Breaking Water Waves
Finite element analysis and high Reynolds number limit

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Model description

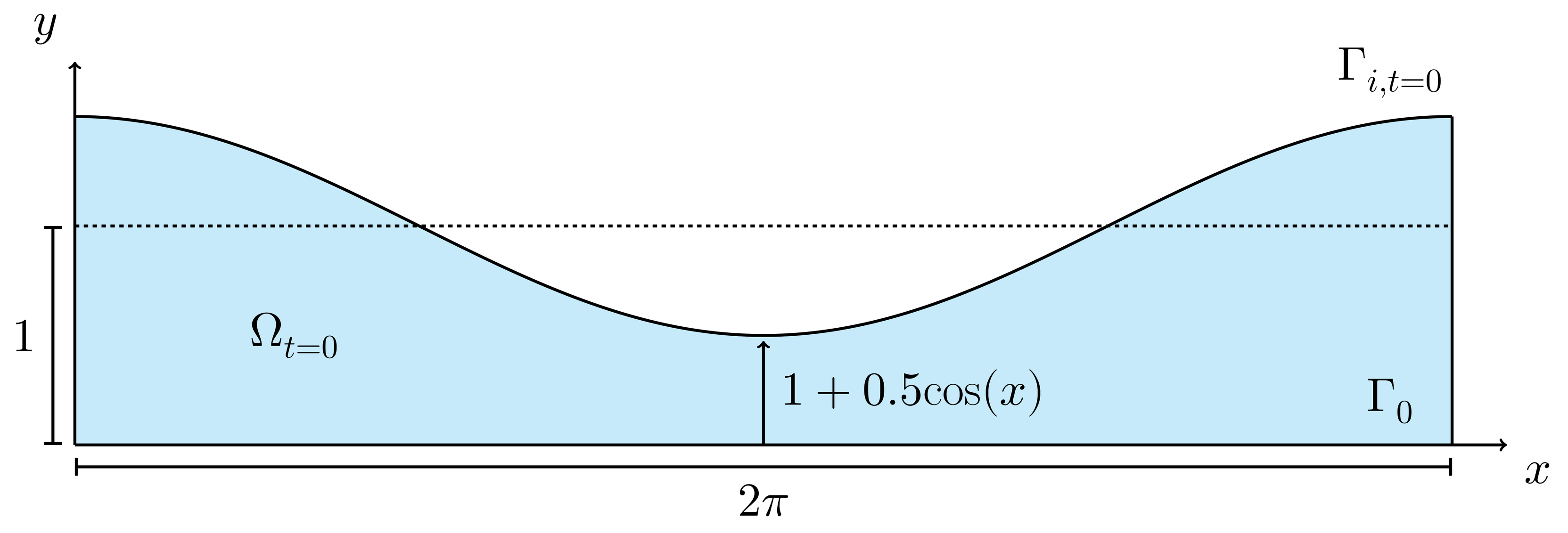

Figure 1. Geometry of the initial domain.

We solve (numerically) the non-dimensional incompressible **Navier-Stokes** equation in the moving domain Ω_t. **Navier** conditions are used at the bottom Γ_0 while **Stress-free** conditions are used on the water-air interface,

$$
\left\{
\begin{aligned}
\frac{\partial \boldsymbol{u}}{\partial t} + (\boldsymbol{u}\cdot\boldsymbol{\nabla})\boldsymbol{u} + \boldsymbol{\nabla} p - \frac{1}{\mathrm{Re}}\Delta \boldsymbol{u} &= -\hat{\boldsymbol{y}} && \text{in } (0,T)\times\Omega_t \\
\boldsymbol{\nabla}\cdot\boldsymbol{u} &= 0 && \text{in } (0,T)\times\Omega_t \\
\boldsymbol{u}\cdot\hat{\boldsymbol{n}} &= 0 && \text{on } (0,T)\times\Gamma_0 \\
\hat{\boldsymbol{t}}\cdot\left(\boldsymbol{\nabla}\boldsymbol{u} + (\boldsymbol{\nabla}\boldsymbol{u})^t\right)\cdot\hat{\boldsymbol{n}} &= 0 && \text{on } (0,T)\times\Gamma_0 \\
-p\hat{\boldsymbol{n}} + \frac{1}{\mathrm{Re}}\left(\boldsymbol{\nabla}\boldsymbol{u} + (\boldsymbol{\nabla}\boldsymbol{u})^t\right)\cdot\hat{\boldsymbol{n}} &= 0 && \text{on } (0,T)\times\Gamma_{i,t} \\
\boldsymbol{u}(0,\cdot) &= \boldsymbol{\nabla}\phi_0 && \text{in } \Omega_0
\end{aligned}
\right. .
$$

The **initial potential** ϕ_0 is a harmonic function whose value on the interface is given by a finite amplitude extension of the linear wave solution,

$$
\frac{\partial \phi_0}{\partial \hat{\boldsymbol{n}}}(x) = \boldsymbol{u}(0,x)\cdot\hat{\boldsymbol{n}} = a\sqrt{gk\tanh(kh_0)}\cdot\begin{bmatrix} (\tanh kh_0)^{-1}\cos kx \\ \sin kx \end{bmatrix}.
$$

The interface is advected with the fluid.

