X) \geq C\frac{\mu(\Gamma)\ell_{\min}(\Gamma)\lambda_1(\Gamma)}{d_{\max}}.$$

Here C is the constant provided by Theorem 2.16.

It would be interesting to define an appropriate suitable notion of conformal invariance for metric graphs, in the spirit of [42].

2.4. **Examples of Cayley graphs with large eigenvalues.** The basic example is the example of a family of Cayley graphs of fixed valence which form a family of expanders, i.e., such that the first non-trivial eigenvalue of the Laplacian of graphs in the family is lower bounded by a constant. Consider e.g. a finite index subgroup G of $\mathrm{SL}_n(\mathbb{Z})$ for $n \geq 3$. Then G satisfies Kazhdan (T) property, and as a consequence, for a fixed symmetric set of generators S for G, the family of Cayley graphs $\mathrm{Cay}(H\backslash G; S)$ where H runs over all finite index subgroup of G form a family of expanders [46].

Example 2.18. Let X be a smooth curve over a number field k of genus at least two. There exists an infinite family of étale covers $X_i \to X$ such that the Cayley graphs $\mathrm{Cay}(X_i/X; S)$, for S a (profinite) generating set for $\pi_1^{\text{ét}}(X)$, form a family of expanders with sizes tending to infinity. This is because the topological fundamental group of $X_{\mathbb{C}}$ has a quotient which is isomorphic to $\mathrm{SL}_3(\mathbb{Z})$. By Yang-Li-Yau, $\gamma(X_i)$ tends to infinity.

The following recent result of Pyber-Szabó [53] (see also [18]) provides a rich class of examples of Cayley graphs with large eigenvalues. For earlier results of similar type see [37, 35].

Let m be an integer and consider a family of subgroups G_p of $GL_m(\mathbb{F}_p)$ indexed by all but finitely many prime numbers p. Let S_p, $S_p = S_p^{-1}$, be a generating set for G_p of order at most a constant s, for any p. Consider the family of Cayley graphs $\mathrm{Cay}(G_p; S_p)$.

Theorem 2.19 (Pyber-Szabó [53]). *If the groups G_p are non-trivial perfect groups generated by their elements of order p, then $\lambda_1(\mathrm{Cay}(G_p;S_p))|\mathrm{Cay}(G_p;S_p)| \to \infty$, when p tends to infinity. More precisely, $\lambda_1(\mathrm{Cay}(G_p;S_p)) >> \frac{1}{\log|G_p|^A}$ for some constant A.*

For a survey of recent results see e.g. [55].

3. Applications in Algebraic and Arithmetic Geometry

In this section we discuss some applications of the materials of the previous sections in algebraic and arithmetic geometry.

3.1. **Brill-Noether theorem.** Let X be a smooth proper curve of genus g over a field κ. Brill-Noether theory studies the geometry of the space W^r_d of divisors of a given degree d which move in a linear system of dimension at least a given integer r. The main theorem of Brill-Noether theory, in rank one, is the Brill-Noether theorem, proved by Griffiths and Harris, which asserts that

Theorem 3.1. *([36]) Let g, and r and d as above. Define $\rho = g-(r+1)(g-d+r)$. Then for a generic curve X,*

(i) *If $\rho < 0$, then there is no divisor of degree d and rank at least r on X.*
(ii) *If $\rho \geq 0$, then W^r_d has dimension* $\min\{g,\rho\}$.

We show how to prove (i), which is the more difficult part of the theorem, by using divisor theory on graphs and by essentially following [24] (note that the presentation is slightly different from [24]).

Since the assertion is an open property, it will be enough to prove the existence of a smooth proper curve of genus g satisfying (i). By Baker's specialization lemma, it will be enough to show the existence of a metric graph Γ of genus g such that there is no divisor of degree d and rank at least r on Γ provided that $\rho < 0$. The simplest graphs for which we can write down explicitly the whole divisor theory will do the job: these are cycles and, more generally, (generic) chains of cycles.

3.1.1. *Rank of divisors on a generic chain of cycles.* It is possible to provide a formula for the rank of divisors on a metric graph Γ obtained as a connected sum of two metric graphs Γ_1 and Γ_2, c.f. [4]. This is done as follows. Consider two metric graphs Γ_1 and Γ_2, and suppose that two distinguished points $v_1 \in \Gamma_1$ and $v_2 \in \Gamma_2$ are given. Recall first that the direct sum of (Γ_1, v_1) and (Γ_2, v_2), denoted $\Gamma = \Gamma_1 \vee \Gamma_2$, is the metric graph obtained by identifying the points v_1 and v_2 in the disjoint union of Γ_1 and Γ_2. Denote by $v \in \Gamma$ the image of v_1 and v_2 in $\widetilde{\Gamma}$. (By abuse of notation, we will use v to denote both v_1 in Γ_1 and v_2 in Γ_2.) We refer to $v \in \widetilde{\Gamma}$ as a cut-vertex and to $\Gamma = \Gamma_1 \vee \Gamma_2$ as the decomposition corresponding to the cut-vertex v. There is an addition map $\mathrm{Div}(\Gamma_1) \oplus \mathrm{Div}(\Gamma_2) \to \mathrm{Div}(\Gamma)$ which associates to any pair of divisors D_1 and D_2 in $\mathrm{Div}(\Gamma_1)$ and $\mathrm{Div}(\Gamma_2)$ the divisor, $D_1 + D_2$ on Γ defined by pointwise addition of coefficients in D_1 and D_2. Let $r_1(\cdot)$, $r_2(\cdot)$, and $r(\cdot) = r_{\Gamma_1 \vee \Gamma_2}(\cdot)$ be the rank functions in Γ_1, Γ_2, and Γ, respectively. For any non-negative integer s, denote by $\eta_{v,D_1}(s)$, or simply $\eta(s)$, the smallest integer n such that $r_1\big(D_1 + n(v)\big) = s$. Then for any divisor D_2 in $\mathrm{Div}(\Gamma_2)$, we have

$$r(D_1 + D_2) = \min_{s \in \mathbb{N} \cup \{0\}} \Big\{ s + r_2\big(D_2 - \eta(s)\,(v)\big) \Big\}. \tag{1}$$

Formula (1) is very handy for dealing with metric graphs which are direct sum of simple graphs such as cycles.

