A NUMERICAL STUDY
OF THE
VANISHING VISCOSITY LIMIT
IN THE THEORY OF
WATER WAVES

RÉUNION DE RENTRÉE DE L'ÉQUIPE D'ANALYSE - DMA

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Joint work with Emmanuel Dormy

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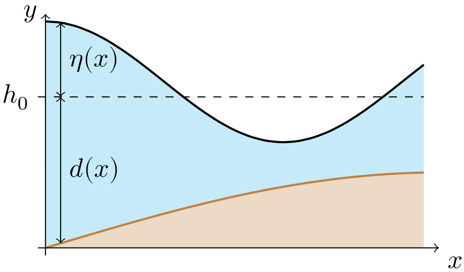
A plenty of models

Shallow Water

$$\begin{cases} \partial_t \eta + \nabla \cdot (\eta \boldsymbol{u}) = -\nabla \cdot (d\boldsymbol{u}) \\ \partial_t \boldsymbol{u} + \boldsymbol{u} \cdot \nabla \boldsymbol{u} = -g \nabla \eta \end{cases}$$



Korteweg-de Vries-Johnson

$$\partial_t \phi + \frac{3}{2} d^{-\frac{7}{4}} \phi \partial_x \phi + \frac{1}{6} d^{\frac{1}{2}} \partial_{xxx} \phi = 0$$

And **many** others: Serre-Green-Naghdi, Boussinesq, Camassa-Holm, ...

The Water Waves equations

All these models stem from the **Water Waves (WW)** equations (Lannes (2013))

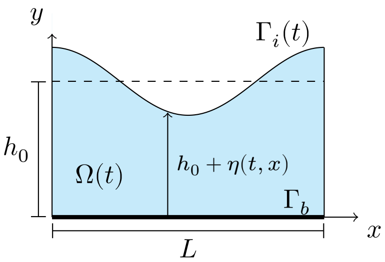
$$\begin{cases} \partial_t \eta = G[\eta, d]\psi \\ \partial_t \psi = -g\eta - \frac{1}{2}|\nabla\psi|^2 + \frac{1}{2}\frac{(G[\eta,d]\psi + \nabla\eta\cdot\nabla\psi)^2}{1+|\nabla\eta|^2} \end{cases}$$

also known as the Zakharov (1968) - Craig-Sulem (1993) formulation of the Water Waves problem.

The **Dirichlet-Neumann** operator $G[\eta, d]\psi$ is computed by solving the following problem

$$\begin{cases} \Delta\phi = 0 & \text{in } \Omega(t) \\ \phi = \psi & \text{on } \Gamma_i(t) \\ \partial_n \phi = 0 & \text{on } \Gamma_b \end{cases}$$

and then taking $G[\eta, d]\psi = \partial_n \phi\big|_{y=h_0+\eta(t,x)}$.



Free-surface Euler's equations

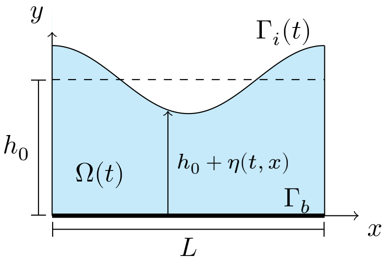
The **Water Waves** equations, themselves, are derived from **Euler's equations** with a **free surface**,

$$\begin{cases} \partial_t \boldsymbol{u} + \boldsymbol{u} \cdot \nabla \boldsymbol{u} + \nabla p &= 0 \quad \text{in } \Omega(t) \\ \nabla \cdot \boldsymbol{u} &= 0 \quad \text{in } \Omega(t) \\ p &= 0 \quad \text{on } y = h_0 + \eta(t,x) \\ u_n &= 0 \quad \text{on } y = h_0 - d(x) \\ \partial_t \eta + u_y - u_x \partial_x \eta &= 0 \end{cases}$$

by **ASSUMING irrotationality** of the flow,

$$\boldsymbol{u} = \nabla \phi$$

but is it a well-motivated **hypothesis**?



