Breaking Water Waves: Mathematical description and new numerical results

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What is a Breaking Wave?

Two points of view

Mathematics

Definition. A function

$$u \in C^0([0,T), C^1(\mathbb{R}))$$

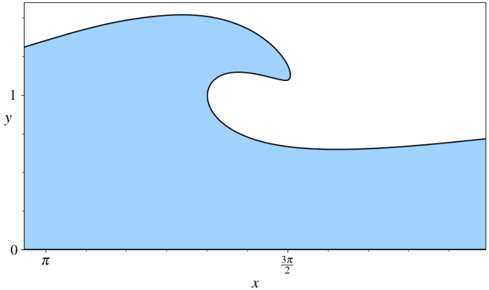
develops a **wave breaking** as $t \to T$ if

$$\limsup_{t \to T} \left\{ \sup_{x \in \mathbb{R}} \left| \partial_x u(t,x) \right| \right\} = +\infty$$

while

$$\sup_{(t,x) \in [0,T) \times \mathbb{R}} \left| u(t,x) \right| < +\infty$$

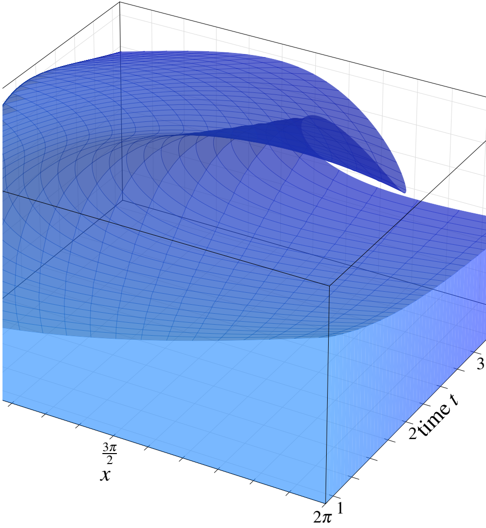
Experiments



The interface is a multivalued function.

Breaking Water Waves

1. **Mathematical Wave Breaking**
   Global well-posedness of the KdV equation; Breaking in the Camassa-Holm setting.
2. **The Water Wave problem and Wave Breaking**
   Introduction to the Water Waves problem; Incorporating Wave Breaking in 2d.
3. **New Numerical Results**
   Free-surface Navier-Stokes equation; Numerical methods; High Reynolds number limit; Vorticity sheet diffusion below the free-surface; Absence of finite-time singularity.



Wave Breaking in two equations

Korteweg-de Vries

Camassa-Holm

Motivation

Korteweg-de Vries (1895)

$$\partial_t u + u\partial_x u + \partial_{xxx} u = 0$$

Globally well-posed in $H^2(\mathbb{R})$, hence no wave breaking.

Camassa-Holm (1993)

$$\partial_t (u - \partial_{xx} u) + u\partial_x u - \partial_x u \partial_{xx} u - u\partial_{xxx} u = 0$$

If the solution is not global, it exhibits wave breaking!

Can we motivate these equations from the general water waves problem?

General Framework

Incompressible Euler's equations in Ω_t,

$$\begin{aligned} \partial_t \boldsymbol{u} + (\boldsymbol{u} \cdot \nabla)\boldsymbol{u} &= -\nabla p + \boldsymbol{g} \\ \nabla \cdot \boldsymbol{u} &= 0 \end{aligned}$$

Free-surface $\gamma(t, s)$ advection,

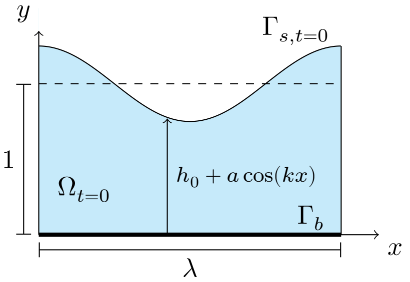
$$\partial_t \gamma(t, s) = \boldsymbol{u}(t, \gamma(t, s))$$

Dynamic condition at the surface,

$$p - p_a = \sigma\kappa(s)$$

Impenetrability at the bottom,

$$\boldsymbol{u} \cdot \boldsymbol{n} = 0$$



Non-dimensionalisation

Following Johnson (1997), we redefine the quantities

$$x \to \lambda x \qquad u_x \to \sqrt{gh_0}\, u_x$$

$$y \to h_0 y \qquad u_y \to \frac{h_0}{\lambda}\sqrt{gh_0}\, u_y$$

$$t \to \frac{\lambda}{\sqrt{gh_0}} t \qquad p \to p_a + g(h_0 - y) + gh_0 p$$

and define the parameters

$$\epsilon = \frac{a}{h_0} \text{ the amplitude} \qquad \delta = \frac{h_0}{\lambda} \text{ the shallowness}$$

We also suppose a single-valued interface (for now!)

$$\gamma(t, s) = \begin{bmatrix} x \\ 1 + \varepsilon \eta(t, x) \end{bmatrix}$$

Non-dimensional equations

Euler's equations,

$$\partial_t u_x + \boldsymbol{u} \cdot \nabla u_x = -\partial_x p$$
$$\delta^2 \left(\partial_t u_y + \boldsymbol{u} \cdot \nabla u_y \right) = -\partial_y p$$
$$\partial_x u_x + \partial_y u_y = 0$$

and boundary conditions,

$$p = \varepsilon \eta \quad \& \quad u_y = \varepsilon \left(\partial_t \eta + u_x \partial_x \eta \right) \qquad \text{at} \quad y = 1 + \varepsilon \eta$$
$$u_y = 0 \qquad \text{at} \quad y = 0$$

First limitting case

$\delta \to 0$, irrotational

Setting $\delta = 0$ (shallow-water scaling) allows to find the pressure,

$$\left.\begin{array}{rcll} p & = & \varepsilon\eta & \text{at } y = 1 + \varepsilon\eta \\ \partial_y p & = & 0 & \end{array}\right\} \Longrightarrow p(t,x,y) = \varepsilon\eta(t,x)$$

Assuming irrotationality allows to find the horizontal velocity,

$$\left.\begin{array}{rcll} u_y & = & 0 & \text{at } y = 0 \\ \partial_x u_y - \partial_y u_x & = & 0 & \end{array}\right\} \Longrightarrow u_x(t,x,y) = u_x(t,x)$$

In that case,

$$u_y = \varepsilon\partial_t\eta + \varepsilon u_x \partial_x \eta \quad \text{at } y = 1 + \varepsilon\eta \qquad \Longrightarrow \qquad u_y = \varepsilon\frac{\partial_t\eta + u_x\partial_x\eta}{1+\varepsilon\eta}y$$

leading to the **Shallow-Water equations**

$$\begin{aligned} \partial_t h + \partial_x(h u_x) = 0 \\ \partial_t u_x + u_x\partial_x u_x + \partial_x h = 0 \end{aligned}$$

Second limitting case

$\varepsilon = \delta^2$, travelling waves

We now suppose that $\varepsilon = \delta^2$ (very specific regime!) and that the solution is a function of the variables

$$\xi = x - t \qquad \text{and} \qquad \tau = \varepsilon t$$

We also suppose that the solution can be represented as a parameter asymptotic expansion,

$$\eta(t, x; \varepsilon) = \sum_{n=0}^{+\infty} \varepsilon^n \eta_n(t, x) \qquad \text{and similarly for} \quad u_x, u_y, p$$

then η_0 must satisfy

$$2\partial_\tau \eta_0 + 3\eta_0 \partial_\xi \eta_0 + \frac{1}{3}\partial_{xxx}\eta_0 = 0$$

the Korteweg-de Vries (KdV) equation. More details in Constantin et al. (2013).

