Numerical study of the vanishing viscosity limit in (Breaking) Water Waves

1st European Fluid Dynamics Conference

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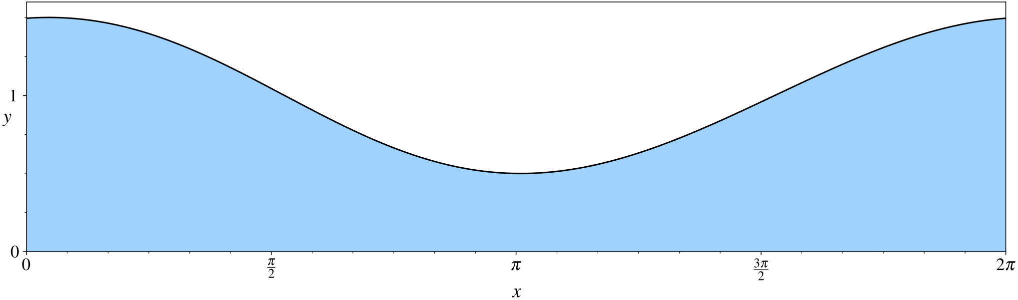
October 14, 2024

$\mathrm{Re} = 10^6$ result



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Numerical method

Navier-Stokes

Lagrangian advection

Finite Element Methods

Viscous Water Waves

Navier-Stokes equation

Incompressible, non-dimensional, **Navier-Stokes** equation in $\Omega(t)$:

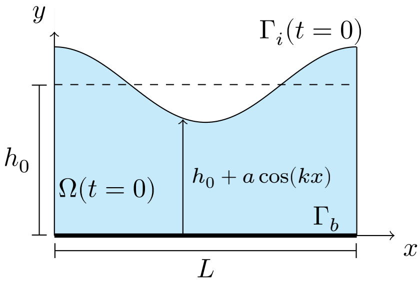
$$\begin{cases} \partial_t \boldsymbol{u} + (\boldsymbol{u} \cdot \nabla)\boldsymbol{u} & = & -\nabla p + \frac{1}{Re}\Delta \boldsymbol{u} + \boldsymbol{g} \\ \nabla \cdot \boldsymbol{u} & = & 0 \end{cases}$$

Navier boundary conditions on Γ_b,

$$\boldsymbol{u} \cdot \boldsymbol{n} = 0 \quad ; \quad \boldsymbol{t} \cdot \left[\nabla \boldsymbol{u} + (\nabla \boldsymbol{u})^t\right] \cdot \boldsymbol{n} = 0$$

Stress-free boundary condition on $\Gamma_i(t)$,

$$p\boldsymbol{n} - \frac{1}{Re} \cdot \left[\nabla \boldsymbol{u} + (\nabla \boldsymbol{u})^t\right] \cdot \boldsymbol{n} = 0$$



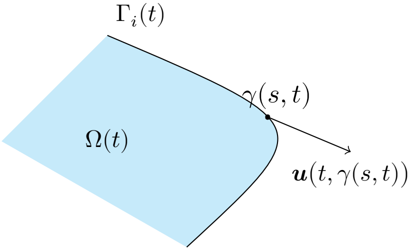
Interface advection

Lagrangian scheme

Interface is a **parametrised curve** $\gamma(s,t) \in \mathbb{R}^2$ whose evolution is given by

$$\frac{\partial \gamma}{\partial t}(s,t) = \boldsymbol{u}\Big(t, \gamma(s,t)\Big)$$

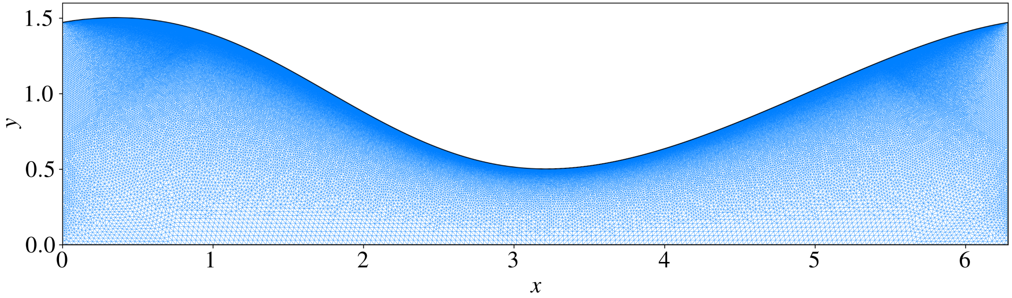
i.e. points on the interface have the same velocity as the fluid particles.



