Effects of viscosity on Water Waves

Séminaire doctoral du LAMFA - Université Picardie Jules Verne

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May 22, 2024

Free-surface Navier-Stokes equation

$$\partial_t \boldsymbol{u} + (\boldsymbol{u} \cdot \nabla)\boldsymbol{u} + \nabla p = \frac{1}{\mathrm{Re}} \Delta \boldsymbol{u} + \boldsymbol{y}$$

$$\nabla \cdot \boldsymbol{u} = 0$$

with stress-free boundary conditions on the interface $\Gamma_s(t)$,

$$p\boldsymbol{n} - \frac{2}{\mathrm{Re}} \left[\nabla \boldsymbol{u} + (\nabla \boldsymbol{u})^t \right] \cdot \boldsymbol{n} = 0$$

and Navier conditions at the bottom Γ_b,

$$\boldsymbol{u} \cdot \boldsymbol{n} = 0$$

$$\cdot \left[\nabla \boldsymbol{u} + (\nabla \boldsymbol{u})^t \right] \cdot \boldsymbol{n} = 0$$

Weak formulation

Function space

$$\mathbf{H}^1_{\Gamma_b}(\Omega_t) = \left\{ \boldsymbol{v} \in \left(H^1(\Omega_t)\right)^2 : \boldsymbol{v} \cdot \boldsymbol{n} = 0 \text{ on } \Gamma_b \right\}$$

We do not assume the incompressibility, $\nabla \cdot \boldsymbol{u} = 0$, directly in the function space as it would not work in finite element.

Find $\boldsymbol{u} \in \mathcal{C}^1([0,T); \mathbf{H}^1_{\Gamma_b}(\Omega_t))$ and $p \in L^\infty([0,T), L^2(\Omega_t))$ such that

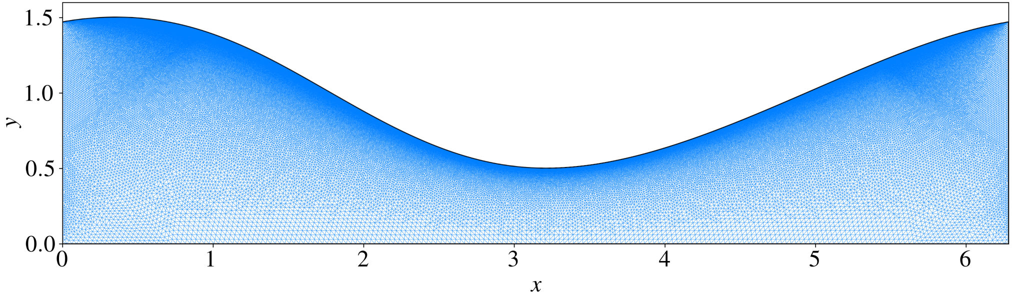
$$\boxed{\int_{\Omega_t} \boldsymbol{v} \cdot \partial_t \boldsymbol{u} + \boldsymbol{v} \cdot (\boldsymbol{u} \cdot \nabla)\boldsymbol{u} + \frac{2}{\mathrm{Re}} \mathbb{S}(\boldsymbol{v}) : \mathbb{S}(\boldsymbol{u}) - p \nabla \cdot \boldsymbol{v} + q \nabla \cdot \boldsymbol{u} - \boldsymbol{v} \cdot \boldsymbol{g} = 0}$$

for all $\boldsymbol{v} \in \mathbf{H}^1_{\Gamma_b}(\Omega_t)$ and $q \in L^2(\Omega_t)$, at all time $t \in (0,T)$.

Finite Element

We use `FreeFem` (c++ finite element library, Hecht, 2012) for

- mesh generation and advection
- matrices generation
- Multi-threading using a PETSc interface



Initial mesh with $4\,000$ points on the interface, leading to $\approx 200\,000$ triangles, $\approx 10^6$ degrees of freedom.