Third limitting case

Another possible regime for travelling waves

If we now suppose that the solution is a function of

$$\xi = \sqrt{\varepsilon}(x - t) \qquad \tau = \varepsilon^{\frac{3}{2}} t$$

and if we suppose the following scaling

$$u_y \to \sqrt{\varepsilon} u_y$$

then, applying the same Parameter Asymptotic Expansion as before, we get the following equation for $u_{x0} = u$,

$$\partial_\tau u - \varepsilon\delta^2 \partial_{\xi\xi\tau} u + 2\partial_\xi u + 3\varepsilon u \partial_\xi u = \varepsilon^2 \delta^2 \left[2\partial_\xi u \partial_{\xi\xi} u + u \partial_{\xi\xi\xi} u \right]$$

the Camassa-Holm (CH) equation.

A bit artificial?

The KdV equation

Local existence

$$\left.\begin{array}{rcll} \partial_t u - 6u\partial_x u + \partial_{xxx} u & = & 0 & \text{in } [0,T) \times \mathbb{R} \\ u(0,\cdot) & = & u_0 & \text{in } \mathbb{R} \end{array}\right\} \tag{KdV}$$

Theorem. Let $u_0 \in H^2(\mathbb{R})$. Then there exists $T > 0$ and a function

$$u \in C^0([0,T); H^2(\mathbb{R})) \cap C^1([0,T); H^{-1}(\mathbb{R}))$$

solution of (KdV). Furthermore the principle of continuation holds,

$$\sup_{t\in[0,T)} \left\| u(t,\cdot) \right\|_{H^2(\mathbb{R})} < +\infty \qquad \Longrightarrow \qquad T = +\infty$$

For a proof, see Kato (1975) using dissipative operators, or Bona & Smith (1975) using a regularisation technique.

The KdV equation

Global existence

There exists infinitely many conserved quantities for KdV. Three of them are usefull here,

$$I_0 = \int_{\mathbb{R}} u^2(t,x)\,dx$$

$$I_1 = \int_{\mathbb{R}} \left[(\partial_x u)^2 + 2u^3\right] dx$$

$$I_2 = \int_{\mathbb{R}} \left[(\partial_{xx} u)^2 - 10u(\partial_x u)^2\right] dx$$

Indeed, we can find a constant $C = C(I_0, I_1, I_2)$ such that for all $t \in [0, T)$,

$$\left\| u(t,\cdot) \right\|_{H^2(\mathbb{R})} < C$$

Hence, by the principle of continuation, $T = +\infty$. Full details in Constantin et al. (2013).

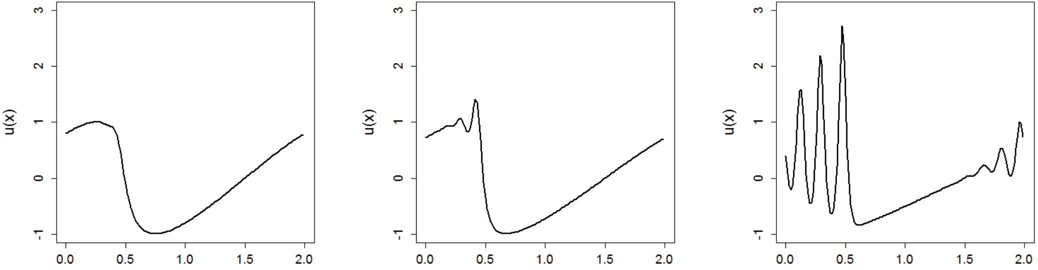
The KdV equation

Can it (wave) break?

$$\left.\begin{array}{rcll} \partial_t u - 6u\partial_x u + \partial_{xxx} u & = & 0 & \text{in } [0,T) \times \mathbb{R} \\ u(0,\cdot) & = & u_0 & \text{in } \mathbb{R} \end{array}\right\} \qquad \text{(KdV)}$$

The solution of (KdV) lying in $H^2(\mathbb{R})$ at all time $t > 0$, it cannot (wave) break.

Intuitively, this is due to the dispersion.

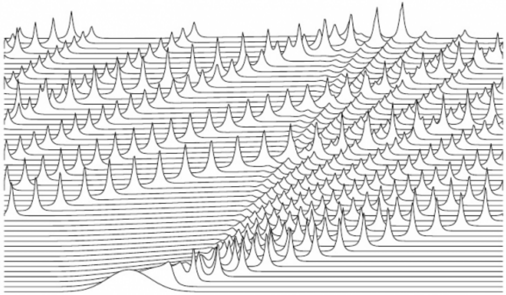


The Camassa-Holm equation

$$\left.\begin{array}{rcll} \partial_t(u - \partial_{xx} u) + 3u\partial_x u - 2\partial_x u \partial_{xx} u - u\partial_{xxx} u & = & 0 & \text{in } [0,T) \times \mathbb{R} \\ u(0,\cdot) & = & u_0 & \text{in } \mathbb{R} \end{array}\right\} \qquad \text{(CH)}$$

introduced in Camassa & Holm (1993) because it provides peaked solitons solutions (as the extreme Stokes wave?),

$$u(t,x) = c \exp\Big(- \big|x - ct\big|\Big)$$



→ Crisan & Holm (2018)

The Camassa-Holm equation

Local existence

$$
\left.\begin{array}{rcll}
\partial_t(u-\partial_{xx}u) + 3u\partial_x u - 2\partial_x u\partial_{xx}u - u\partial_{xxx}u & = & 0 & \text{in } [0,T)\times\mathbb{R} \\
u(0,\cdot) & = & u_0 & \text{in } \mathbb{R}
\end{array}\right\} \tag{CH}
$$

Now let $p = u - \partial_{xx}u = Qu$ and then (CH) becomes

$$\partial_t p + 2p\partial_x(Q^{-1}p) + (Q^{-1}p)\partial_x p = 0$$

Theorem. (Constantin & Escher (1998)) Let $u_0 \in H^3(\mathbb{R})$. Then there exists $T > 0$ and a function

$$u \in C^0([0,T); H^3(\mathbb{R})) \cap C^1([0,T); L^2(\mathbb{R}))$$

solution of (CH). Furthermore the principle of continuation holds,

$$\sup_{t\in[0,T)} \left\|u(t,\cdot)\right\|_{H^3(\mathbb{R})} < +\infty \qquad \Longrightarrow \qquad T = +\infty$$

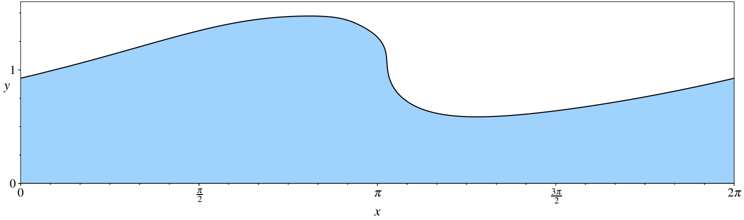
The Camassa-Holm equation

Wave breaking

$$p = Qu = u - \partial_{xx} u \qquad\qquad \partial_t p + 2p\partial_x(Q^{-1}p) + (Q^{-1}p)\partial_x p = 0$$

Theorem. (Constantin & Escher (1998)) If $T < +\infty$, then

$$\liminf_{t \to T} \left\{ \inf_{x \in \mathbb{R}} \partial_x u(t,x) \right\} = -\infty$$



The Camassa-Holm equation

Wave breaking

Idea of proof:

1. Suppose there exists $K > 0$ s.t. $\partial_x u(t,x) > -K$ for all (t,x).

The Camassa-Holm equation

Wave breaking

Idea of proof:

1. Suppose there exists $K > 0$ s.t. $\partial_x u(t,x) > -K$ for all (t,x).
2. Let the conserved quantity

$$J(t) = \left\|u(t,\cdot)\right\|^2_{H^1(\mathbb{R})} = J(0) = J$$

The Camassa-Holm equation

Wave breaking

Idea of proof:

1. Suppose there exists $K > 0$ s.t. $\partial_x u(t,x) > -K$ for all (t,x).
2. Let the conserved quantity

$$J(t) = \left\|u(t,\cdot)\right\|_{H^1(\mathbb{R})}^2 = J(0) = J$$

3.

$$\frac{d}{dt}\left\|p(t,\cdot)\right\|_{H^1(\mathbb{R})}^2 \leqslant C\left(K, \|u_0\|_{H^1(\mathbb{R})}\right)\left\|p(t,\cdot)\right\|_{H^1(\mathbb{R})}^2$$

The Camassa-Holm equation

Wave breaking

Idea of proof:

1. Suppose there exists $K > 0$ s.t. $\partial_x u(t,x) > -K$ for all (t,x).
2. Let the conserved quantity

$$J(t) = \left\|u(t,\cdot)\right\|_{H^1(\mathbb{R})}^2 = J(0) = J$$

3.

$$\frac{d}{dt}\left\|p(t,\cdot)\right\|_{H^1(\mathbb{R})}^2 \leqslant C\big(K, \|u_0\|_{H^1(\mathbb{R})}\big)\left\|p(t,\cdot)\right\|_{H^1(\mathbb{R})}^2$$

4. Use Grönwall,

$$\left\|p(t,\cdot)\right\|_{H^1(\mathbb{R})}^2 \leqslant e^{CT}\left\|p(0,\cdot)\right\|_{H^1(\mathbb{R})}^2$$

5. contradiction with the principle of continuation. $\square$

The Camassa-Holm equation

Existence of breaking "waves"

Theorem. (Constantin & Escher (1998)) Assume $u_0 \in H^3(\mathbb{R})$ odd and $\partial_x u_0(0) < 0$, then

$$T < \frac{1}{2\left|\partial_x u_0(0)\right|}$$

Idea of proof:

1. Let $g(t) = \partial_x u(t,0)$
2. Then
$$g'(t) \leqslant -\frac{1}{2} g^2(t)$$
3. and so
$$\frac{1}{g(t)} \geqslant \frac{1}{g(0)} + \frac{t}{2}$$

$\square$

The Camassa-Holm equation

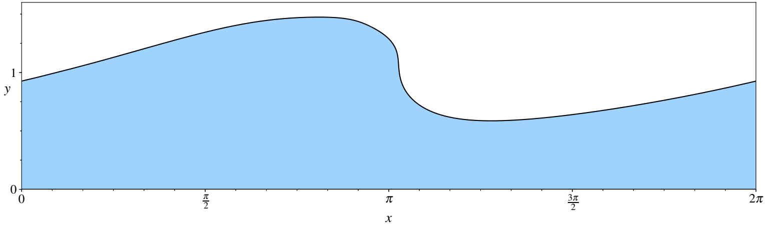
Global solutions

$$p = Qu = u - \partial_{xx} u \qquad\qquad \partial_t p + 2p\partial_x (Q^{-1} p) + (Q^{-1} p)\partial_x p = 0$$

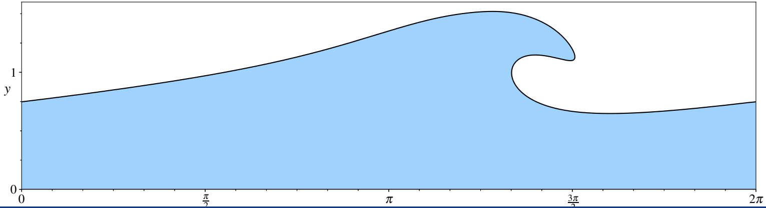
Theorem. (Constantin & Escher (1998)) If $p_0 \in H^1(\mathbb{R}) \cap L^1(\mathbb{R})$ doesn't change sign, then

$$T = +\infty$$

"Real-life" wave breaking



$t = 1.85$



$t = 3.1$

The "generalised" Water Waves problem

The Water Waves problem
... with parametrised interface

Coming back to Euler's equations

$$\begin{aligned} \partial_t \boldsymbol{u} + (\boldsymbol{u} \cdot \nabla)\boldsymbol{u} &= -\nabla p + \boldsymbol{g} \\ \nabla \cdot \boldsymbol{u} &= 0 \end{aligned}$$

with boundary conditions at the free surface,

$$p = p_a + \text{surface tension}$$
$$\partial_t \gamma(t,s) = \boldsymbol{u}(t, \gamma(t,s))$$

and at the bottom (if any),

$$\boldsymbol{u} \cdot \boldsymbol{n} = 0$$

We make two **assumptions**:

no viscosity $\quad \mathrm{Re} = +\infty \qquad\qquad$ no vorticity $\quad \omega = \nabla^\perp \cdot \boldsymbol{u} = 0$

Irrotationality

The Laplace equation

This allows to write

$$\boldsymbol{u} = \nabla\phi$$

so that Euler's equations (and Boundary Conditions) yield a set of equations for ϕ

$$\begin{aligned}
\Delta\phi &= 0 && \text{in the fluid} \\
\partial_t\phi + \frac{1}{2}\left|\nabla\phi\right|^2 + g\gamma_2 + p - p_a &= 0 && \text{at the free surface} \\
\partial_n\phi &= 0 && \text{at the bottom}
\end{aligned}$$

The Dirichlet-Neumann operator

Losing one space dimension

Let ψ the trace of ϕ on the interface,

$$\psi(t,s) = \phi\big(t, \gamma(t,s)\big)$$

and introduce the Dirichlet-Neumann **operator**,

$$G[\gamma]\psi = \partial_n \phi\Big|_{\boldsymbol{x}=\gamma(t,s)}$$

Computing $G[\gamma]\psi$ amounts to solve the problem

$$\begin{cases} \Delta\phi & = & 0 & \text{in the fluid} \\ \phi & = & \psi & \text{at the surface} \\ \partial_n\phi & = & 0 & \text{at the bottom} \end{cases}$$

and then taking the trace of $\partial_n \phi$ on the interface.

This allows to rewrite the water waves problem as a function of γ and ψ only! Here is how to do it (with a parametrised interface).

Interface equation

We can expand the $\nabla\phi$ vector in $(\boldsymbol{n}, \boldsymbol{s})$ to find the equation for the evolution of ϕ,

$$
\begin{aligned}
\partial_t \gamma(t,s) &= \boldsymbol{u}(t, \gamma(t,s)) \\
&= \nabla\phi(t, \gamma(t,s)) \\
&= \Big[\boldsymbol{n} \cdot \nabla\phi(t,\gamma(t,s))\Big]\boldsymbol{n} + \Big[\boldsymbol{s} \cdot \nabla\phi(t,\gamma(t,s))\Big]\boldsymbol{s} \\
&= \Big[\partial_n \phi(t,\gamma(t,s))\Big]\boldsymbol{n} + |\partial_s\gamma|^{-1}(t,s)\Big[\partial_s\phi(t,\gamma(t,s))\Big]\boldsymbol{s} \\
&= G[\gamma]\psi(t,s)\boldsymbol{n} + \frac{\partial_s\psi}{|\partial_s\gamma|}(t,s)\boldsymbol{s}
\end{aligned}
$$

where s and $\boldsymbol{n}$ are functions of γ,

$$
\boldsymbol{s} = \frac{\partial_s\gamma}{|\partial_s\gamma|} \qquad\qquad \boldsymbol{n} = \boldsymbol{s}^{\perp}
$$

Equation for ψ

Setting $p = p_a$, we obtain from Bernoulli's equation on the free surface,

$$\begin{aligned}\partial_t \psi(t,s) &= -g\gamma_2(t,s) + \frac{1}{2}\left|\nabla\phi\right|^2(t,\gamma(t,s)) \\ &= -g\gamma_2(t,s) + \frac{1}{2}\left[\left(G[\gamma]\psi\right)^2 + \left(\frac{\partial_s\psi}{|\partial_s\gamma|}\right)^2\right]\end{aligned}$$

If we were to add an external pressure term and the surface tension effects, we would instead have

$$\partial_s\psi + g\gamma_2 - \frac{1}{2}\left[\left(G[\gamma]\psi\right)^2 + \left(\frac{\partial_s\psi}{|\partial_s\gamma|}\right)^2\right] = p_{\mathrm{ext}}(t,s) + \sigma\kappa(t,s)$$

where σ is the surface tension coefficient and κ the surface curvature.