Problem formulation

Navier-Stokes
Lagrangian advection
Numerical methods

Viscous Water Waves

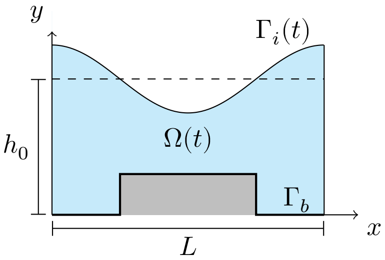
Nondimensionalization

Nondimensional quantities are defined as follows

$$\boldsymbol{x} \to h_0 \boldsymbol{x}$$
$$\boldsymbol{u} \to \sqrt{g h_0} \cdot \boldsymbol{u}$$
$$p \to \rho g h_0 \cdot p$$

This allows to define the **Reynolds number** Re,

$$\mathrm{Re} = \frac{\rho h_0 \sqrt{g h_0}}{\mu}$$



Viscous Water Waves

Navier-Stokes equation

Incompressible, non-dimensional, **Navier-Stokes** equation in $\Omega(t)$:

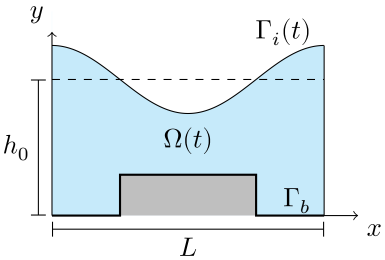
$$\begin{cases} \partial_t \boldsymbol{u} + (\boldsymbol{u} \cdot \nabla)\boldsymbol{u} &= -\nabla p + \dfrac{1}{\mathrm{Re}} \Delta \boldsymbol{u} + \boldsymbol{g} \\ \nabla \cdot \boldsymbol{u} &= 0 \end{cases}$$

Navier boundary conditions on Γ_b,

$$\boldsymbol{u} \cdot \boldsymbol{n} = 0 \quad ; \quad \boldsymbol{t} \cdot \left[\nabla \boldsymbol{u} + (\nabla \boldsymbol{u})^t\right] \cdot \boldsymbol{n} = 0$$

Stress-free boundary condition on $\Gamma_i(t)$,

$$p\boldsymbol{n} - \frac{1}{\mathrm{Re}} \cdot \left[\nabla \boldsymbol{u} + (\nabla \boldsymbol{u})^t\right] \cdot \boldsymbol{n} = 0$$



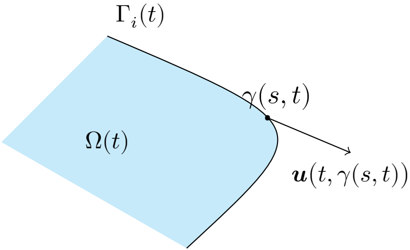
Interface advection

Lagrangian scheme

Interface is a **parametrised curve** $\gamma(s,t) \in \mathbb{R}^2$ whose evolution is given by

$$\frac{\partial \gamma}{\partial t}(s,t) = \boldsymbol{u}\Big(t, \gamma(s,t)\Big)$$

i.e. points on the interface have the same velocity as the fluid particles.



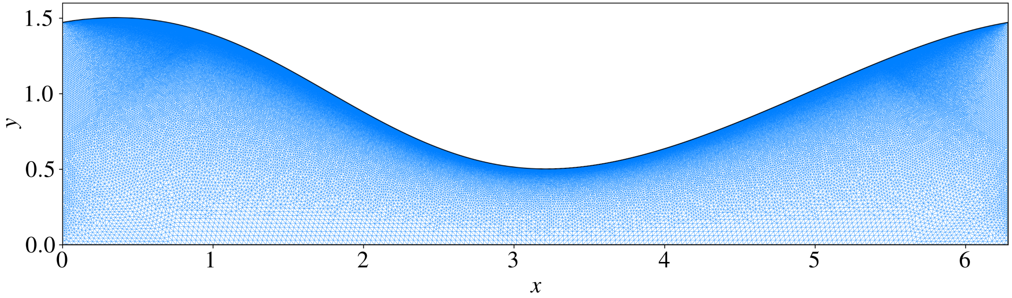
The interface contains all points parametrised by γ,

$$\Gamma_i(t) = \bigcup_s \{\gamma(s,t)\}$$

Finite Elements discretization

We use the `FreeFem` finite elements library Hecht (2012) for

- Mesh generation and handling
- Matrices computations and handling
- Interface with PETSc