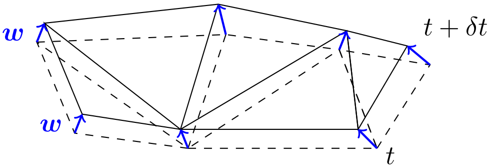
Mesh advection

Let $\boldsymbol{w}$ the mesh velocity. It is computed at each time step solving the following problem numerically

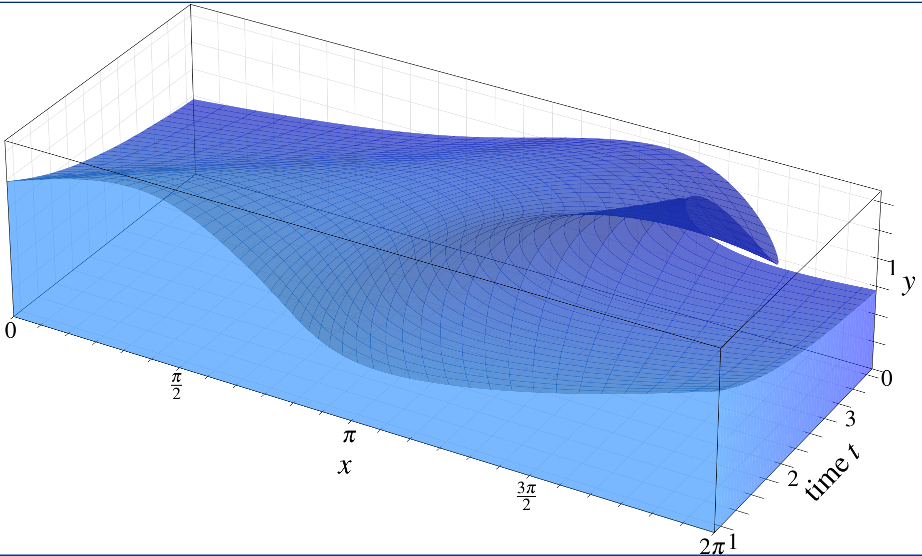
$$\begin{cases} \Delta \boldsymbol{w} &= 0 & \text{in } \Omega_t \\ \boldsymbol{w} &= \boldsymbol{u} & \text{on } \Gamma_{s,t} \\ \boldsymbol{w} &= 0 & \text{on } \Gamma_b \end{cases}$$

Hence, points on the interface $\Gamma_s(t)$ are advected in a lagrangian manner.

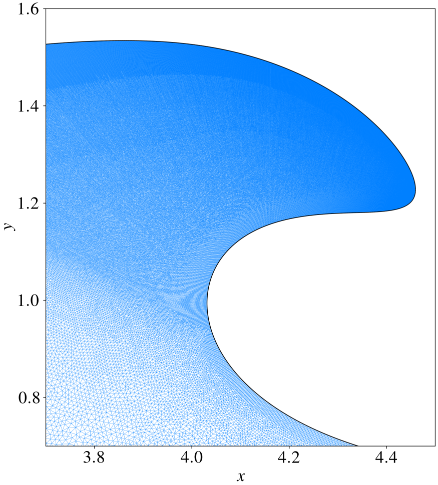
This is called the Arbitrary Lagrangian-Eulerian (ALE) method.

