A numerical study of the viscous breaking water waves problem and the limit of vanishing viscosity

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Abstract. We have developped a finite-element based numerical scheme to study the water-air interface evolution of viscous water waves in the presence of a plunging jet. The results are compared with the inviscid case (figure 2). No finite-time singularity seems to appear as the minimum radius of curvature becomes Reynolds-independant for $\mathrm{Re} > 10^4$ with a non-vanishing limit (figure 3). The dissipation effects are located in the vicinity of the moving interface (figure 4), in a boundary layer of size $\delta \sim \mathrm{Re}^{-\frac{1}{2}}$. In the particular case of (real units) water waves, the effects of viscosity become non-negligible at length-scales smaller than the capillary length, at which surface tension becomes the prevailing regularizing mechanism.

The viscous water waves problem

The non-dimensionnal Navier-Stokes equations with stress-free boundary conditions at the top $\Gamma_{i,t}$ and Navier condition at the bottom Γ_0 (figure 1),

$$\begin{cases} \dfrac{\partial \boldsymbol{u}}{\partial t} + \left(\boldsymbol{u}\cdot\boldsymbol{\nabla}\right)\boldsymbol{u} + \boldsymbol{\nabla} p - \dfrac{1}{\mathrm{Re}}\Delta \boldsymbol{u} = -\hat{\boldsymbol{y}} & \text{in } (0,T)\times\Omega_t \\ \boldsymbol{\nabla}\cdot\boldsymbol{u} = 0 & \text{in } (0,T)\times\Omega_t \\ \boldsymbol{u}\cdot\hat{\boldsymbol{n}} = 0 & \text{on } (0,T)\times\Gamma_0 \\ \hat{\boldsymbol{t}}\cdot\bar{\varepsilon}(\boldsymbol{u})\cdot\hat{\boldsymbol{n}} = 0 & \text{on } (0,T)\times\Gamma_0 \\ -p\hat{\boldsymbol{n}} + \dfrac{1}{\mathrm{Re}}\bar{\varepsilon}(\boldsymbol{u})\cdot\hat{\boldsymbol{n}} = 0 & \text{on } (0,T)\times\Gamma_{i,t} \\ \boldsymbol{u}(0,\cdot) = \boldsymbol{u}_0 & \text{in } \Omega_0 \end{cases}, \qquad (1)$$

with Re the Reynold number, $\bar{\varepsilon}(\boldsymbol{u}) = \boldsymbol{\nabla u} + (\boldsymbol{\nabla u})^t$ twice the stress tensor, $\hat{\boldsymbol{t}}$ the unit tangential vector and $\hat{\boldsymbol{n}}$ the unit normal vector. The initial condition is a finite amplitude extension of the first-order two-dimensional solution of the inviscid water waves problem.