The "generalised" Water Waves problem

$$\begin{cases} \partial_t \gamma & = & \boldsymbol{n}\, G[\gamma]\psi + \boldsymbol{s}\, \dfrac{\partial_s \psi}{|\partial_s \gamma|} \\ \partial_t \psi & = & -g\gamma_2 + \dfrac{1}{2}\left[\left(G[\gamma]\psi\right)^2 + \left(\dfrac{\partial_s \psi}{|\partial_s \gamma|}\right)^2 \right] \end{cases}$$

$\rightarrow$ 2 equations for the interface advection now!

Sanity check

Link with usual Water Waves problem

If γ is a graph,

$$\gamma(t,s) \to \gamma(t,x) = \begin{bmatrix} x \\ \eta(t,x) \end{bmatrix}$$

(This part is false!) Then we recover the usual water waves problem,

$$\begin{cases} \partial_t \eta &= \sqrt{1+(\partial_x \eta)^2}\, G[\gamma]\psi & = \mathfrak{G}[\eta]\psi \\ \partial_t \psi &= -g\eta + \dfrac{1}{2}\left[(G[\gamma]\psi)^2 + \dfrac{(\partial_x \psi)^2}{1+(\partial_x \eta)^2} \right] & = -g\eta - \dfrac{1}{2}\dfrac{(\mathfrak{G}[\eta]\psi + \partial_x \eta \partial_x \psi)^2}{1+(\partial_x \eta)^2} - \dfrac{1}{2}(\partial_x \psi)^2 \end{cases}$$

where $\mathfrak{G}[\eta]\psi = G[\gamma]\psi\sqrt{1+(\partial_s \eta)^2}$ is the usual Dirichlet-Neumann operator.
This parametrisation only allows upward moving fluid particle! The good way to recover the usual WW equations is done through the inverse function theorem.

Hamiltonian structure

Is it still there?

We define the Hamiltonian for the ”generalised” water waves problem as

$$H[\gamma,\psi] = \frac{1}{2}\int_\gamma \psi G[\gamma]\psi\, ds + \frac{1}{2}\int_\gamma \gamma_2^2 \frac{\partial_s \gamma_1}{|\partial_s \gamma|}\, ds$$

Then we can check that

$$\begin{cases} \boldsymbol{n}\cdot\partial_t\gamma & = & \mathrm{grad}_{\delta\psi} H \\ \partial_t\psi & = & -\mathrm{grad}_{\boldsymbol{n}\cdot\delta\gamma} H \end{cases}$$

$\rightarrow$ All the details are available in Craig (2017).

The $\mathrm{grad}_{\delta x}$ corresponds to the functional derivative, computed by introducing a perturbation δx in a functional,

$$H[x+\delta x] = H[x] + \int \mathrm{grad}_{\delta x} H \cdot \delta x$$

The "generalised" Water Waves problem

$$\begin{cases} \partial_t \gamma & = & \boldsymbol{n}\, G[\gamma]\psi + \boldsymbol{s}\, \dfrac{\partial_s \psi}{|\partial_s \gamma|} \\ \partial_t \psi & = & -g\gamma_2 + \dfrac{1}{2}\left[\left(G[\gamma]\psi\right)^2 + \left(\dfrac{\partial_s \psi}{|\partial_s \gamma|}\right)^2 \right] \end{cases}$$

Viscous Breaking Waves

Adding viscosity
and vorticity

Free-surface Navier-Stokes equations

$$\partial_t \boldsymbol{u} + (\boldsymbol{u} \cdot \nabla)\boldsymbol{u} = -\nabla p + \frac{1}{\mathrm{Re}} \Delta \boldsymbol{u} + \boldsymbol{g}$$
$$\nabla \cdot \boldsymbol{u} = 0$$

Stress-free boundary conditions at the interface,

$$p\boldsymbol{n} - \frac{1}{\mathrm{Re}} \underbrace{\left[\frac{\nabla \boldsymbol{u} + (\nabla \boldsymbol{u})^t}{2} \right]}_{\mathbb{S}(\boldsymbol{u})} \cdot \boldsymbol{n} = \text{surface tension}$$

Navier conditions at the bottom,

$$\boldsymbol{u} \cdot \boldsymbol{n} = 0$$
$$\boldsymbol{s} \cdot \mathbb{S}(\boldsymbol{u}) \cdot \boldsymbol{n} = 0$$

Initial condition
Linear wave

We know many irrotational solutions of the non-viscous water wave problem over a flat bottom (e.g. Stokes waves, Gertsner waves, solitary waves).

Linear waves of (small) amplitude a,

$$\gamma(t,x) = a\cos(kx-\omega t) \qquad \text{with} \qquad \omega = \sqrt{gk\tanh(kh_0)}$$
$$\phi(t,x,y) = \frac{a\omega}{k}\frac{\cosh(ky)}{\sinh(kh_0)}\cdot\sin(kx-\omega t) + \mathcal{O}(ka)$$

The velocity is then

$$\boldsymbol{u}(t,x,y) = \nabla\phi = \frac{a\omega}{\sinh(kh_0)}\cdot\begin{bmatrix}\cosh(ky)\cos(kx-\omega t)\\ \sinh(ky)\sin(kx-\omega t)\end{bmatrix}$$

Initial condition

Laplace problem

Initial cosine interface $\gamma(0, x) = a\cos(kx)$.

Initial irrotational velocity $\boldsymbol{u}_0 = \nabla\phi$ with ϕ solution of the Laplace problem

$$\Delta\phi = 0 \qquad \partial_n\phi = 0 \text{ on } \Gamma_b \qquad \partial_n\phi = \boldsymbol{u}_0 \cdot \boldsymbol{n} \text{ given on } \Gamma_{s,0}$$

We know that

$$\boldsymbol{n}(x) = \frac{1}{\sqrt{1 + (\partial_x\gamma)}} \begin{bmatrix} -\partial_x\gamma \\ 1 \end{bmatrix}$$

and we choose

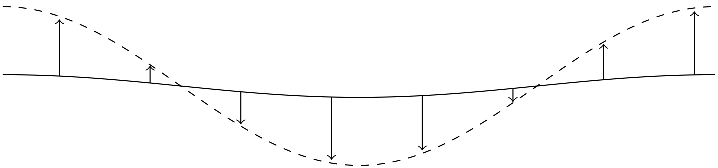
$$\boldsymbol{u}_0\Big|_{y=\gamma(0,x)} = a\omega \begin{bmatrix} \tanh^{-1}(kh_0)\cos(kx) \\ \sin(kx) \end{bmatrix}$$

i.e. putting $y = h_0$ in the linear wave solution.

Initial condition

Finite amplitude extension

$$\boldsymbol{u}_0\big|_{y=\gamma(0,x)} = \frac{a\omega}{\sinh(kh_0)} \cdot \begin{bmatrix} \cosh(ky)\cos(kx) \\ \sinh(ky)\sin(kx) \end{bmatrix} \quad \longrightarrow \quad a\omega \begin{bmatrix} \tanh^{-1}(kh_0)\cos(kx) \\ \sin(kx) \end{bmatrix}$$



$\boldsymbol{u}_0$ evaluated for $y = h_0$ (small amplitude) and mapped on $y = \gamma(0, x)$ (finite amplitude).