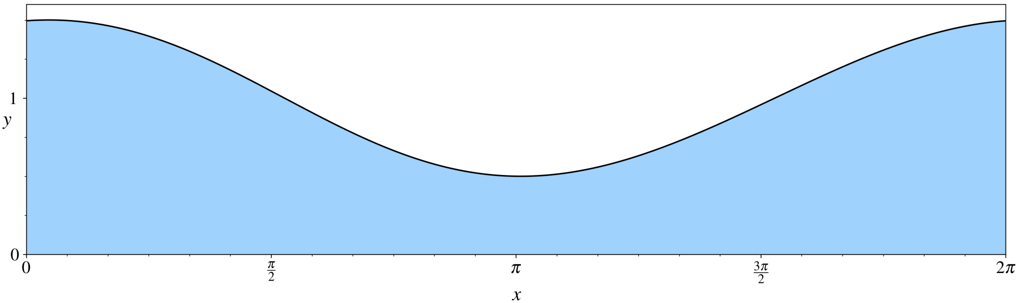


$4\,000$ points on the interface, initially $\approx 200\,000$ triangles, $\approx 10^6$ degrees of freedom.

The $\mathrm{Re} \to +\infty$ limit

Flat bottom topography

$\mathrm{Re} = 10^6$ **result**



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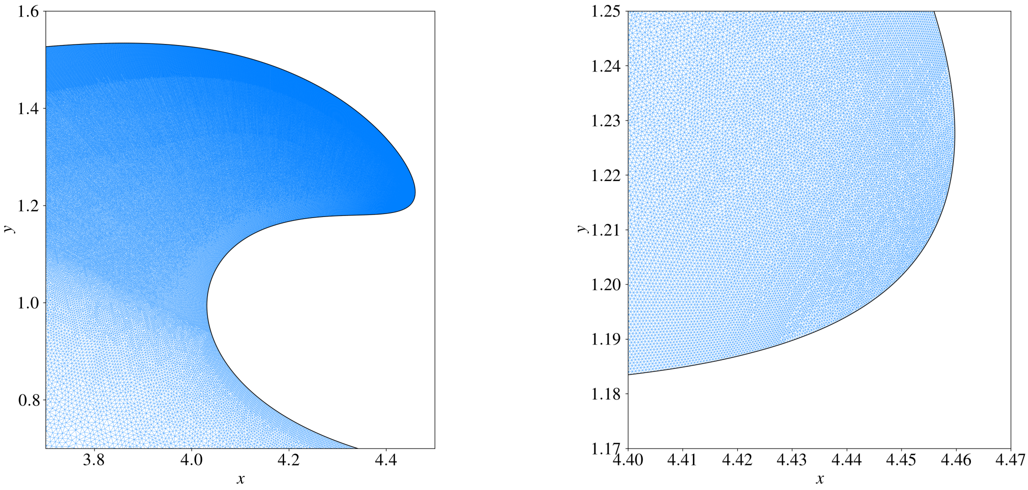
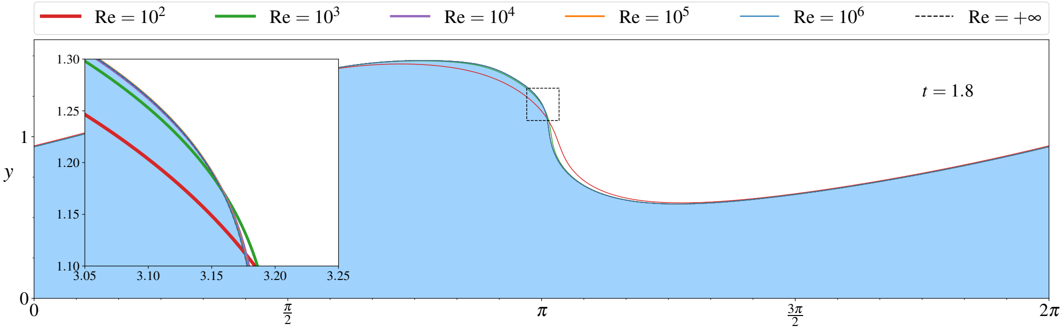
Mesh at $\mathrm{Re} = 10^6$