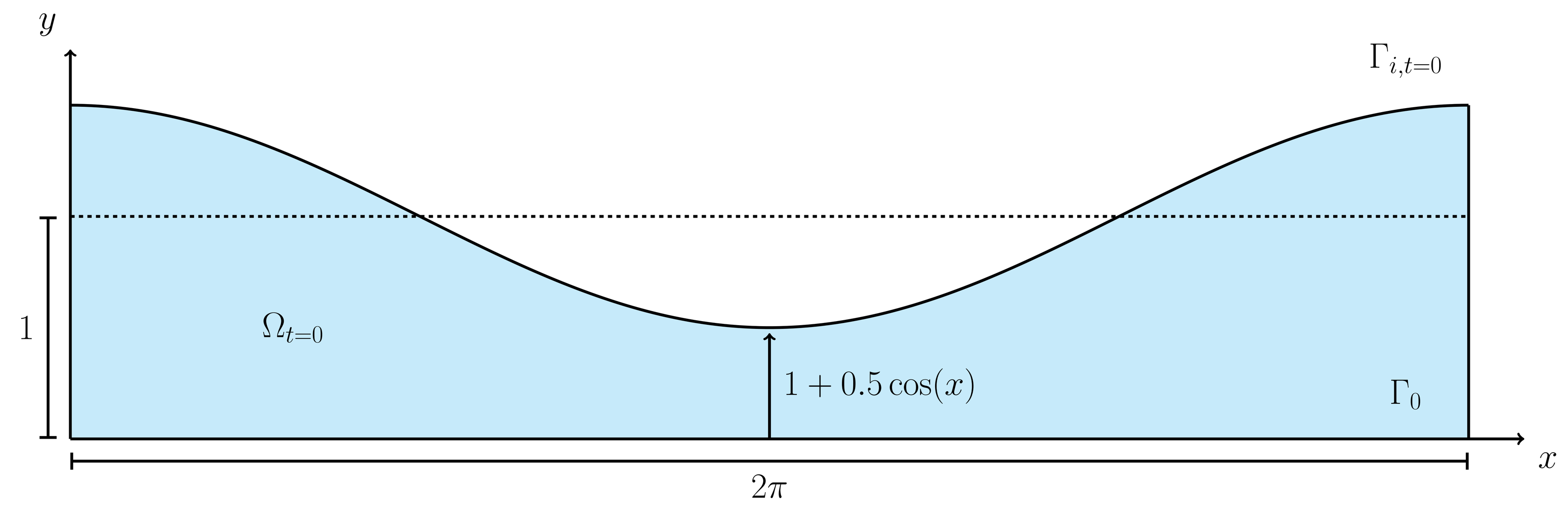

Figure 1. Geometry of the initial domain.