This is a purely arbitrary way of constructing the initial condition.

Weak formulation

Function space

$$\mathbf{H}^1_{\Gamma_b}(\Omega_t) = \left\{ \boldsymbol{v} \in \left(H^1(\Omega_t)\right)^2 : \boldsymbol{v} \cdot \boldsymbol{n} = 0 \text{ on } \Gamma_b \right\}$$

We do not suppose $\nabla \cdot \boldsymbol{u} = 0$ in the function space (would not work with finite elements).

Find $\boldsymbol{u} \in \mathcal{C}^1([0,T);\mathbf{H}^1_{\Gamma_b}(\Omega_t))$ and $p \in L^\infty([0,T), L^2(\Omega_t))$ such that

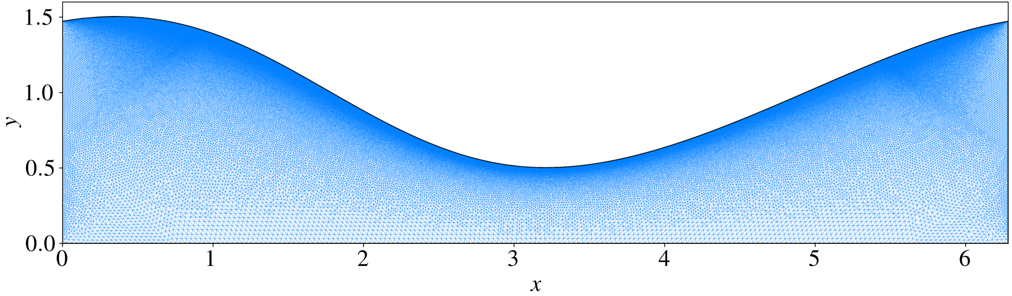
$$\boxed{\int_{\Omega_t} \boldsymbol{v} \cdot \partial_t \boldsymbol{u} + \boldsymbol{v} \cdot (\boldsymbol{u} \cdot \nabla)\boldsymbol{u} + \frac{2}{\mathrm{Re}} \mathbb{S}(\boldsymbol{v}) : \mathbb{S}(\boldsymbol{u}) - p \nabla \cdot \boldsymbol{v} + q \nabla \cdot \boldsymbol{u} - \boldsymbol{v} \cdot \boldsymbol{g} = 0}$$

for all $\boldsymbol{v} \in \mathbf{H}^1_{\Gamma_b}(\Omega_t)$ and $q \in L^2(\Omega_t)$, at all time $t \in (0,T)$.

Finite Elements discretization

We use the `FreeFem` finite elements library (Hecht, 2012) for

- Mesh generation and handling
- Matrices computations and handling
- Interface with PETSc



$4\,000$ points on the interface, initially $\approx 200\,000$ triangles, $\approx 10^6$ degrees of freedom.

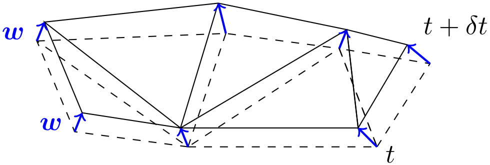
Mesh advection scheme

Let $\boldsymbol{w}$ the mesh velocity. At each time step, we numerically solve the problem

$$\left\{\begin{array}{rcll} \Delta \boldsymbol{w} & = & 0 & \text{in } \Omega_t \\ \boldsymbol{w} & = & \boldsymbol{u} & \text{on } \Gamma_{s,t} \\ \boldsymbol{w} & = & 0 & \text{on } \Gamma_b \end{array}\right.$$

And each point of the mesh is advected according to $\boldsymbol{w}$. Points on the interface are thus purely Lagrangian!

This is called the Arbitrary Lagrangian Eulerian method (ALE).



Parallelization and timings

MPI domain decomposition with graph partitionner. PETSc matrices and solvers.

We use a geometric multigrid solver for fast convergence using a large number of MPI processes.

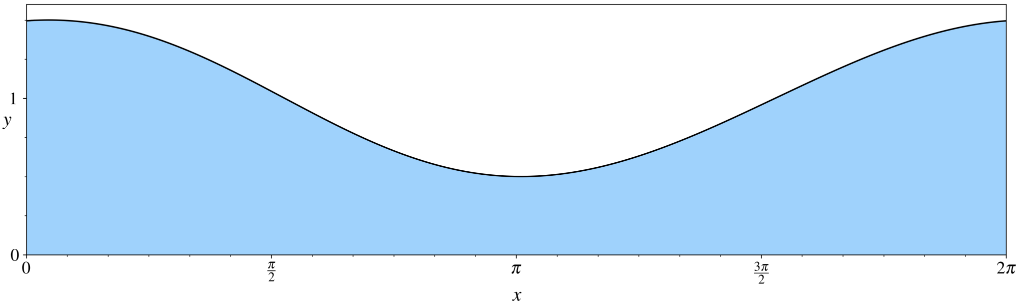
At $\mathrm{Re} = 10^6$ (between 1 and 3 million unknowns), convergence in $\sim 5 \pm 2$ GMRES iterations $\to \sim 20$ seconds on 48 CPU cores.

Main computational limitations due to FreeFem memory management and spurious behaviors in mesh handling.

From 3 days ($\mathrm{Re} = 10^2$, $3\,500$ points on the interface) to 2 weeks of computations ($\mathrm{Re} = 10^6$, $4\,000$ points on the interface) for one 2d simulation.

Clearly this method is not efficient enough to handle 3d, unless we decrease the precision.

$\mathrm{Re} = 10^6$ **result**



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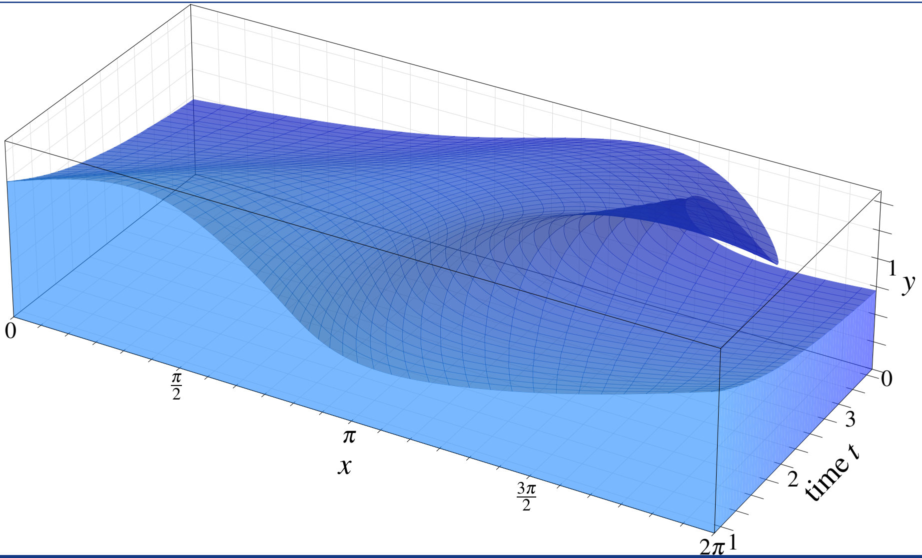
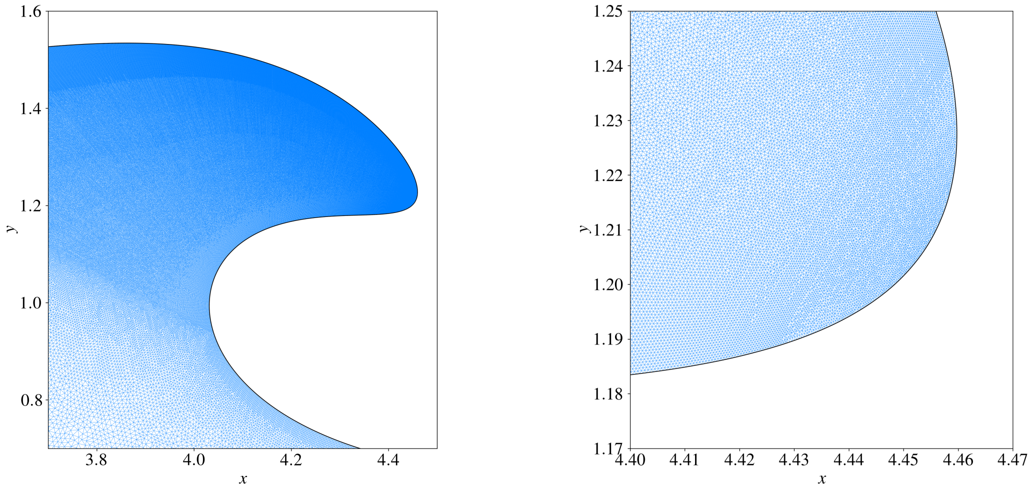
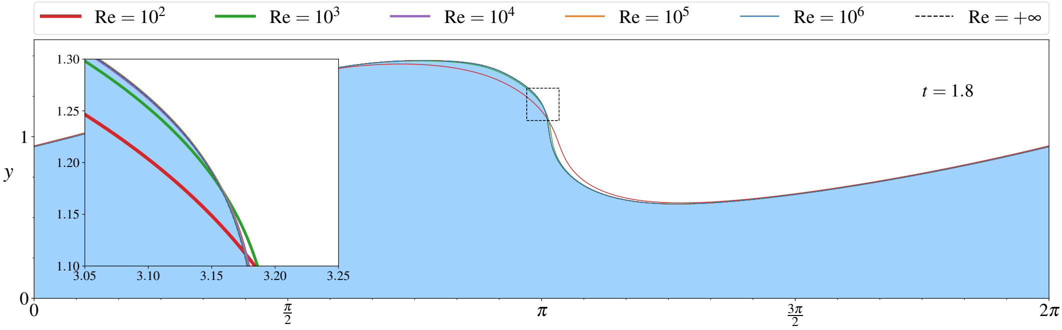
$\mathrm{Re} = 10^6$ **result**