Consider a metric graph Γ which is a chain of cycles C_i, $i = 1, \dots, g$, of length $\ell_1, \dots, \ell_g$, respectively. Denote the cut vertices by $v_1, \dots, v_{g-1}$, so that each v_i lies in both C_i and C_{i+1}. In addition chose points $v_0 \neq v_1$ and $v_g \neq v_{g-1}$ on C_1 and C_g respectively, so that the corresponding graph with vertex set $v_0, \dots, v_g$ is a simple graph model of Γ. At some point later we will suppose that for any cycle C_i in Γ, $1 \leq i \leq g$, the two vertices v_{i-1} and v_i which lie on C_i are *generically* located on C_i. We will refer to such Γ as a *generic chain of cycles*

Consider now a divisor $D = \sum_{i=1}^{g} D_i$ of degree d on Γ, so that each D_i is a divisor on C_i, and D_i has support in $C_i \setminus \{v_{i-1}\}$ for $i \geq 2$ (note that this decomposition is unique). We are interested in determining how large the rank of D can be.

Consider the cut-vertex v_i, $i \in \{1, \dots, g-1\}$, in Γ, and denote by $\Gamma_{i,1}$ and $\Gamma_{i,2}$ the two metric graphs which contain v_{i-1} and v_{i+1}, respectively, in the decomposition of Γ associated to v_i, i.e., $\Gamma_{i,1} \vee_{v_i} \Gamma_{i,2} = \Gamma$. Denote by $D_{i,1}$ and $D_{i,2}$ the restriction of D to $\Gamma_{i,1}$ and $\Gamma_{i,2}$ respectively, and let $\eta_i : \mathbb{N} \cup \{0\} \to \mathbb{Z}$ be the function defined above for the cut-vertex v_i in Γ. We have the following relation coming from (1):

$$r_\Gamma(D) = \min_{s \geq 0}\{s + r_{i,2}(D_{i,2} - \eta_i(s)(v_i))\}, \tag{2}$$

where $r_{i,2}$ denotes the rank function on $\Gamma_{i,2}$.

It follows that the rank of D is determined as soon as the functions η_i are determined. Indeed, the functions η_i satisfy similar recursive equations between them. Suppose we already know the function $\eta_i : \mathbb{N} \cup \{0\} \to \mathbb{Z}$. To determine η_{i+1}, we consider in the metric graph $\Gamma_{i+1,1}$ the cut-vertex v_i whose removal gives the metric graphs $\Gamma_{i,1}$ and the cycle C_{i+1}. By definition, $\eta_{i+1}(s)$ is the smallest integer satisfying $r_{i+1,1}\big(D_{i+1,1} + \eta_{i+1}(s)(v_{i+1})\big) = s$. The recursive relation satisfied by the left-hand side of this equation gives

$$s = r_{i+1,1}\big(D_{i+1,1} + \eta_{i+1}(s)(v_{i+1})\big) = \min_{t \geq 0}\Big\{t + r_{C_{i+1}}\big(D_{i+1} + \eta_{i+1}(s)(v_{i+1}) - \eta_i(t)(v_i)\big)\Big\}. \tag{3}$$

It follows that the function η_{i+1} can be calculated from the values of η_i and the rank function on C_{i+1}. As a consequence, once we know η_n we can determine the rank of D.

3.1.2. *Brill-Noether theory on a generic chain of cycles.* Consider now a divisor $D = \sum_{i=1}^{g} D_i$ on Γ of degree d, that we suppose to be v_0-reduced [50, 3]. This means that

- (i) For $i \geq 2$, each D_i is effective of degree at most one.
- (ii) Among all divisors D in the linear equivalence class of D, D_1 has the maximum coefficient at v_0.

Note that each divisor has a unique v_0-reduced divisor in its linear equivalence class.

Denote by d_0 the coefficient of D at v_0. We now present a criterion for the rank $r_\Gamma(D)$ to be at least r. It will be convenient to define $\eta_0(s) = s - d_0$, and define η_g by

$$\eta_g(s) = \min_n\Big\{n \in \mathbb{Z} \,:\, r_\Gamma\big(D + n(v_g)\big) = s\Big\},$$

so that $\eta_g(r) \leq 0$ if and only if $r_\Gamma(D) \geq r$. Note that recursive Equation (3) remains valid for $i = g$, with the definition $\Gamma_{g+1,1} = \Gamma$ (so $r_{g+1,1} = r_\Gamma$ and $D_{g+1,1} = D$).

In taking the minimum in (3), the values of any function η_i over $t \geq r+1$ are automatically larger than r. In addition, the recursive equation relating η_{i+1} to η_i does not involve the value of η_i on larger integers. Therefore, if we are just interested in knowing whether or not $r_\Gamma(D) \geq r$, we can restrict all functions η_i to the set $\{0, \dots, r\}$ and consider the values $\eta_i(0), \dots, \eta_i(r)$.