Interface for different values of Re



$\mathrm{Re} = +\infty$ simulations (i.e. Euler solution) computed with the numerical methods of Dormy & Lacave (2024).

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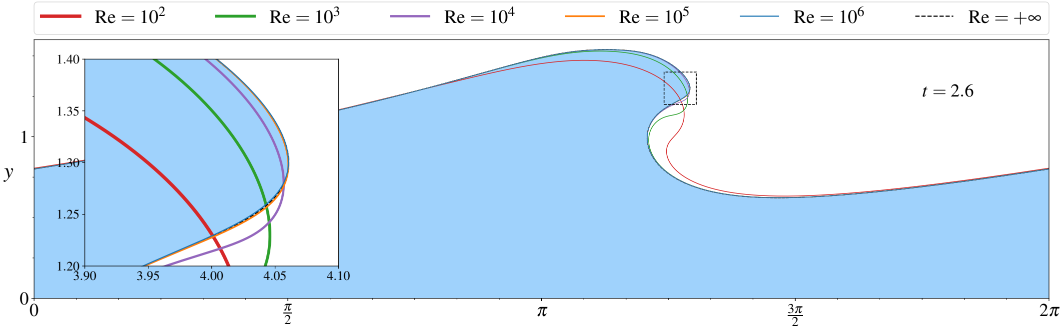

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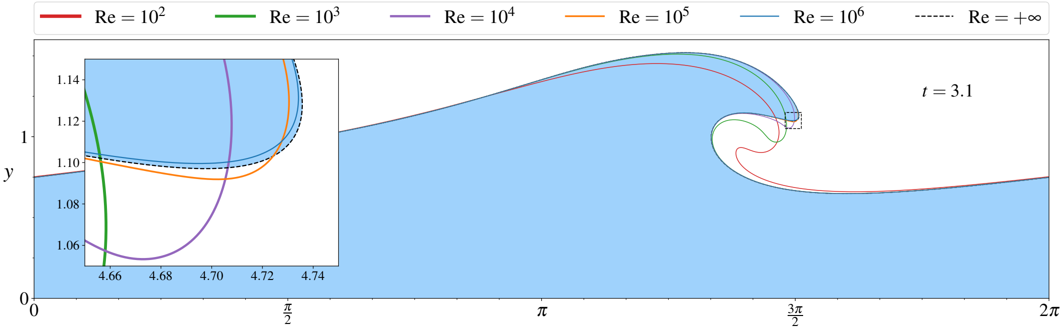

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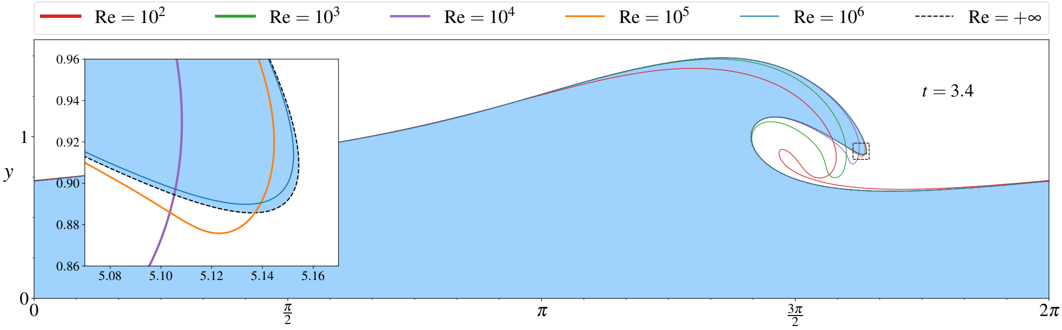