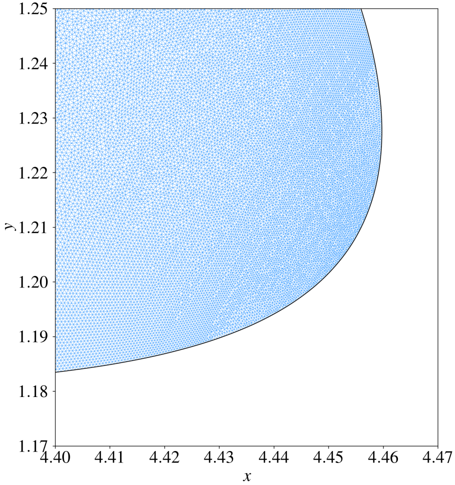


Results with $\mathrm{Re} = 10^6$

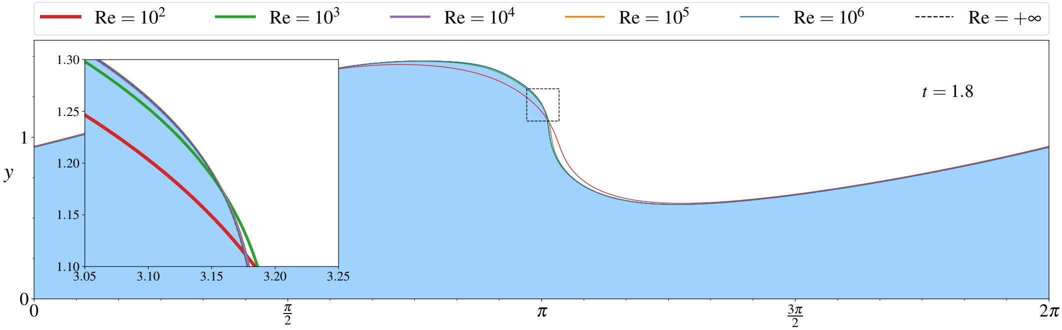


Mesh at $\mathrm{Re} = 10^6$



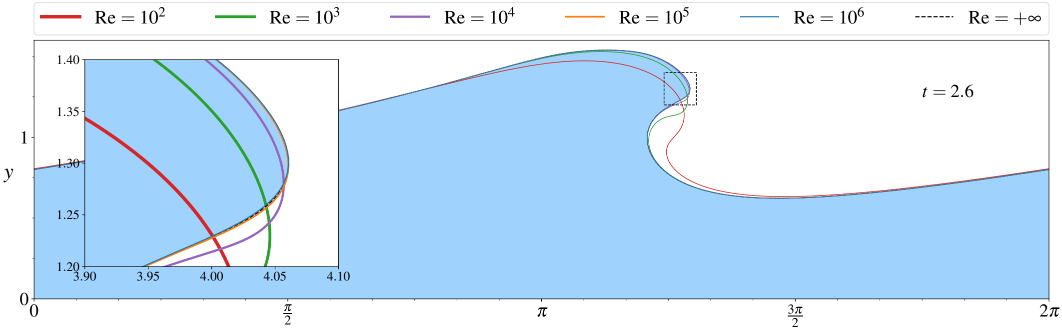


Increasing the Reynolds number Re



The simulation $\mathrm{Re} = +\infty$ (i.e. the Euler solution) has been computed with the code of Dormy & Lacave (2024).

Increasing the Reynolds number Re



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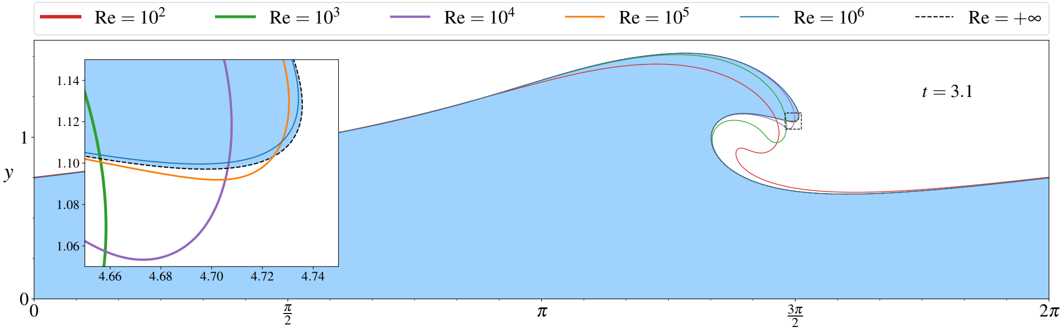

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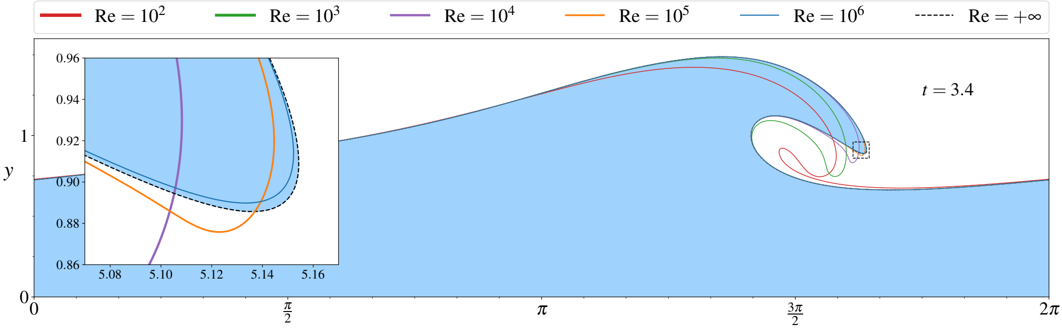