Where does the viscous dissipation happen?

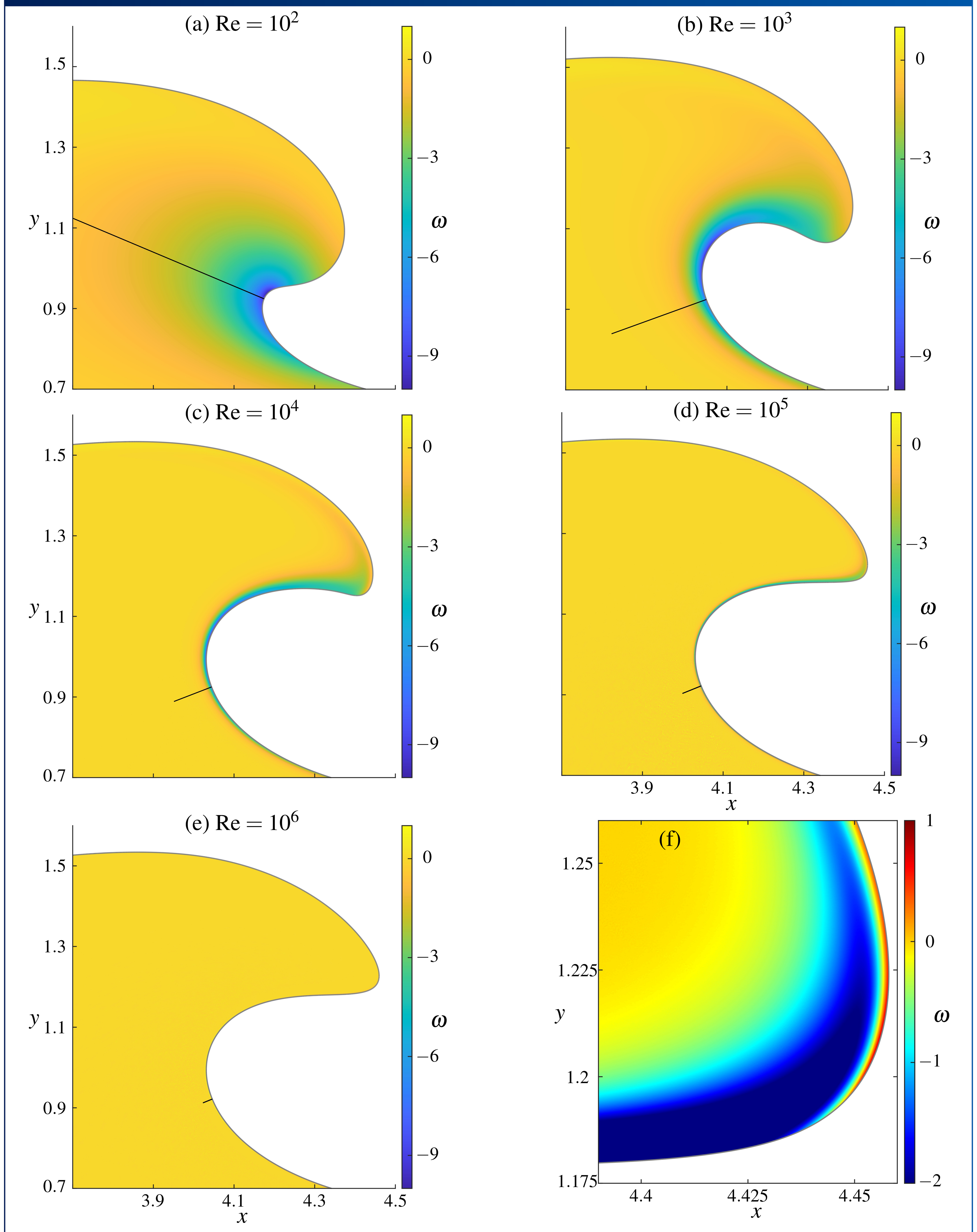

Figure 4. (a-e) The **vorticity** $\omega = \partial_x u_y - \partial_y u_x$ at time $t = 2.9$. We see that the viscous dissipation is happening in a boundary layer below the surface of size $\delta \approx \mathrm{Re}^{-\frac{1}{2}}$. Solid black lines correspond to vorticity cuts (not shown here). (f) A zoom on a positive vorticity region in the $\mathrm{Re} = 10^5$ case.