Finite Elements discretization

We use the `FreeFem` finite elements library Hecht (2012) for

- Mesh generation and handling
- Matrices computations and handling
- Interface with PETSc



$4\,000$ points on the interface, initially $\approx 200\,000$ triangles, $\approx 10^6$ degrees of freedom.

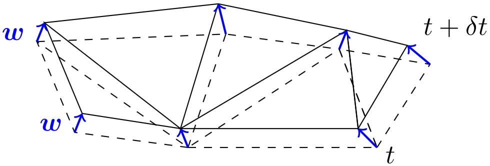
Mesh advection scheme

Let $\boldsymbol{w}$ the **velocity of the mesh**. At each time step, we numerically solve the problem

$$\begin{cases} \Delta \boldsymbol{w} = 0 & \text{in } \Omega_t \\ \boldsymbol{w} = \boldsymbol{u} & \text{on } \Gamma_i(t) \\ \boldsymbol{w} = 0 & \text{on } \Gamma_b \end{cases}$$

And each point of the mesh is advected with velocity $\boldsymbol{w}$. Points on the interface are thus **purely Lagrangian**!

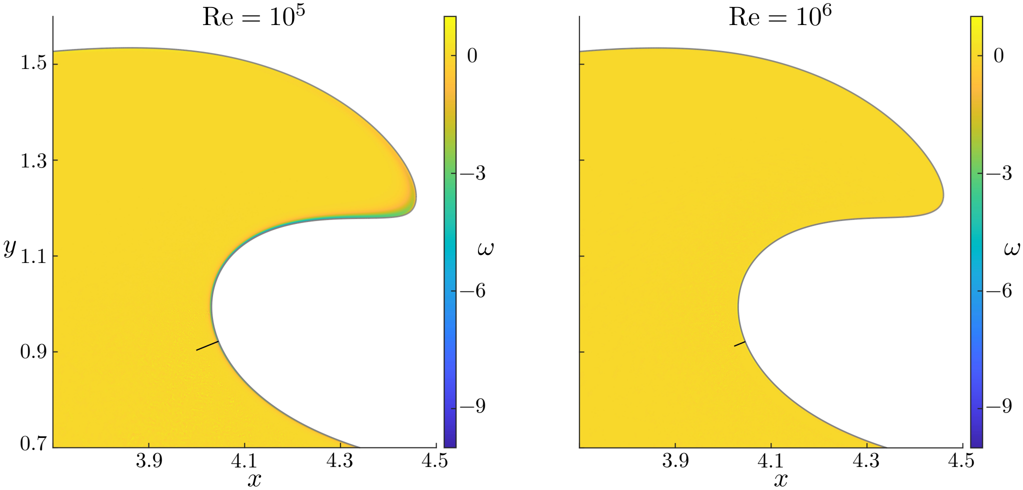
This is called the **Arbitrary Lagrangian Eulerian** method (ALE).



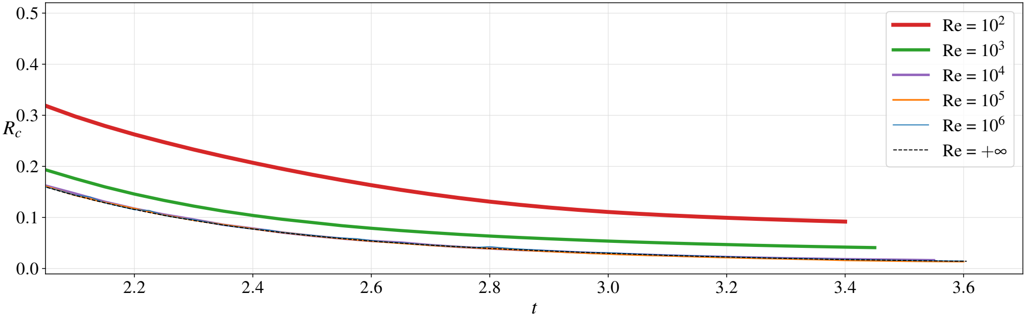
Viscosity and free surfaces

A general theory of interface regularization due to viscosity

Viscous dissipation



Maximum curvature of the interface

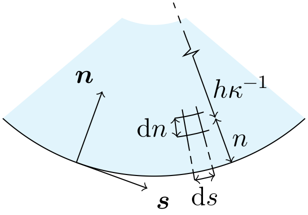


where $R_C = \kappa^{-1}$ is the curvature radius.

An equation for the Curvature evolution

In Longuet-Higgins (1953) is derived an **equation for the curvature evolution**.