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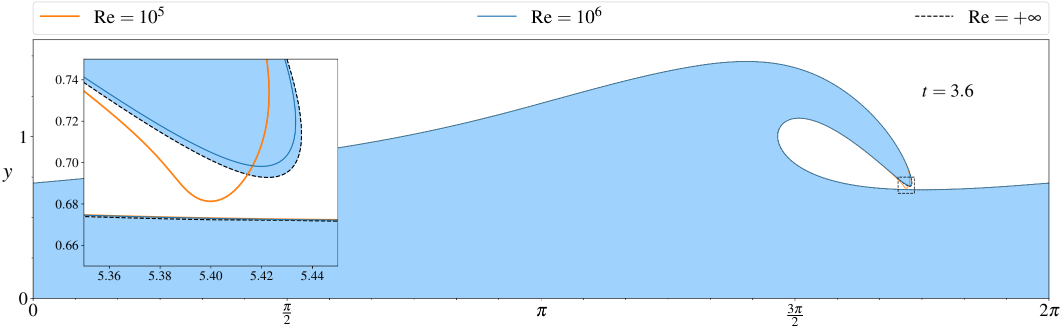

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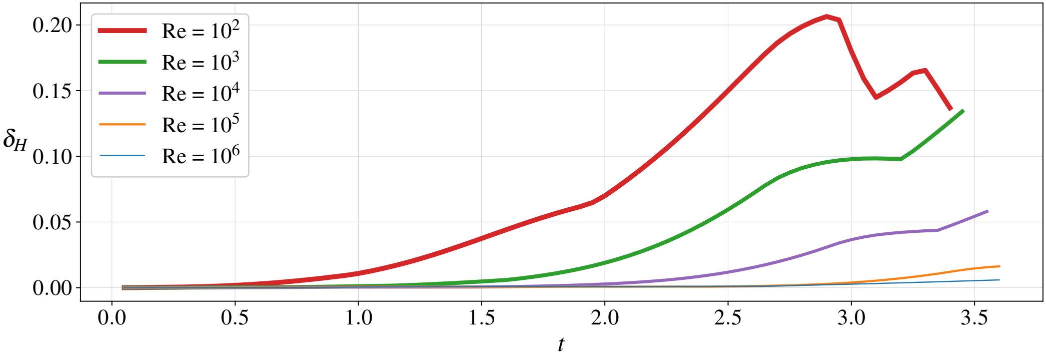
Convergence

Haussdorff distance

$$\delta_H(\gamma_1,\gamma_2) = \max\left\{\tilde{\delta}_H(\gamma_1,\gamma_2), \tilde{\delta}_H(\gamma_2,\gamma_1)\right\} \quad \textbf{where} \quad \tilde{\delta}_H(\gamma_1,\gamma_2) = \max_{s_1}\min_{s_2}\left|\gamma_1(s_1) - \gamma_2(s_2)\right|$$



Energy considerations

Link with vorticity

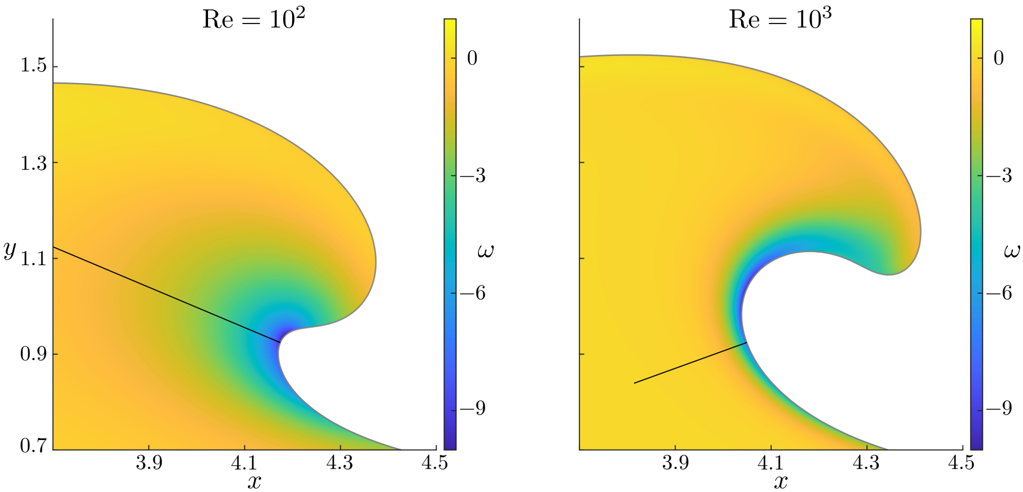
The **kinetic energy equation** can be obtained multiplying Navier-Stokes eq. by $\boldsymbol{u}$,