Evolution of a Water Wave at $\mathrm{Re} = 10^6$

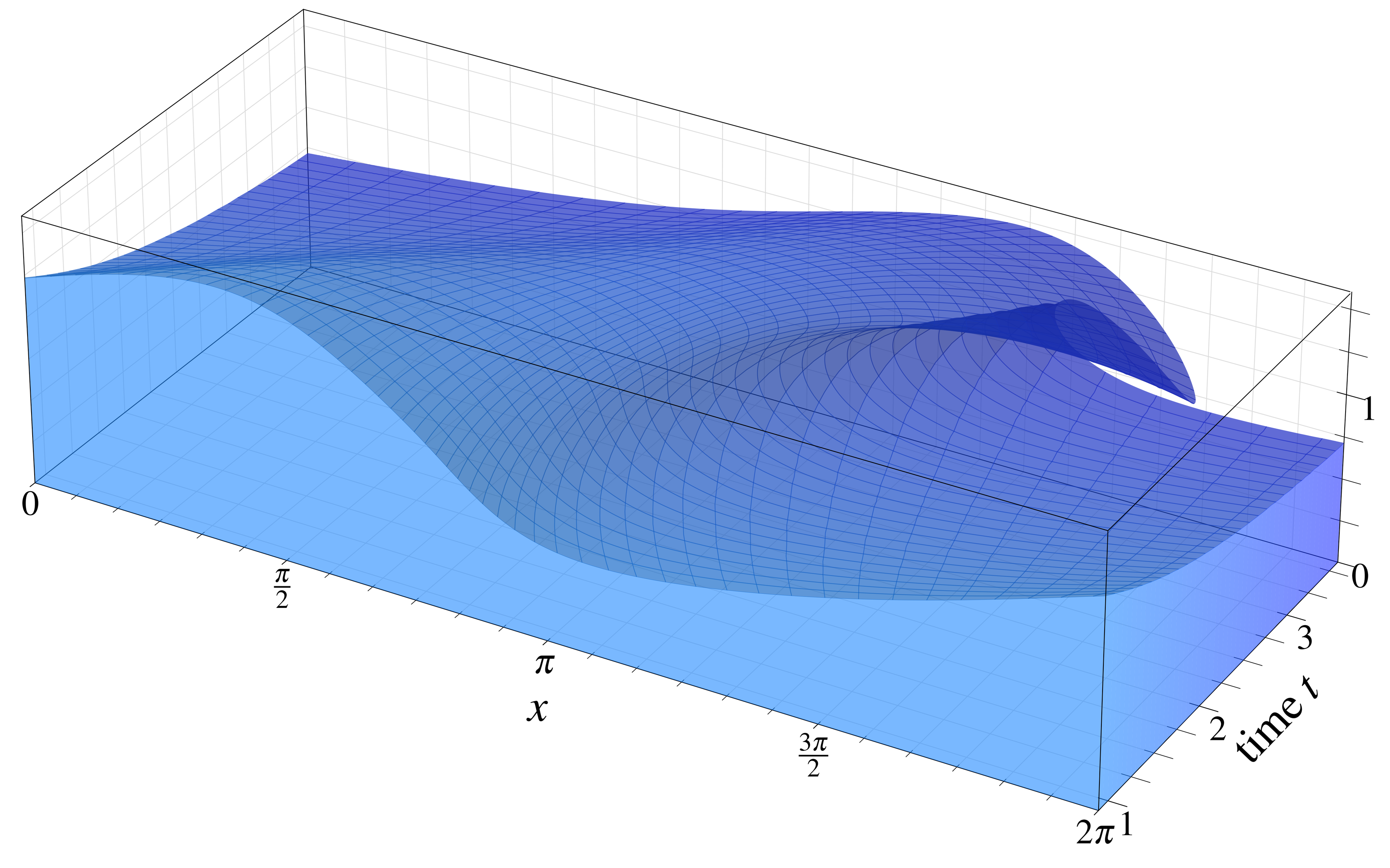

Figure 2. Evolution of a wave of initial amplitude $a = 0.5$ at $\mathrm{Re} = 10^6$.