The following proposition summarizes the basic properties of η_i, and gives a necessary and sufficient condition for $r_\Gamma(D) \geq r$ in terms of the values of $\eta_i(r)$.

Proposition 3.2. *Let r be a non-negative integer. Then:*

1. *For $i = 0$, we have $\eta_0(s) = s - d_0$ for all s.*
2. *For each i, we have $\eta_i(0) < \eta_i(1) < \cdots < \eta_i(r)$.*
3. *$r_\Gamma(D) \geq r$ if any only if for any $i = 0, \dots, g$ we have $\eta_i(r) \leq 0$.*

We now make the recursive equation (3) more explicit to relate the values of η_{i+1} to the values of η_i. For fixed i, (3) tells us that

$$r_{C_{i+1}}(D_{i+1} + \eta_{i+1}(s)(v_{i+1}) - \eta_i(t)(v_i)) \geq s - t$$

for every $t \leq s$, with equality for some value of t. Moreover, the inequalities for $t \leq s-2$ are implied by the inequality for $t = s-1$. Indeed, since $\eta_i(s-1) - \eta_i(t) \geq s - 1 - t$ and C_{i+1} is of genus one, the inequality for $s-1$ implies that

$$\begin{aligned} r_{C_{i+1}}\Big(D_{i+1} + \eta_{i+1}(s)(v_{i+1}) - \eta_i(t)(v_i)\Big) &\geq r_{C_{i+1}}\Big(D_{i+1} + \eta_{i+1}(s)(v_{i+1}) - \eta_i(s-1)(v_i)\Big) \\ &\qquad + s - t - 1 \\ &\geq s - t. \end{aligned}$$

Therefore the minimum in (3) is achieved for $t = s$ or $t = s-1$. For these two values of t we have

$$r_{C_{i+1}}\Big(D_{i+1} + \eta_{i+1}(s)(v_{i+1}) - \eta_i(s)(v_i)\Big) \geq 0, \text{ and} \tag{4}$$

$$r_{C_{i+1}}\Big(D_{i+1} + \eta_{i+1}(s)(v_{i+1}) - \eta_i(s-1)(v_i)\Big) \geq 1, \tag{5}$$

and $\eta_{i+1}(s)$ is defined in such a way that one of the two above inequalities is an equality.

The following cases can happen:

(a) $D_{i+1} = 0$.
(b) $D_{i+1} = (z_{i+1})$ for a point $z_{i+1} \in C_{i+1} \setminus \{v_i\}$.

In case (a), equations (4) and (5) tell us that $\eta_{i+1}(s) = \eta_i(s) + 1$ for any $0 \leq s \leq r$.

In case (b), since C_{i+1} is of genus one, we have $\eta_{i+1}(s) \in \{\eta_i(s), \eta_i(s) - 1\}$. In addition, $\eta_{i+1}(s) = \eta_i(s) - 1$ if and only if

(1) $(z_{i+1}) + (\eta_i(s) - 1)(v_{i+1}) - \eta_i(s)(v_i) \sim 0$ in C_{i+1} (by Equation (4)); and
(2) $\eta_i(s-1) \leq \eta_i(s) - 2$ (by Equation (5)).

Since, by genericity assumption, v_i and v_{i+1} are generically located on C_{i+1}, Relation (1) above can be satisfied by at most one value of $0 \leq s \leq r$. In other words, we have $\eta_{i+1}(t) = \eta_i(t)$ for all $0 \leq t \leq r$ except possibly for one value of s satisfying properties (1) and (2) above, for which we will have $\eta_{i+1}(s) = \eta_i(s) - 1$.

Consider now $\eta_0, \dots, \eta_n$ as vectors of $\mathbb{Z}^{r+1}$ with basis $e_0, \dots, e_r$ and define

$$\mathcal{A} := \{\alpha(0)e_0 + \cdots + \alpha(r)e_r \mid \alpha(0) < \cdots < \alpha(r) \leq 0\} \subset \mathbb{Z}^{r+1}.$$

A *lingering lattice path* in $\mathcal{A}$ of length g is a sequence of vectors $\alpha_0, \dots, \alpha_g$ in $\mathcal{A}$ such that for each i exactly one of the following holds:

- $\alpha_{i+1} = \alpha_i + \sum_{s=0}^{r} e_s$
- $\alpha_{i+1} = \alpha_i - e_s$ for some $0 \leq s \leq r$
- $\alpha_{i+1} = \alpha_i$.

A lingering lattice path is called *of type* (d, d_0) if $\alpha_0 = \sum_s (s - d_0)e_s$ and the number of i such that $\alpha_{i+1} = \alpha_i - e_s$, for some s, is $d - d_0$. By the above discussion, each v_0-reduced divisor D of degree d and rank at least r defines a lingering path $\eta_0, \dots, \eta_g$ in $\mathcal{A}$ of type (d, d_0) with $d = \deg(D)$ and d_0 the coefficient of v_0.