$$\partial_t \left(\frac{\boldsymbol{u}^2}{2} \right) = \boldsymbol{g} \cdot \boldsymbol{u} - \boldsymbol{u} \cdot \nabla p + \frac{1}{\mathrm{Re}} \left[\nabla \cdot (\boldsymbol{u}^{\perp} \omega) - \omega^2 \right]$$

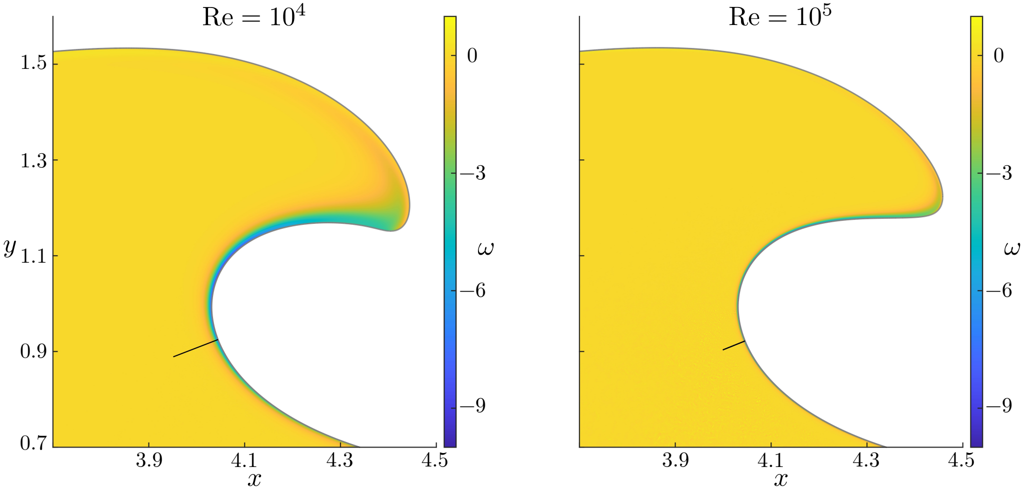
where $\boldsymbol{u}^{\perp} = [-u_y, u_x]$ and $\omega = \nabla^{\perp} \cdot \boldsymbol{u}$ is the **vorticity**.

This shows that fluids **dissipates energy** in **the support of** ω!