Convergence to the Euler solution

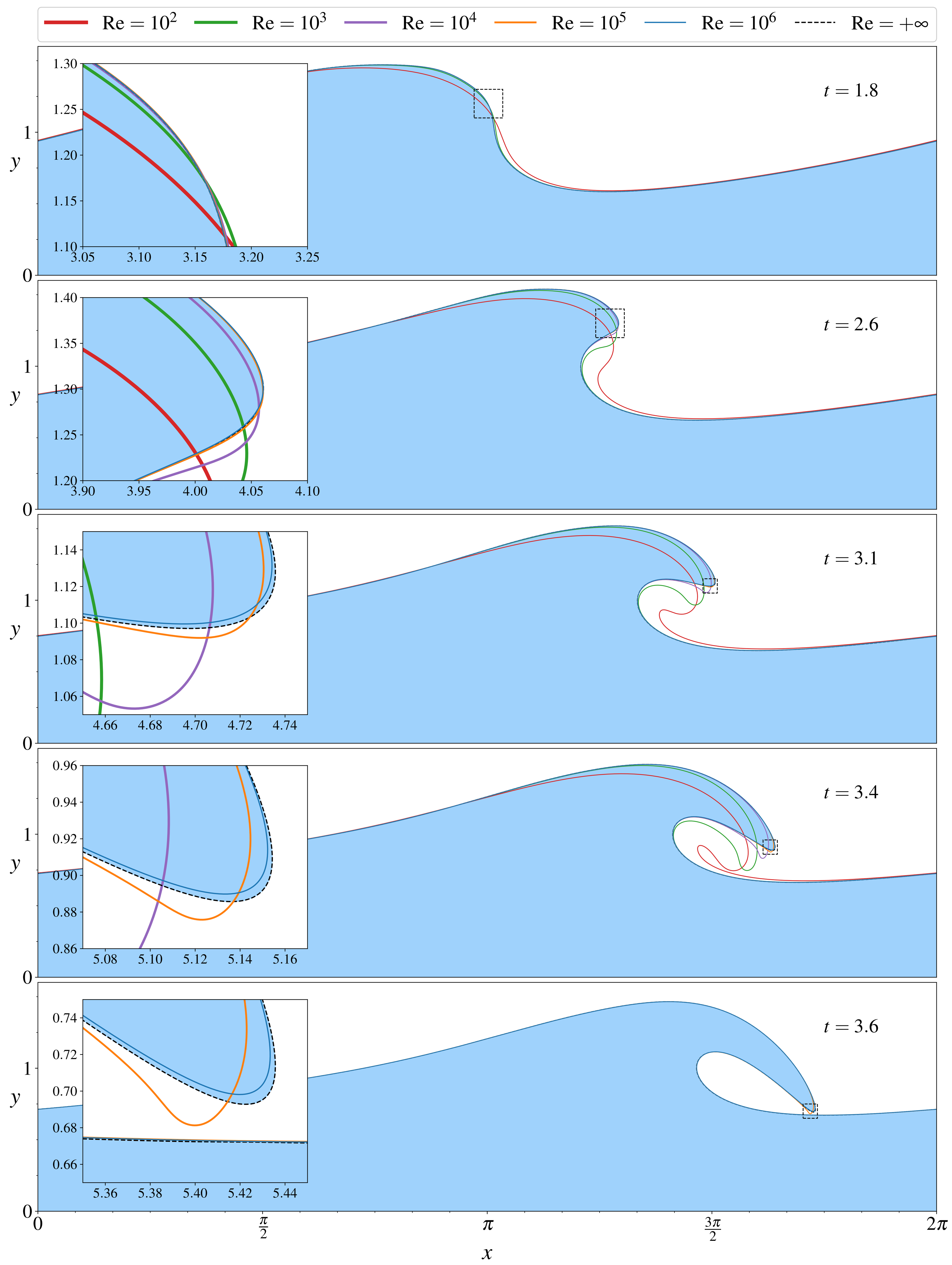

Figure 2. The shape of the interface for different values of the Reynolds number and at different time with emphasis on the crest. The $\mathrm{Re} = +\infty$ corresponds to the Euler solution obtained from [1]. The shaded region corresponds to the Euler domain.

Curvature at the tip of the wave

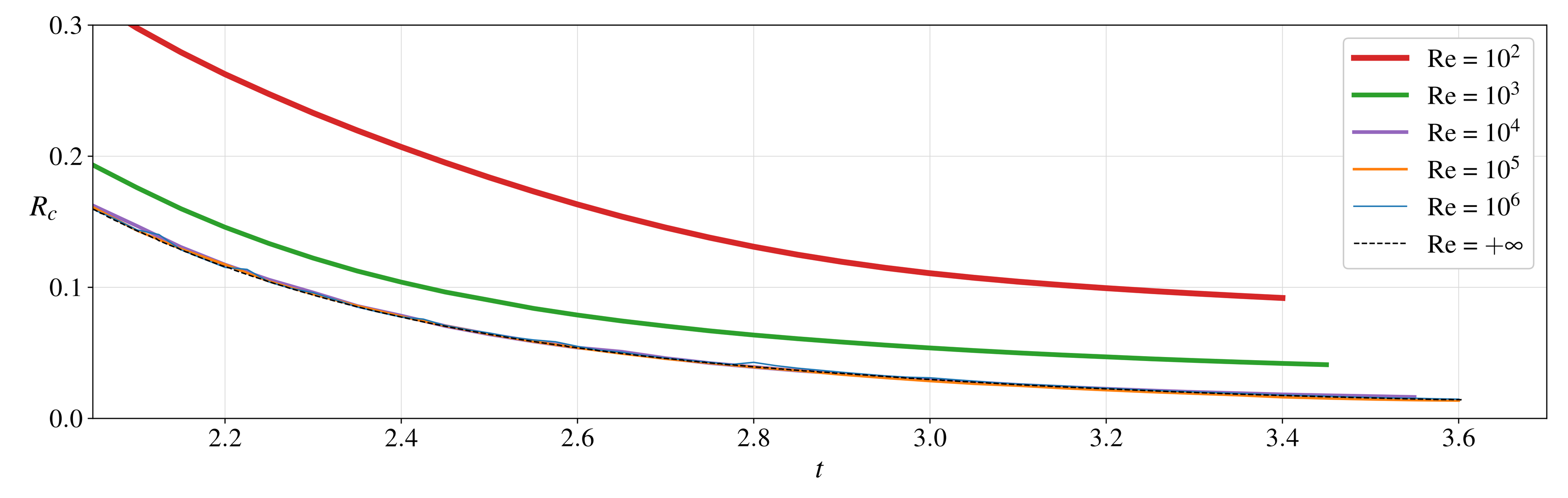

Figure 3. Evolution of the minimum radius of curvature of the interface (*i.e.* at the tip) for different values of the Reynolds number. The $\mathrm{Re} = +\infty$ curve is computed from the Euler solution obtained using the code from [1].