Let D be a v_0-reduced divisor which gives the lingering lattice path η. Since the degree of D is d and the coefficient of v_0 is d_0, there are exactly $g - d + d_0$ indices i with $D_i = 0$ and so $\eta_{i+1} = \eta_i + \sum_s e_s$. The coordinate $\eta_g(r)$ is thus equal to

$$\eta_0(r) + g - d + d_0 - a_r = r - d_0 + g - d + d_0 - a_r = r + g - d - a_r,$$

where a_r is the number of indices i such that $\eta_{i+1} = \eta_i - e_r$. This shows that $a_r \geq r + g - d$. By the definition of $\mathcal{A}$, and since $\eta_i \in \mathcal{A}$ for each i, the number a_s of indices i such that $\eta_{i+1} = \eta_i - e_s$ is at least a_r, i.e., $a_s \geq a_r$. so the number of indices i with $\eta_{i+1} = \eta_i$ is at most $g - (r+1)a_r - n + d - d_0$. A simple calculation shows that this is at most $\rho + r - d_0$, where $\rho = g - (r+1)(g - d + r)$. In particular, if $\rho < 0$, this number would be negative, which implies there is no divisor of degree d and rank at least r on Γ.

Theorem 3.3. *([24]) Let Γ be a generic chain of cycles of genus g. If $\rho < 0$, there is no divisor of degree d and rank at least r on Γ.*

As we noted, this theorem implies part (1) of Griffiths-Harris Theorem 3.1.

3.2. **Improved Chabauty-Coleman.** In this section, we discuss a recent theorem due to Katz and Zureick-Brown [40]; the presentation follows [4].

Let K be a number field and suppose X is a smooth, proper, geometrically integral curve over K of genus $g \geq 2$. Let J be the Jacobian of X, which is an abelian variety of dimension g defined over K. If the Mordell-Weil rank r of $J(K)$ is less than g, Coleman [25] adapted an old method of Chabauty to prove that if $p > 2g$ is a prime which is unramified in K and $\mathfrak{p}$ is a prime of good reduction for X lying over p, then $\#X(K) \leq \#\bar{X}(\mathbf{F}_\mathfrak{p}) + 2g - 2$. Here $\bar{X}$ denotes the special fiber of a smooth proper model for X over the completion $\mathcal{O}_\mathfrak{p}$ of $\mathcal{O}_K$ at $\mathfrak{p}$ and $\mathbf{F}_\mathfrak{p} = \mathcal{O}_K/\mathfrak{p}$. Stoll [58] improved this bound by replacing $2g - 2$ with $2r$. Lorenzini and Tucker [45] (see also [48]) proved the same bound as Coleman without assuming that X has good reduction at $\mathfrak{p}$; in their bound, $\bar{X}(\mathbf{F}_\mathfrak{p})$ is replaced by $\bar{\mathfrak{X}}^{\mathrm{sm}}(\mathbf{F}_\mathfrak{p})$ where $\mathfrak{X}$ is a proper regular model for X over $\mathcal{O}_\mathfrak{p}$ and $\bar{\mathfrak{X}}^{\mathrm{sm}}$ is the smooth locus of the special fiber of $\mathfrak{X}$. Katz and Zureick-Brown combine the improvements of Stoll and Lorenzini-Tucker by proving:

Theorem 3.4 ([40])**.** *Let K be a number field and suppose X is a smooth, proper, geometrically integral curve over K of genus $g \geq 2$. Suppose the Mordell-Weil rank r of $J(K)$ is less* *than g, and that $p > 2g$ is a prime which is unramified in K. Let $\mathfrak{p}$ be a prime of $\mathcal{O}_K$ lying over p and let $\mathfrak{X}$ be a proper regular model for X over $\mathcal{O}_\mathfrak{p}$. Then*

$$\#X(K) \leq \#\bar{\mathfrak{X}}^{\mathrm{sm}}(\mathbf{F}_\mathfrak{p}) + 2r.$$

In order to explain the main new idea in the paper of Katz and Zureick-Brown, we first quickly recall the basic arguments used by Coleman, Stoll, and Lorenzini-Tucker. (See [48] for a highly readable and more detailed overview.) Assume first that we are in the setting of Coleman's paper, so that $r < g$, $p > 2g$ is a prime which is unramified in K, and X has good reduction at the prime $\mathfrak{p}$ lying over p. Fix a rational point $P \in X(K)$ (if there is no such point, we are already done!). Coleman associates to each regular differential ω on X over $K_\mathfrak{p}$ (the $\mathfrak{p}$-adic completion of K) a "definite p-adic integral" $\int_P^Q \omega \in K_\mathfrak{p}$. If V_{chab} denotes the vector space of all ω such that $\int_P^Q \omega = 0$ for all $Q \in X(K)$, Coleman shows that $\dim V_{\mathrm{chab}} \geq g - r > 0$. Locally, p-adic integrals are obtained by formally integrating a power series expansion for ω with respect to a local parameter. Using this observation and an elementary Newton polygon argument, Coleman proves that

$$\#X(K) \leq \sum_{\widetilde{Q} \in \bar{X}(\mathbf{F}_\mathfrak{p})} \left(1 + n_{\widetilde{Q}}\right),$$

where $n_{\widetilde{Q}}$ is the minimum over all nonzero ω in V_{chab} of $\mathrm{ord}_{\widetilde{Q}}\widetilde{\omega}$; here $\widetilde{\omega}$ denotes the reduction of a suitable rescaling $c\omega$ of ω to $\bar{X}$, where the scaling factor is chosen so that $c\omega$ is regular and non-vanishing along the special fiber $\bar{X}$. If we choose any nonzero $\omega \in V_{\mathrm{chab}}$, then the fact that the canonical divisor class on $\bar{X}$ has degree $2g-2$ gives

$$\sum_{\widetilde{Q} \in \bar{X}(\mathbf{F}_\mathfrak{p})} n_{\widetilde{Q}} \leq \sum_{\widetilde{Q} \in \bar{X}(\mathbf{F}_\mathfrak{p})} \mathrm{ord}_{\widetilde{Q}}\widetilde{\omega} \leq 2g - 2,$$

which yields Coleman's bound.