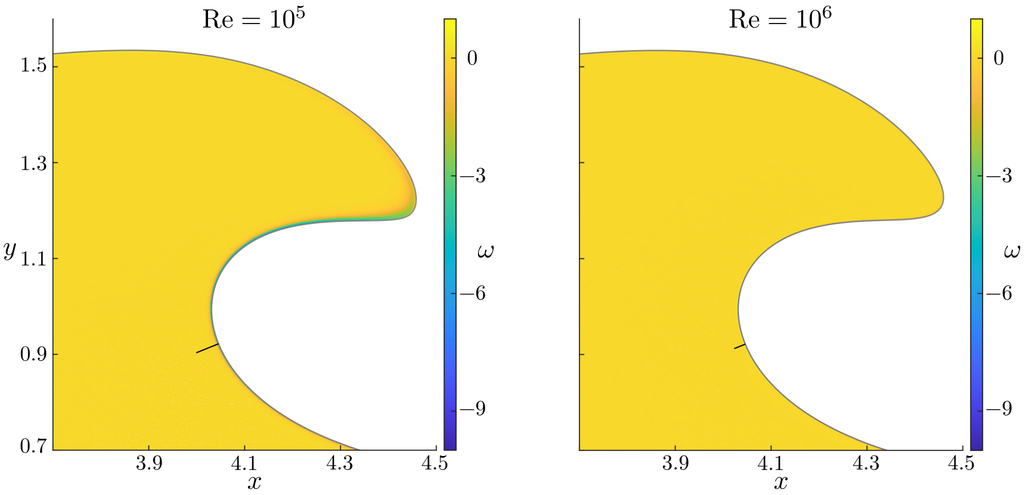
Viscous dissipation



Viscous dissipation



Viscous dissipation

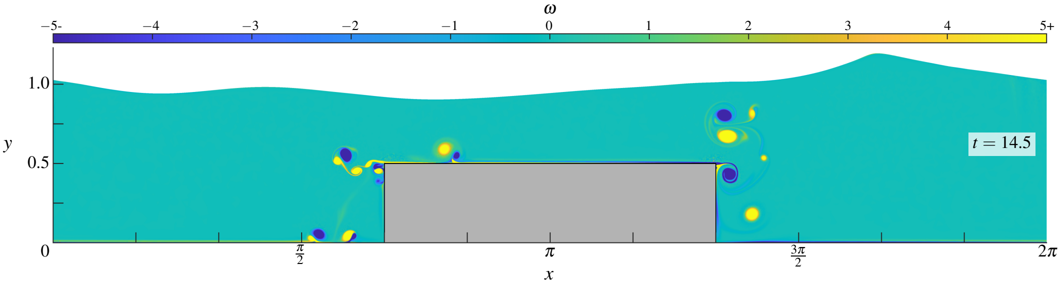


The $\mathrm{Re} \to +\infty$ limit

What if there is a rock lying on the bottom?

Vortices

Re $= 10^5$



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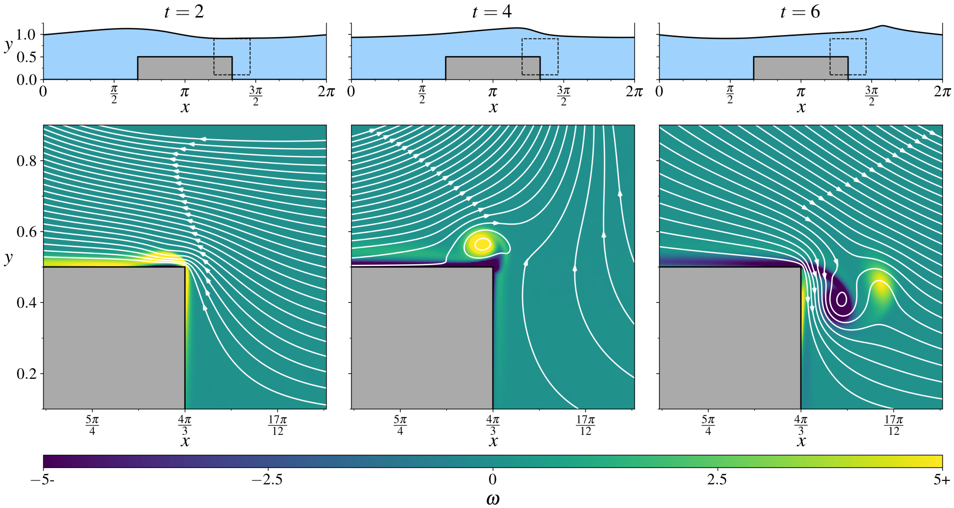
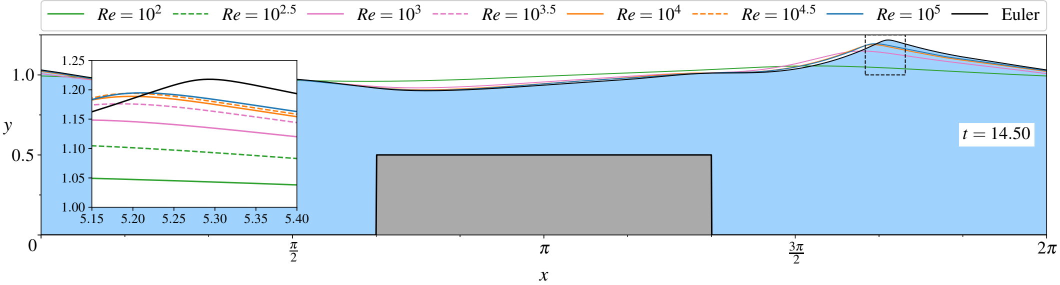
Streamlines