Introduce the Frenet frame (s, n), i.e. the tangential and normal coordinate w.r.t. the (time-dependant) curve $\gamma(t, s)$.



A point $\gamma(t, s)$ on the curve is advected by a vector field $\boldsymbol{u}(s, n) = (u_s, u_n)$.

The curvature $\kappa(t, s)$ obeys

$$\partial_t \kappa = \partial_{ss} u_n - u_s \partial_s \kappa + \kappa^2 u_n$$

Stream function

Expressing the **incompressibility condition** in (s, n) coordinates yields

$$\nabla \cdot \boldsymbol{u} = \partial_s u_s + \partial_n (h u_n) = 0 \qquad \text{with} \qquad h = 1 - \kappa^{-1} n$$

This motivates the definition of a **stream function** ψ s.t.

$$u_s = -\partial_n \psi \qquad \text{and} \qquad u_n = h^{-1} \partial_s \psi$$

The vorticity is

$$\omega = h^{-1} \Big(\partial_s u_n - \partial_n (h u_s) \Big) = -\Delta \psi$$

We **suppose** that the velocity $\boldsymbol{u}$ can be expressed as an irrotational part and a viscous part

$$\boldsymbol{u} = \nabla \phi + \nabla^{\perp} \psi_{Re}$$

Asymptotic expansion

We know (and observed) that the typical size δ of the boundary layer is

$$\delta = \frac{1}{\sqrt{Re}}$$

We therefore suppose that the deppendance of ψ in the viscosity Re is of the form

$$\psi_{Re}(t,s,n) = \psi_0(t,s,n) + \delta\psi_1(t,s,n) + \delta^2\psi_2(t,s,n) + \mathcal{O}(\delta^3)$$

Since the viscous effects seem to vanish as $Re \to +\infty$, we get $\psi_0 = 0$.

The typical variation length of ψ in the normal direction is δ. Hence we can assume that

$$\psi_{Re}(t,s,n) \equiv \psi_{Re}(t,s,n\delta^{-1})$$

so each normal derivative decreases the order of the expansion by 1.

Effects of viscosity on the curvature

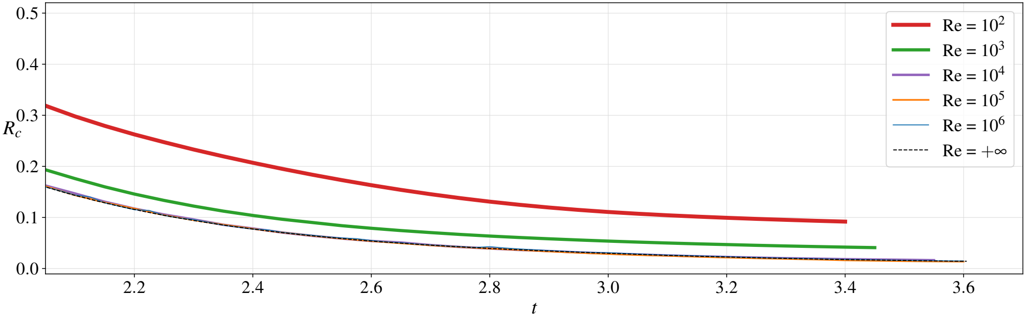
Inserting the expansion in the voticity equation, we find that the leading term is of order $\mathcal{O}(\delta^{-1})$. However we observed that the vorticity seem to behave as $\mathcal{O}(1)$. Therefore $\psi_1 = 0$.

Inserting the expansion in the equation for κ, we get

$$\partial_t \kappa = \text{irrotational part } + \underbrace{\partial_{ss}(h^{-1}\partial_s \psi_{Re})}_{O(\delta^2)} - u_s \partial_s \kappa + \underbrace{\kappa^2 \partial_s \psi_{Re}}_{O(\kappa^2 \delta^2)}$$

Interpretation: The effects of viscosity appear in time $\mathcal{O}(1)$ when the curvature is of order $\mathcal{O}(\delta^{-2})$

Maximum curvature of the interface



where $R_C = \kappa^{-1}$ is the curvature radius.

Is irrotationality well motivated?