Convergence to the inviscid solution

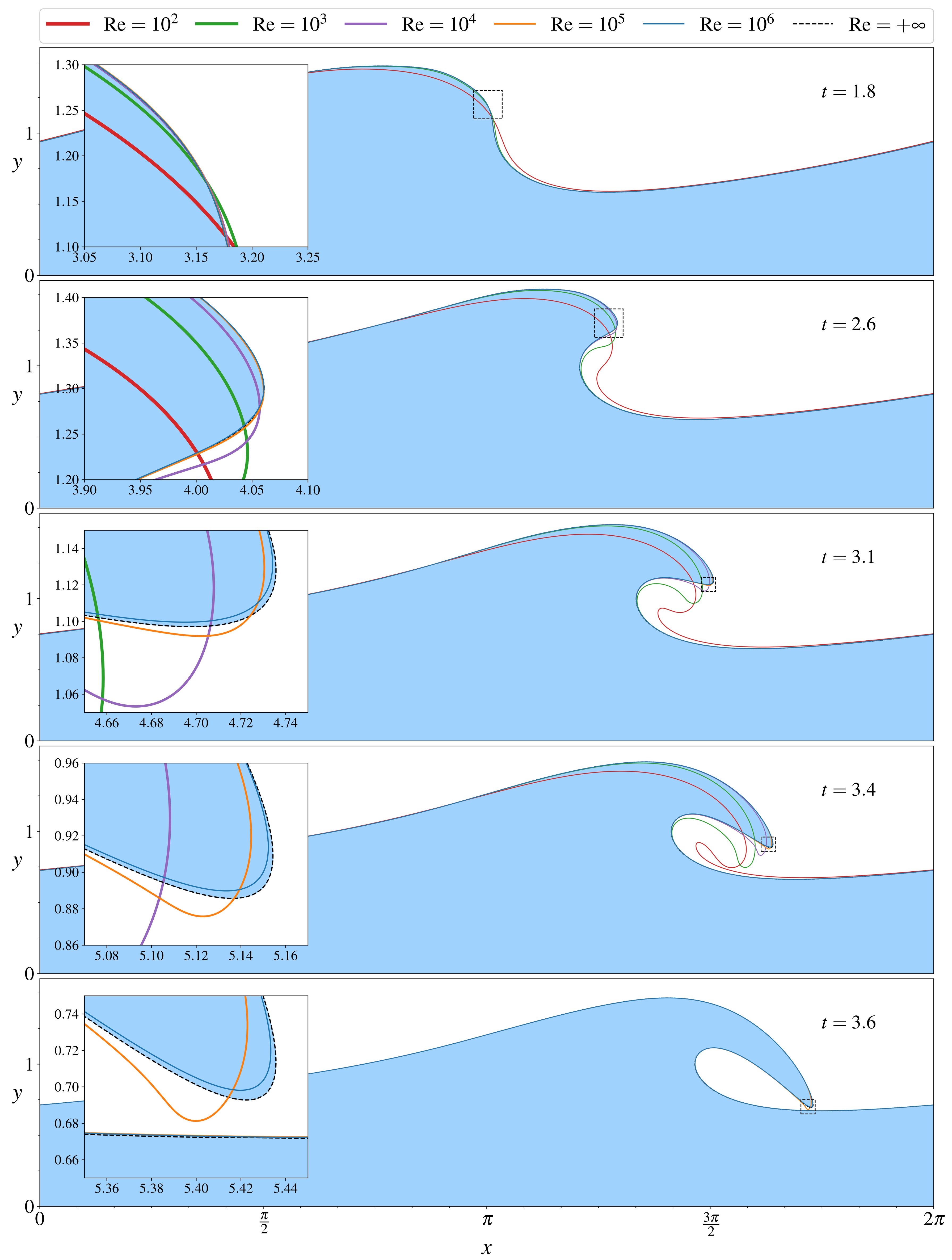

Figure 3. Influence of the **viscosity** on the shape of the wave as time increases. The $\mathrm{Re} = +\infty$ simulation has been achieved using an Euler-based code (see [2] for details).

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This work makes use of the FreeFEM finite-element library [3].

[1] Riquier, A. & Dormy, E. (2024) **Numerical study of a viscous breaking water wave and the limit of vanishing viscosity**, J. Fluid Mech. **984**(R5).

[2] Dormy, E. & Lacave, C. (2024) **Inviscid Water-Waves and interface modeling**, accepted in Quarterly of Applied Mathematics, available online.

[3] Hecht, F. (2012) **New development in freefem++**, J. Numer. Math., **20**(3-4).