Boundary layer

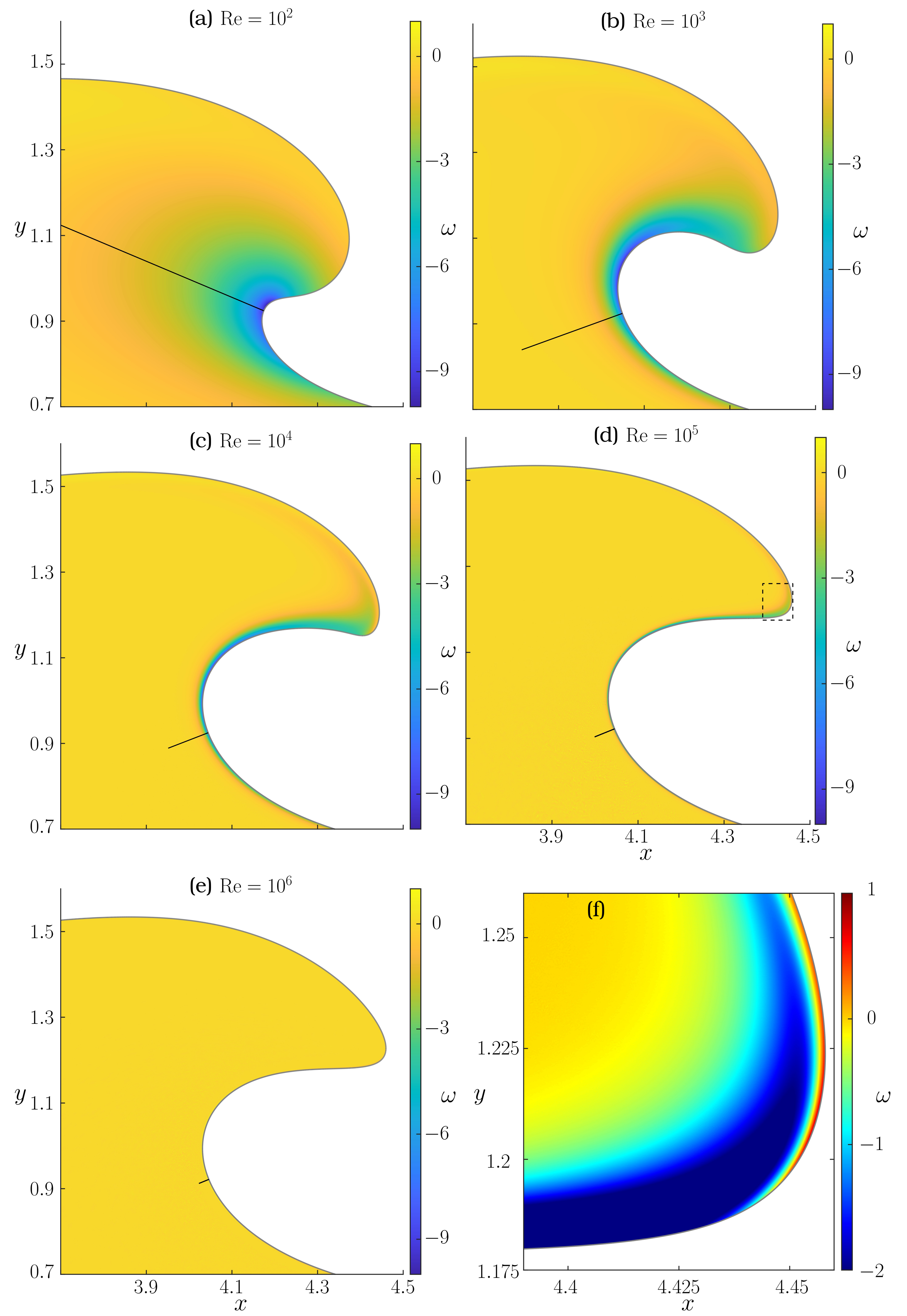

Figure 4. (a-e) The vorticity $\omega = \partial_x u_y - \partial_y u_x$ near the crest for different values of the Reynolds number at time $t = 2.9$. (f) A zoom on the tip of the wave for the $\mathrm{Re} = 10^5$ case (dashed rectangle in (d)). The color legend has been truncated from below to guarantee overall color coherence. The black lines correspond to cuts made along the normal at $y = 1$ used to investigate the scaling in $\mathrm{Re}^{-\frac{1}{2}}$ of the boundary layer size (not shown).

This work has been carried out using the FreeFem c++ finite-element library [2].

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References

[1] Emmanuel Dormy & Christophe Lacave (2023), ***Inviscid water-waves and interface modeling***, arXiv:2306.02363.

[2] Frédéric Hecht (2012), ***New development in freefem++***. *J. Numer. Math.*, **20**(3-4), 251-265.