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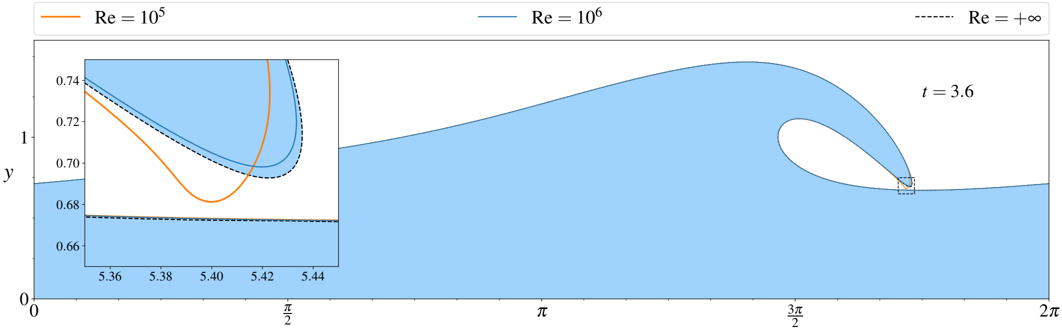

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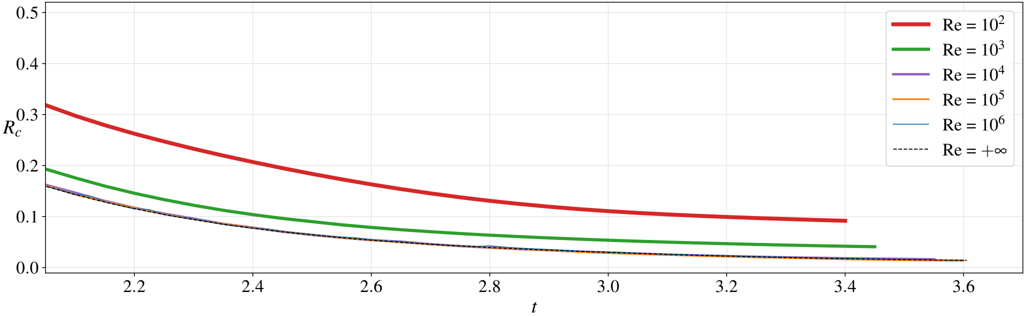
Time evolution of the maximum curvature



where $R_C = \kappa^{-1}$ is the curvature radius.

Energy dissipation

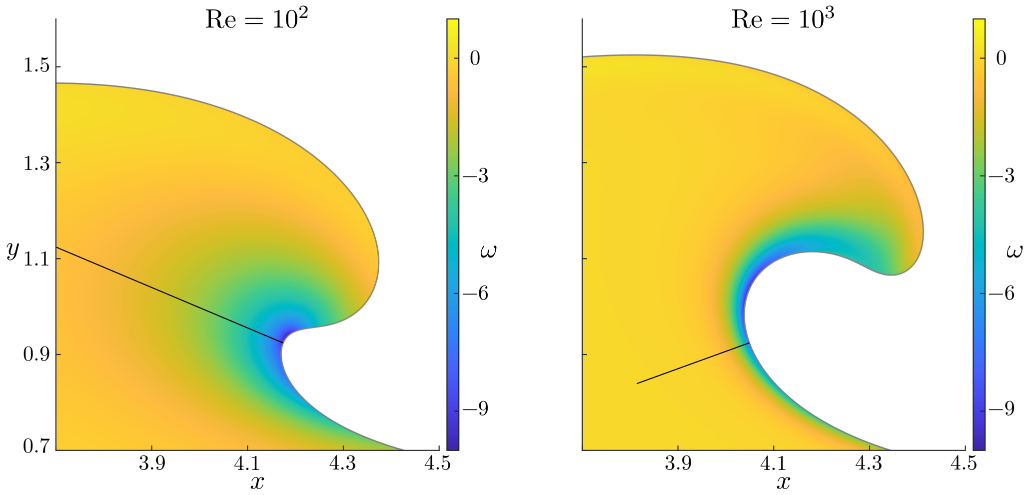
Local equation

If we multiply Navier-Stokes by $\boldsymbol{u}$, we easily get a local equation of the evolution of the kinetic energy