Stoll observed that one could do better than this by adapting the differential ω to the point $\widetilde{Q}$ rather than using the same differential ω on all residue classes. Define the *Chabauty divisor*

$$D_{\mathrm{chab}} = \sum_{\widetilde{Q} \in \bar{X}(\mathbf{F}_\mathfrak{p})} n_{\widetilde{Q}}(\widetilde{Q}).$$

Then D_{chab} and $K_{\bar{X}} - D_{\mathrm{chab}}$ are both equivalent to effective divisors, so by Clifford's inequality (applied to the smooth proper curve $\bar{X}$) we have $r(D_{\mathrm{chab}}) := h^0(D_{\mathrm{chab}}) - 1 \leq \frac{1}{2}\mathrm{deg}(D_{\mathrm{chab}})$. On the other hand, by the semicontinuity of h^0 under specialization we have $h^0(D_{\mathrm{chab}}) \geq \dim V_{\mathrm{chab}} \geq g - r$. Combining these inequalities gives

$$\sum_{\widetilde{Q} \in \bar{X}(\mathbf{F}_\mathfrak{p})} n_{\widetilde{Q}} \leq 2r$$

which leads to Stoll's refinement of Coleman's bound.

Lorenzini and Tucker observed that one can generalize Coleman's bound to the case of bad reduction as follows. Let $\mathfrak{X}$ be a proper regular model for X over $\mathcal{O}_\mathfrak{p}$ and note that points of $X(K)$ specialize to $\bar{\mathfrak{X}}^{\mathrm{sm}}(\mathbf{F}_\mathfrak{p})$. One obtains by a similar argument the bound

$$\#X(K) \leq \sum_{\widetilde{Q} \in \bar{\mathfrak{X}}^{\mathrm{sm}}(\mathbf{F}_\mathfrak{p})} \left(1 + n_{\widetilde{Q}}\right), \tag{6}$$

where $n_{\widetilde{Q}}$ is the minimum over all nonzero ω in V_{chab} of $\mathrm{ord}_{\widetilde{Q}}\widetilde{\omega}$; here $\widetilde{\omega}$ denotes the reduction of (a suitable rescaling of) ω to the unique irreducible component of the special fiber of $\mathfrak{X}$ containing $\widetilde{Q}$ and $\dim V_{\mathrm{chab}} \geq g - r > 0$ as before. Choosing a nonzero $\omega \in V_{\mathrm{chab}}$ as in Coleman's bound, the fact that the relative dualizing sheaf for $\mathfrak{X}$ has degree $2g-2$ gives the Lorenzini-Tucker bound.

In order to combine the bounds of Stoll and Lorenzini-Tucker, we see that it is natural to form the Chabauty divisor

$$D_{\mathrm{chab}} = \sum_{\widetilde{Q} \in \bar{\mathfrak{X}}^{\mathrm{sm}}(\mathbf{F}_{\mathfrak{p}})} n_{\widetilde{Q}}(\widetilde{Q})$$

and try to prove, using some version of semicontinuity of h^0 and Clifford's inequality, that its degree is at most $2r$. This is the main technical innovation of Katz and Zureick-Brown, so we state it as a theorem:

Theorem 3.5 (Katz–Zureick-Brown). *The degree of D_{chab} is at most $2r$.*

Combining Theorem 3.5 with (6) yields Theorem 3.4. As noted by Katz and Zureick-Brown, if one makes a base change from $K_{\mathfrak{p}}$ to an extension field K' over which there is a regular semistable model $\mathfrak{X}'$ for X dominating the base change of $\mathfrak{X}$, then the corresponding Chabauty divisors satisfy $D'_{\mathrm{chab}} \geq D_{\mathrm{chab}}$. (Here D'_{chab} is defined relative to the K'-vector space $V'_{\mathrm{chab}} = V_{\mathrm{chab}} \otimes_K K'$; one does not want to look at the Mordell-Weil group of J over extensions of K.) In order to prove Theorem 3.5, we may therefore assume that $\mathfrak{X}$ is a regular semistable model for X (and also that the residue field of K' is algebraically closed).

Let $d = \deg(D_{\mathrm{chab}})$. We now explain how to prove that $d \leq 2r$ when $\mathfrak{X}$ is a semistable regular model using the augmented (weighted) version of Baker's specialization lemma from [7], which takes into account the genus of the irreducible components in the semistable model of the curve X.

Sketch of the proof of Theorem 3.5. Let $s = \dim_{K'} V'_{\mathrm{chab}} - 1 \geq g - r - 1 \geq 0$. We can identify V'_{chab} with an $(s+1)$-dimensional space W of rational functions on X in the usual way by identifying $H^0(X, \Omega^1_X)$ with $L(K_X) = \{f \ : \ \mathrm{div}(f) + K_X \geq 0\}$ for a canonical divisor K_X on X. The divisor D_{chab} on $\bar{\mathfrak{X}}^{\mathrm{sm}}$ defines in a natural way a divisor D of degree d on the augmented metric graph Γ, the dual graph of $\mathfrak{X}$, with the genus function which gives the augmentation (in the terminology of [5]).

Denote by K the canonical divisor of the augmented metric graph Γ, which we recall by definition, is $K = \sum_{x \in \Gamma}(2\,\mathrm{val}(x) - 2 + g(x))(x)$. As a corollary of the augmented specialization theorem [7], one sees that the rank of $K - D$ in the augmented metric graph is at least $g - r - 1 \geq 0$. By Clifford's theorem for augmented metric graphs, which is a consequence of the Riemann-Roch theorem, $2r(K - D) \leq \deg(K - D)$, which gives $\deg(D) \leq 2r$. □

3.3. **Rational points and Galois representations.** We give now an overview of the recent applications of Theorem 2.11 to arithmetic geometry over number fields from [32].