Mesh at $\mathrm{Re} = 10^6$

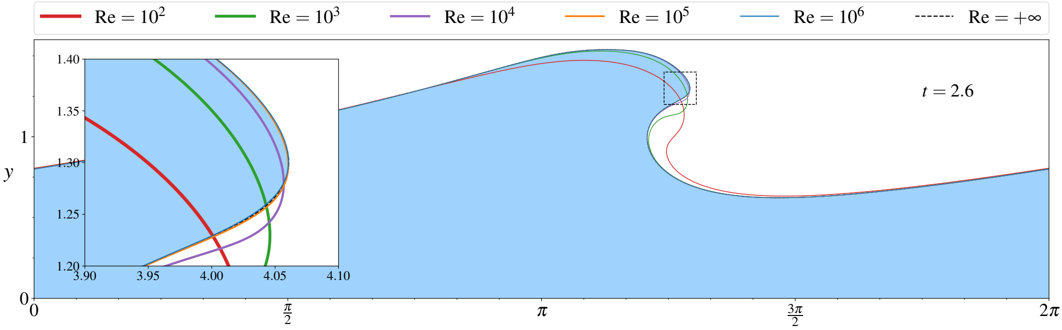


Interface for different values of Re



$\mathrm{Re} = +\infty$ simulations ($\mathrm{i.e.}$ Euler solution) obtained using the code of Dormy & Lacave (2023).

Interface for different values of Re



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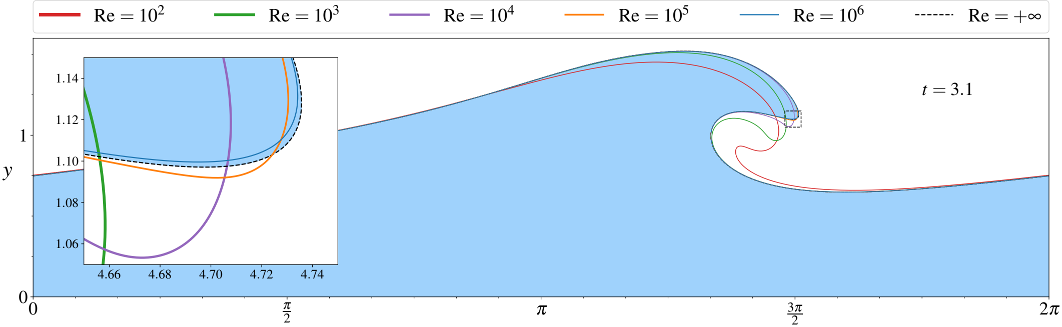

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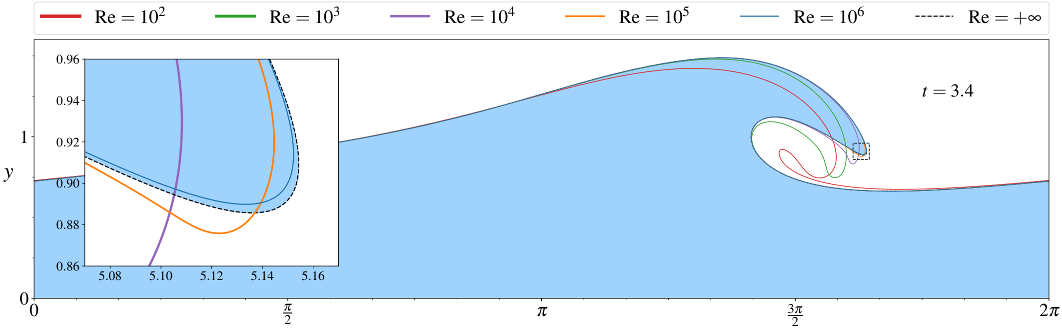

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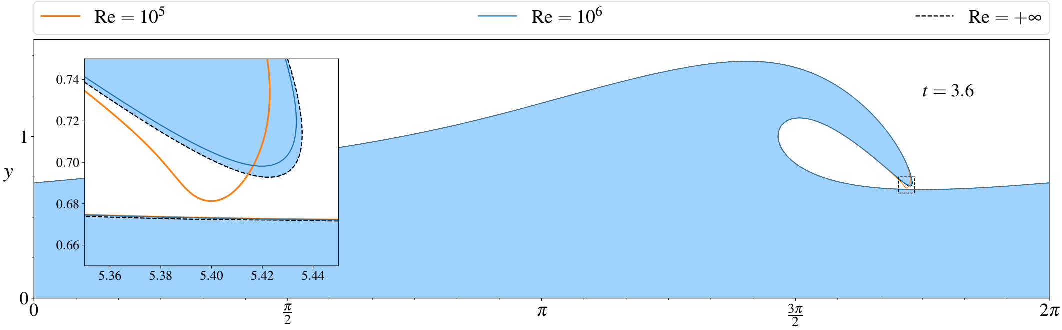
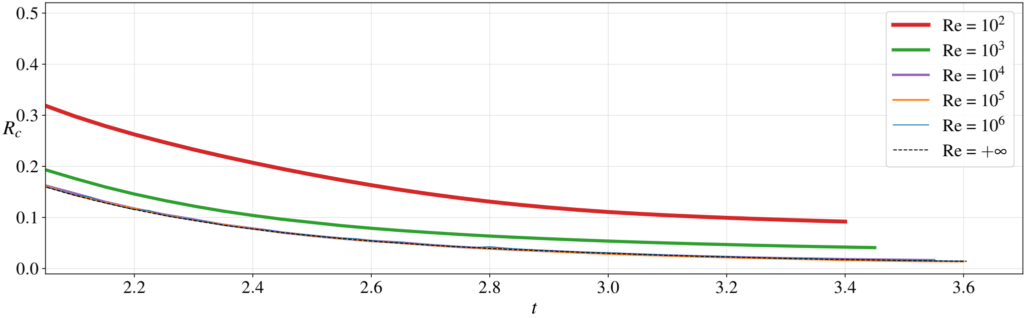


$\mathrm{Re} = +\infty$ simulations (i.e. Euler solution) obtained using the code of Dormy & Lacave (2023).