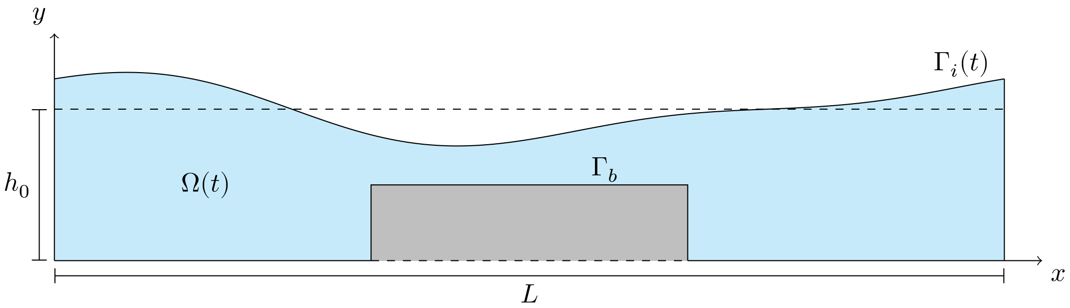
Adding an obstacle on the bottom

Emitting vortices?

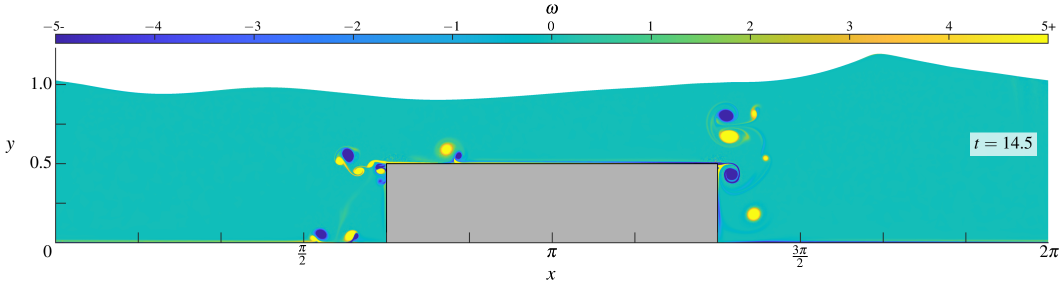
Rectangular step

No slip condition on the bottom,

$$\boldsymbol{u} = 0 \qquad \text{on } \Gamma_b$$



Vortices at $Re = 10^5$



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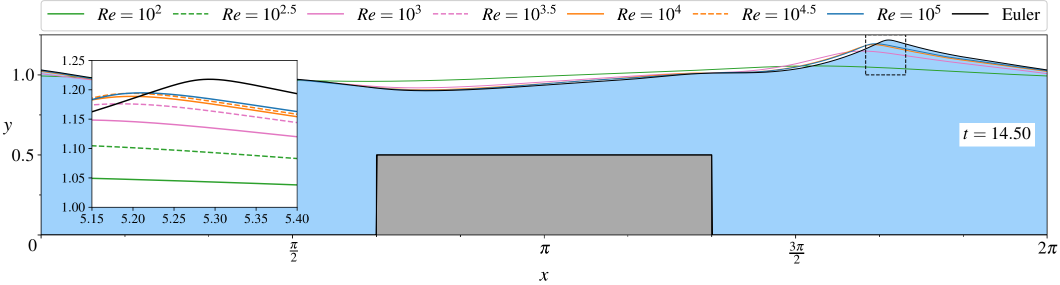
Comparing the interfaces



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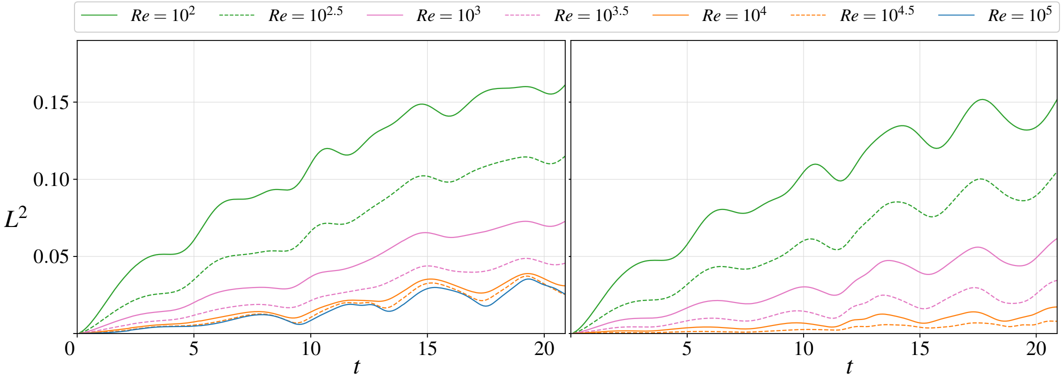
L^2 distances



Distance to Euler

Distance to $Re = 10^5$

Thank you!

More information in A. R. & E. Dormy (2024) **Numerical study of a viscous breaking water wave and the limit of vanishing viscosity**, J. Fluid Mech. (Rapids) **984**, R5.