3.3.1. *Rational points.* Let k be a number field. Let X be a smooth geometrically connected curve over k. Consider a family X_i of étale covers of X defined over k. Consider an Archimedean place of k, an embedding to $\mathbb{C}$, and denote by $X_{i,\mathbb{C}}$ and $X_{\mathbb{C}}$ the corresponding Riemann surfaces associated to X_i and X. The fundamental group $\pi_1(X_{i,\mathbb{C}})$ is a subgroup of $\pi_1(X_{\mathbb{C}})$ (we omit the base points), and fixing a symmetric set of generators S for $\pi_1(X_{\mathbb{C}})$ (i.e., $S = S^{-1}$) allows to define the Cayley graph $\mathrm{Cay}(\pi_1(X_{i,\mathbb{C}})\backslash\pi_1(X_{\mathbb{C}}); S)$ as the quotient of $\mathrm{Cay}(\pi_1(X_{\mathbb{C}}); S)$ by the left action of $\pi_1(X_{i,\mathbb{C}})$ on $\mathrm{Cay}(\pi_1(X_{\mathbb{C}}); S)$. To simplify the notation, we simply write $\mathrm{Cay}(X_i/X; S)$ to denote this finite Cayley graph.

Consider the combinatorial Laplacian of $\mathrm{Cay}(X_i/X; S)$ and let $\lambda_1^{(i)}$ be its first non-trivial eigenvalue.

Theorem 3.6 (Burger [19])**.** *There is a constant $C > 1$ depending only on $X_{\mathbb{C}}$ such that $C^{-1}\lambda_1(X_{i,\mathbb{C}}) \leq \lambda_1^{(i)} \leq C\lambda_1(X_{i,\mathbb{C}})$ for any i.*

Here $X_{i,\mathbb{C}}$ is equipped with a metric of constant curvature.

Proof. By going to the universal cover $\tilde{X}$ and taking a tiling of $\tilde{X}$ obtained by fixing a fundamental domain for the action of $\pi_1(X_{\mathbb{C}})$ on $\tilde{X}$, one can see that each surface $X_{i,\mathbb{C}}$ admits a decomposition into domains isometric to a fixed domain F with piecewise smooth boundary (independent of i) such that the dual complex associated to this tiling is precisely the Cayley graph $\mathrm{Cay}(X_i/X; S)$. The theorem now follows by looking at the discretization functional $\phi : \mathcal{C}^\infty(X_{i,\mathbb{C}}) \to C^0(\mathrm{Cay}(X_i/X; S))$ which sends f to $\phi(f)$ taking a value at a vertex v of $\mathrm{Cay}(X_i/X; S)$ equal to the average of f on the domain corresponding to v in the tiling of $X_{i,\mathbb{C}}$. The inverse of ϕ sends a discrete function defined on vertices of the Cayley graph to a smoothing of the function constant on each domain of the surface $X_{i,\mathbb{C}}$. The ratio between $\lambda_1^{(i)}$ and $\lambda_1(X_{i,\mathbb{C}})$ remains bounded away from zero and infinity, by a non-zero function depending on the first Neumann eigenvalue of the Laplacian operator on F. □

Corollary 3.7. *Assume $\lambda_1^{(i)}|\mathrm{Cay}(X_i/X; S)|$ tends to infinity. Then the gonality of X_i tends to infinity.*

Proof. The volume of $X_{i,\mathbb{C}}$ is $|\mathrm{Cay}(X_i/X; S)|$ times the volume μ of X. By Yang-Li-Yau inequality, $\lambda_1(X_{i,\mathbb{C}})|\mathrm{Cay}(X_i/X; S)|\,\mu \leq 8\pi\gamma(X_i, \mathrm{C})$. Since $\lambda_1^{(i)}|\mathrm{Cay}(X_i/X; S)|$ tends to infinity, and $\lambda_1^{(i)}$ is within a constant factor of $\lambda_1(X_{i,\mathbb{C}})$, it follows that $\gamma(X_{i,\mathbb{C}})$ tends to infinity and the result follows. □

Theorem 3.8 ([32])**.** *Let X_i/X be a family of étale covers of X. Assume that*

$$\lambda_1^{(i)}|\mathrm{Cay}(X_i/X; S)| \to \infty.$$

For any d, the set

$$\bigcup_{k_1 : [k_1 : k] \leq d} X_i(k_1)$$

is finite for all but finitely many i.

Proof. Under the hypothesis of the theorem, the gonality $\gamma(X_i)$ of X_i tends to infinity so there is N_d such that for $i \geq N_d$, $\gamma(X_i) > 2d$. By Faltings-Frey theorem [33], the set $\bigcup_{k_1 : [k_1 : k] \leq d} X_i(k_1)$ is finite for any $i \geq N_d$. □

3.3.2. *Galois representations.*

Theorem 3.9 (Ellenberg-Hall-Kowalski [32]). *Let k be a number field and X/k a smooth geometrically connected algebraic curve. Let $\mathcal{A} \to X$ be a principally polarized abelian scheme over X of dimension $g \geq 1$, defined over k, and let*

$$\rho : \pi_1(X_{\mathbb{C}}) \to \mathrm{Sp}_{2g}(\mathbb{Z})$$

be the associated monodromy representation. For any finite extension k_1/k and a rational point $t \in X(k_1)$, let

$$\overline{\rho}_{t,\ell} : \mathrm{Gal}(\overline{k}/k_1) \to \mathrm{Sp}_{2g}(\mathbb{F}_\ell)$$

be the Galois representation associated to the ℓ-torsion points of $\mathcal{A}_t$.