Maximum curvature of the interface



where $R_C = \kappa^{-1}$ is the curvature radius.

Energy considerations

Local and global

A global equation for the evolution of the energy can be obtained from the weak form of the viscous water waves problem, setting $\boldsymbol{v} = \boldsymbol{u}(t, \cdot)$,

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega_t} \boldsymbol{u}^2 + \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Gamma_{s,t}} y^2 n_y \, ds + \int_{\Omega_t} \frac{2}{\mathrm{Re}} \mathbb{S}(\boldsymbol{u}) : \mathbb{S}(\boldsymbol{u}) = 0$$

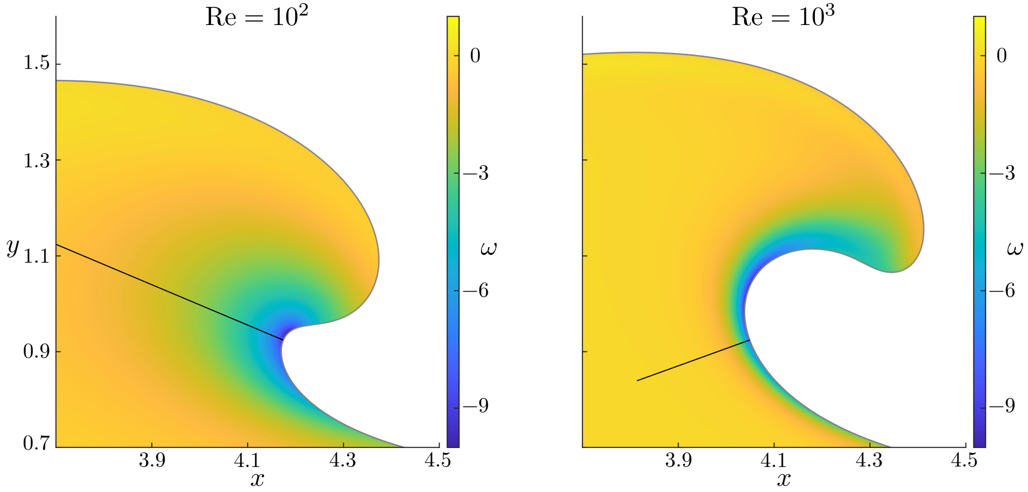
However this does not provide any knowledge about where this dissipation happens...

Instead, we work with the local energy equation, obtained multiplying the Navier-Stokes eq. by $\boldsymbol{u}$,

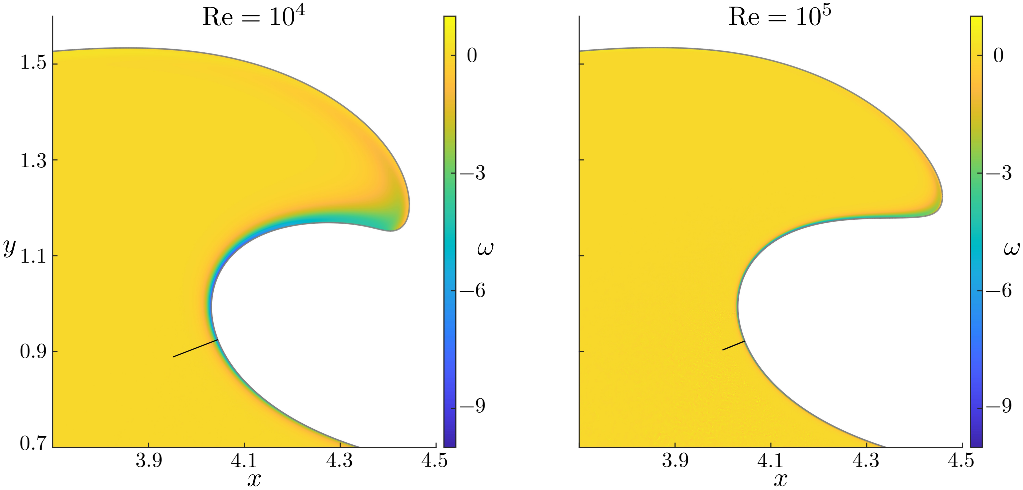
$$\partial_t \left(\frac{\boldsymbol{u}^2}{2} \right) = \boldsymbol{g} \cdot \boldsymbol{u} - \boldsymbol{u} \cdot \nabla p + \frac{1}{\mathrm{Re}} \left[\nabla \cdot (\boldsymbol{u}^{\perp} \omega) - \omega^2 \right]$$

where $\boldsymbol{u}^{\perp} = [-u_y, u_x]$ and $\omega = \nabla^{\perp} \cdot \boldsymbol{u}$ is the vorticity.

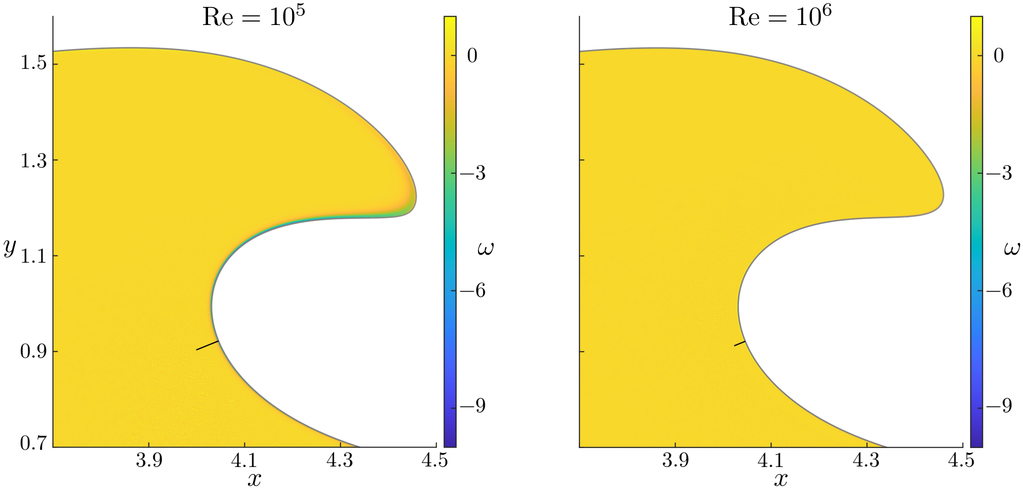
Viscous dissipation



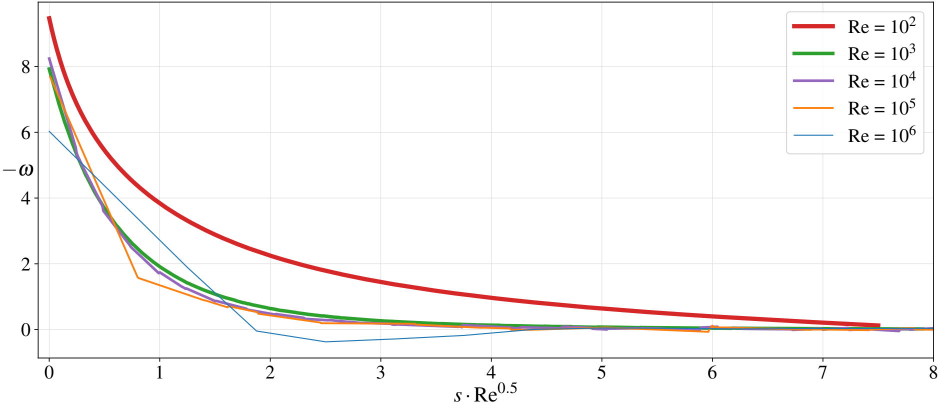
Viscous dissipation



Viscous dissipation



Size of the boundary layer



Exhibits a $\mathrm{Re}^{-\frac{1}{2}}$ scaling (usual in BL theory).