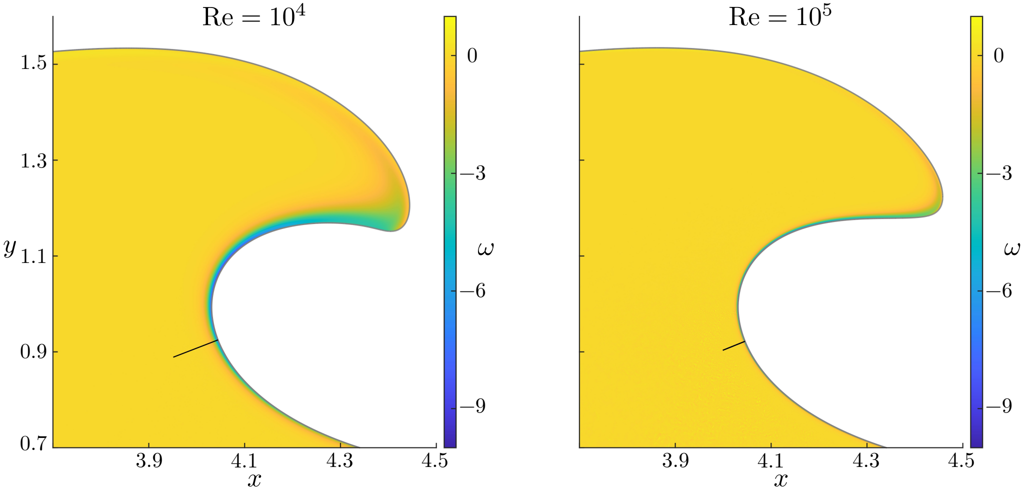
$$\partial_t \left(\frac{\boldsymbol{u}^2}{2} \right) = \boldsymbol{g} \cdot \boldsymbol{u} - \boldsymbol{u} \cdot \nabla p + \frac{1}{\mathrm{Re}} \left[\nabla \cdot (\boldsymbol{u}^{\perp} \omega) - \omega^2 \right]$$

where $\boldsymbol{u}^{\perp} = [-u_y, u_x]$ and $\omega = \nabla^{\perp} \cdot \boldsymbol{u}$ is the vorticity.

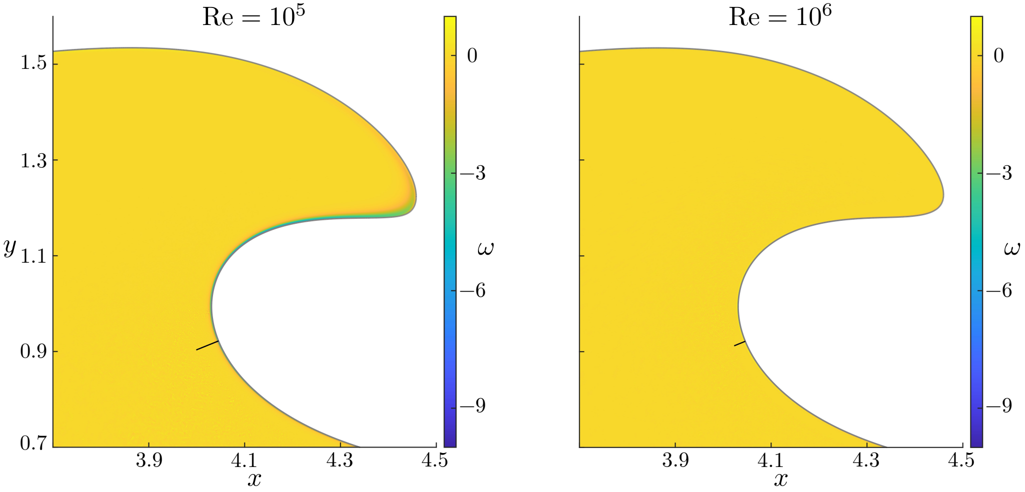
Viscous dissipation



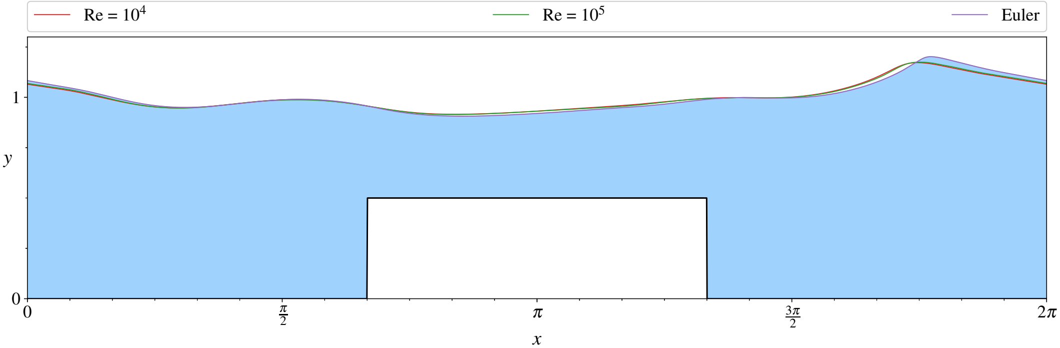
Viscous dissipation



Viscous dissipation



Non-convergence of the Navier-Stokes solution



Thank you!