Assume that the image of ρ is Zariski dense in Sp_{2g}. Then the set

$$\bigcup_{k_1:\ [k_1:k]=d} \{t \in X(k_1) \,|\, \textit{the image of } \overline{\rho}_{t,\ell} \textit{ does not contain } Sp_{2g}(\mathbb{F}_\ell)\}$$

is finite for any $d \geq 1$ and any but finitely many ℓ (depending on d).

Proof. By assumption the image I of ρ is dense in $\mathrm{Sp}_{2g}(\mathbb{Z})$ which implies that the image I_ℓ of the reduction map $I \to \mathrm{Sp}_{2g}(\mathbb{F}_\ell)$ is the whole $\mathrm{Sp}_{2g}(\mathbb{F}_\ell)$ for all but finitely many ℓ. Suppose that for each conjugacy class of a maximal subgroup of $\mathrm{Sp}_{2g}(\ell)$ a fixed representative is designed, and consider all the pairs (ℓ, J) where ℓ is such that $I_\ell = \mathrm{Sp}_{2g}(\mathbb{F}_\ell)$ and $J < \mathrm{Sp}_{2g}(\mathbb{F}_\ell)$ runs over the representatives of the conjugacy classes of maximal subgroups of $\mathrm{Sp}_{2g}(\mathbb{F}_\ell)$. Each such pair (ℓ, J) gives rise to an étale cover $X_{\ell,J} \to X$ with the property that $\mathrm{Cay}(X_{\ell,J}/X; S) = \mathrm{Cay}(J\backslash\mathrm{Sp}_{2g}(\mathbb{F}_\ell); S)$.

In particular, the set of all k_1-rational points t of X such that I_ℓ is not in the image of $\overline{\rho}_{t,\ell}$ lies in the image of k_1-rational points of a pair (ℓ, J) under the map $\pi_{\ell,J}$. So the theorem follows as soon as it is shown that the number of k_1-rational points of the constructed étale covers $X_{\ell,J}$ of X are finite for any fixed $d \geq 1$ and for extensions $[k_1 : k] = d$. For this, it will be enough to show that the family of étale covers $X_{\ell,J}/X$ verifies the condition of Theorem 3.8.

The group $\mathrm{Sp}_{2g}(\mathbb{F}_\ell)$ is perfect for $\ell \geq 5$ and is generated by its elements of order ℓ. In addition each maximal subgroup J of $\mathrm{Sp}_{2g}(\mathbb{F}_\ell)$ is of index at most $\frac{1}{2}(\ell^g - 1)$. By Theorem 2.19, the Cayley graphs $\mathrm{Cay}(\mathrm{Sp}_{2g}(\mathbb{F}_\ell); S)$ have $\lambda_1(\mathrm{Cay}(\mathrm{Sp}_{2g}(\mathbb{F}_\ell); S)) >> \frac{1}{\log|\mathrm{Sp}_{2g}(\mathbb{F}_\ell)|^A}$. The Cayley graph $\mathrm{Cay}(J\backslash\mathrm{Sp}_{2g}(\mathbb{F}_\ell); S)$ is by definition the quotient of $\mathrm{Cay}(\mathrm{Sp}_{2g}(\mathbb{F}_\ell); S)$ under the left action of J, and thus have the same λ_1. An easy calculation now shows that

$$\lambda_1(\mathrm{Cay}(J\backslash\mathrm{Sp}_{2g}(\mathbb{F}_\ell); S))\,|\mathrm{Cay}(J\backslash\mathrm{Sp}_{2g}(\mathbb{F}_\ell); S)| \to \infty$$

when (ℓ, J) runs over all pairs as above with $\ell \geq 5$, which finishes the proof. □

3.4. **Gonality and rational points of bounded degree of Drinfeld modular curves.** In this section, we discuss arithmetic consequences of the combinatorial Yang-Li-Yau inequality from [26]. The main theorem is a linear lower bound in the genus for the gonality of Drinfeld modular curves. This extends the work of Abramovich [1] to positive characteristic case.

3.4.1. *Lower bound on the gonality of X_Γ.* Let K be a function field of genus g over the field of constants $k = \mathbb{F}_q$, of characteristic p. Let ∞ be a fixed place of K of degree δ, and let A be the ring of functions $f \in K$ which have poles at most at ∞.

Let K_∞ be the completion of K at ∞, and denote by $\mathbb{C}_\infty$ the completion of an algebraic closure of K_∞. Let $\Omega = \mathbb{P}^1(\mathbb{C}_\infty) \setminus \mathbb{P}^1(K_\infty) = C_\infty \setminus K_\infty$. The group $\mathrm{GL}_2(K)$ acts by fractional linear transformations on Ω.

Consider now Γ an arithmetic subgroup of $\mathrm{GL}_2(K)$: Γ is a congruent subgroup of $\mathrm{GL}(Y) \subseteq \mathrm{GL}_2(K)$ for a rank-two A-lattice Y in K_∞. This means that Γ contains a subgroup of the form $\mathrm{GL}(Y, \mathfrak{n}) := \ker\{\mathrm{GL}(Y) \to \mathrm{GL}(Y/\mathfrak{n}Y)\}$ for an ideal $\mathfrak{n}$ of A.