Where does this Boundary Layer come from?

The superficial vorticity

We have encountered $\omega = \nabla^\perp \cdot \boldsymbol{u}$ the volume vorticity of the fluid. We can also make sense of the superficial vorticity γ whose definition is made through harmonic theory, as follows.

Let $G(\boldsymbol{x}, \boldsymbol{y})$ the Green's function of the Laplace operator Δ in a domain $\Omega \subset \mathbb{R}^2$. Then the function

$$\psi(\boldsymbol{x}) = \int_\Omega G(\boldsymbol{x}, \boldsymbol{y})\omega(\boldsymbol{y})\,\mathrm{d}\boldsymbol{y}$$

is a solution of the Laplace equation $\Delta\psi = \omega$ in Ω.

If ω is the vorticity, ψ us the stream function such that $\boldsymbol{u} = \nabla^\perp\psi$. But ω can also be a distribution of the form $\gamma\delta_\Gamma$ where Γ is a curve in Ω. In that case,

$$\psi(\boldsymbol{x}) = \int_\Gamma G(\boldsymbol{x}, \boldsymbol{y}(s))\gamma(\boldsymbol{y}(s))\,\mathrm{d}s$$

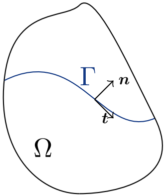
with $\boldsymbol{y}(s)$ a parametrization of the curve Γ. In general, γ depends on the parametrization. It is sometimes called the vortex sheet strength.

Where does this Boundary Layer come from?

The Lundgren-Koumoutsakos theorem

It can also be defined as

$$\gamma(s) = \left[\lim_{z\to 0^+} \boldsymbol{u}(\boldsymbol{y}(s) + z\boldsymbol{n}(s)) - \lim_{z\to 0^-} \boldsymbol{u}(\boldsymbol{y}(s) + z\boldsymbol{n}(s))\right] \cdot \boldsymbol{t}(s)$$



if $\boldsymbol{y}(s)$ is the arc-length parametrization.

The Lundgren & Koumoutsakos (1999) theorem states that if the vorticity is composed of a superficial part γ and a volume part ω, the former diffuses into the latter due to viscous effects

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Gamma \gamma \,\mathrm{d}S + \int_\Gamma \frac{1}{\mathrm{Re}} \nabla\omega \cdot \boldsymbol{n}\,\mathrm{d}S = 0$$

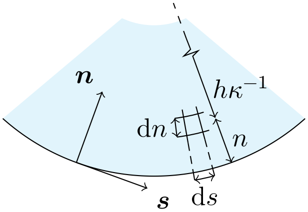
in a way that the total vorticity is conserved,

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Gamma \gamma \,\mathrm{d}S + \frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \omega \mathrm{d}V = 0$$

An equation for the Curvature evolution

In Longuet-Higgins (1953) is derived an equation for the curvature evolution, provided the interface is a $\mathcal{C}^3$ function.

Introduce the Frenet frame (s, n), i.e. the tangential and normal coordinate w.r.t. the (time-dependant) curve $\gamma(t, s)$.



A point $\gamma(t, s)$ on the curve is advected by a vector field $\boldsymbol{u}(s, n) = (u_s, u_n)$.

The curvature $\kappa(t, s)$ obeys

$$\partial_t \kappa = \partial_{ss} u_n - u_s \partial_s \kappa + \kappa^2 u_n$$

The derivation of this PDE is geometric.

Stream function

Expressing the incompressibility condition in (s, n) coordinates yields

$$\nabla \cdot \boldsymbol{u} = \partial_s u_s + \partial_n (h u_n) = 0 \qquad \text{with} \qquad h = 1 - \kappa^{-1} n$$

This motivates the definition of a stream function ψ s.t.

$$u_s = -\partial_n \psi \qquad \text{and} \qquad u_n = h^{-1} \partial_s \psi$$

The vorticity is

$$\omega = h^{-1} \Big(\partial_s u_n - \partial_n (h u_s) \Big) = \Delta \psi$$

We suppose that the velocity $\boldsymbol{u}$ can be expressed as an irrotational part and a viscous part

$$\boldsymbol{u} = \nabla \phi + \nabla^{\perp} \psi_{\mathrm{Re}}$$

Asymptotic expansion

We observed that the typical size δ of the boundary layer is

$$\delta = \frac{1}{\sqrt{\mathrm{Re}}}$$

We therefore suppose that the deppendance of ψ in the viscosity Re is of the form

$$\psi_{\mathrm{Re}}(t,s,n) = \psi_0(t,s,n) + \delta\psi_1(t,s,n) + \delta^2\psi_2(t,s,n) + \mathcal{O}(\delta^3)$$

Since the viscous effects seem to vanish as $\mathrm{Re} \to +\infty$, we get $\psi_0 = 0$.

The typical variation length of ψ in the normal direction is δ. Hence we can assume that

$$\psi_{\mathrm{Re}}(t,s,n) \equiv \psi_{\mathrm{Re}}(t,s,n\delta^{-1})$$

so each normal derivative decreases the order of the expansion by 1.

Effects of viscosity on the curvature

Inserting the expansion in the voticity equation, we find that the leading term is of order $\mathcal{O}(\delta^{-1})$. However we observed that the vorticity seem to behave as $\mathcal{O}(1)$. Therefore $\psi_1 = 0$.

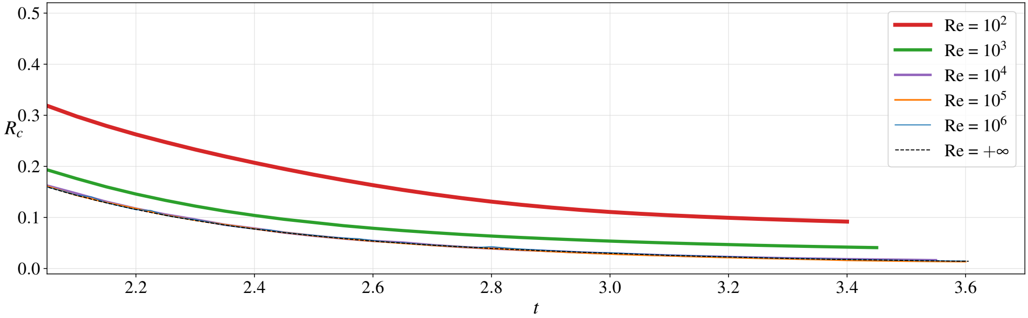
Inserting the expansion in the equation for κ, we get

$$\partial_t \kappa = \text{irrotational part } + \underbrace{\partial_{ss}(h^{-1}\partial_s \psi_{\mathrm{Re}})}_{O(\delta^2)} - \underbrace{\partial_s \kappa \partial_n \psi_{\mathrm{Re}}}_{O(\kappa\delta)} + \underbrace{\kappa^2 h^{-1} \partial_s \psi_{\mathrm{Re}}}_{O(\kappa^2\delta^2)}$$

Interpretation:

- The effects of viscosity appear in time $\mathcal{O}(1)$ when the curvature is of order $\mathcal{O}(\delta^{-1})$
- The effects of viscosity appear in time $\mathcal{O}(\delta^{-2})$ when the curvature is of order $\mathcal{O}(\delta)$

Maximum curvature of the interface



where $R_C = \kappa^{-1}$ is the curvature radius.

Thank you!