The group Γ acts on Ω, and the quotient $\Gamma\backslash\Omega$ is a smooth analytic curve which is the analytification of a smooth affine curve Y_Γ defined over a finite (abelian) extension of K_∞. The Drinfeld modular curve X_Γ is the compactification of Y_Γ obtained by adding a finite number of points, called cusps, to Y_Γ.

Theorem 3.10 ([26])**.** *Let Γ be an arithmetic subgroup of $\mathrm{GL}_2(K)$. There is a constant $c = c(K, \delta)$, such that the gonality $\gamma(X_\Gamma)$ over $\overline{K}$ satisfies*

$$\gamma(X_\Gamma) \geq c\,.\,(g(X_\Gamma) - 1),$$

where $g(X_\Gamma)$ is the genus of X_Γ.

We briefly discuss the proof of this theorem.

Reduction graph of X_Γ. The group Γ acts by automorphisms on the Bruhat-Tits tree $\mathfrak{T}$ of $\mathrm{PGL}_2(K_\infty)$, and the quotient is a finite graph G with a finite set of infinite rays corresponding to the cusps of X_Γ. The Drinfeld curve X_Γ is a Mumford curve with reduction graph over $\mathbb{F}_{q^\delta}$ isomorphic to G.

Maximum valence of G. The Bruhat-Tits tree $\mathfrak{T}$ is a regular tree of valence $q^\delta + 1$. The graph G being the finite part of a quotient of this tree by a subgroup of the automorphism group, it has maximum valence $d_{\max}$ bounded by $q^\delta + 1$.

First non-trivial eigenvalue of the Laplacian of G for $\Gamma = \mathrm{GL}(Y, \mathfrak{n})$. In the case $\Gamma = \mathrm{GL}(Y, \mathfrak{n})$, the Laplacian of G can be described in terms of the projection of the Hecke operator on $\mathfrak{T}$ corresponding to the characteristic function of ∞, and a zero-one matrix corresponding to the infinite rays of the quotient of $\mathfrak{T}$ by $\mathrm{GL}(Y, \mathfrak{n})$. Ramanujan-Petersson conjecture for global function fields, proved by Drinfeld, gives an estimate of the form $\lambda_1 \geq q^\delta - 2q^{\delta/2}$ for the first non-trivial eigenvalue of the Laplacian.

Number of vertices of G for $\Gamma = \mathrm{GL}(Y, \mathfrak{n})$. A direct comparison argument between the two quotient graphs G and G_0 associated to $\mathrm{GL}(Y, \mathfrak{n})$ and $GL(Y)$, respectively, involving the stabilizer of the vertex v_0 of $\mathfrak{T}$ corresponding to the root vertex of $\mathfrak{T}$, leads to a lower bound of the type

$$|G| \geq \frac{1}{q(q^2-1)}[\mathrm{GL}(Y) : \mathrm{GL}(Y, \mathfrak{n})],$$

where $|G|$ is the number of vertices of G.

Gonality of X_Γ for $\Gamma = \mathrm{GL}(Y, \mathfrak{n})$. Combining the above estimates with the combinatorial Yang-Li-Yau inequality, discussed in the previous section, gives the existence of a constant c_0, depending only on q and δ, such that for $\Gamma = \mathrm{GL}(Y, \mathfrak{n})$,

$$\gamma(X_\Gamma) \geq c_0 \, . \, [\mathrm{GL}(Y) : \Gamma]. \tag{7}$$

The bound on the genus is obtained by applying the Riemann-Hurwitz formula to the cover $X_{\mathrm{GL}(Y,\mathfrak{n})} \to X_{\mathrm{GL}(Y)}$, and a careful analysis of the degree of the ramification divisor. Riemann-Hurwitz gives

$$[\mathrm{GL}(Y, \mathfrak{n}) : \mathrm{GL}(Y)] = (g(X_{\mathrm{GL}(Y,\mathfrak{n})}) - 1)\frac{2(q-1)}{2(g(X_{\mathrm{GL}(Y)}) - 1) + R},$$

so it will be essentially enough to give a lower bound on R since $g(X_{\mathrm{GL}(Y)})$ is a constant, depending only on K and δ.

Theorem for general Γ. This follows by looking at the cover $X_{\mathrm{GL}(Y,\mathfrak{n})} \to X_\Gamma$. This gives $\gamma(X_\Gamma) \geq \gamma(X_{\mathrm{GL}(Y,\mathfrak{n})})|\Gamma \cap Z|/[\Gamma : \mathrm{GL}(Y, \mathfrak{n})]$, where $Z \simeq \mathbb{F}_q^*$ is the centralizer of $\mathrm{GL}(Y)$. Combining the theorem for $\mathrm{GL}(Y, \mathfrak{n})$ with Riemann-Hurwitz for the cover $X_{\mathrm{GL}(Y,\mathfrak{n})} \to X_\Gamma$ gives the result for general Γ.

Note that the inequality (7) holds for more general Γ, for a constant $c_0 = c_0(q, \delta)$.

3.4.2. *Rational points of bounded degree.* It is possible to apply the analogue in positive characteristic of Faltings-Frey theorem [56, 22], along with the linear lower bound on the gonality (7) to prove the following theorem.

Suppose that X_Γ is defined over the finite extension L of K.

Theorem 3.11 ([26])**.** *There is a constant $c_0 = c_0(q, \delta)$ such that the set*

$$\bigcup_{L' : \;\; [L':L] \leq \frac{1}{2}(c_0[\mathrm{GL}(Y):\Gamma]-1)} X_\Gamma(L')$$

is finite.

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