Re $= 10^4$



Comparing the interfaces



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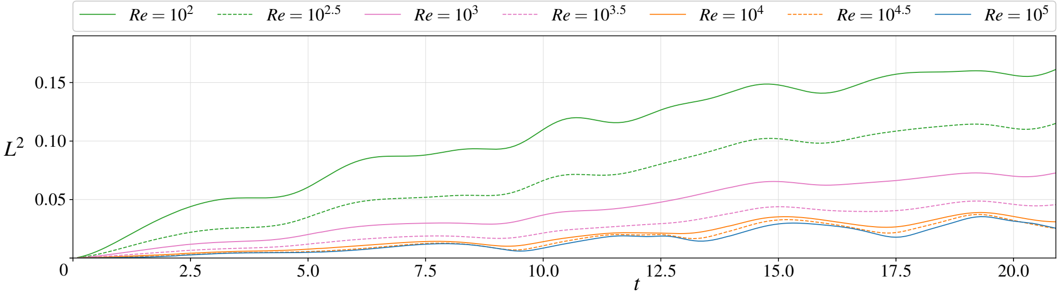
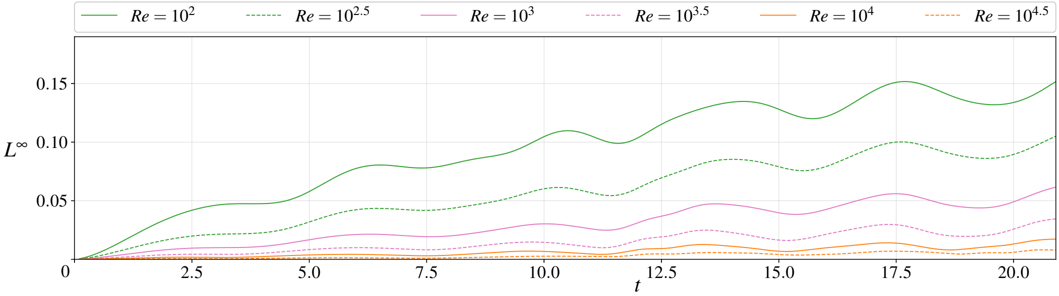
Convergence



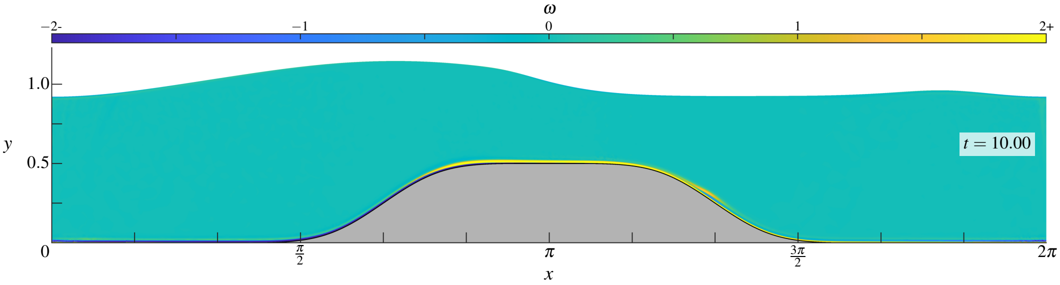
L^2 distance to **Euler's solution**

Convergence

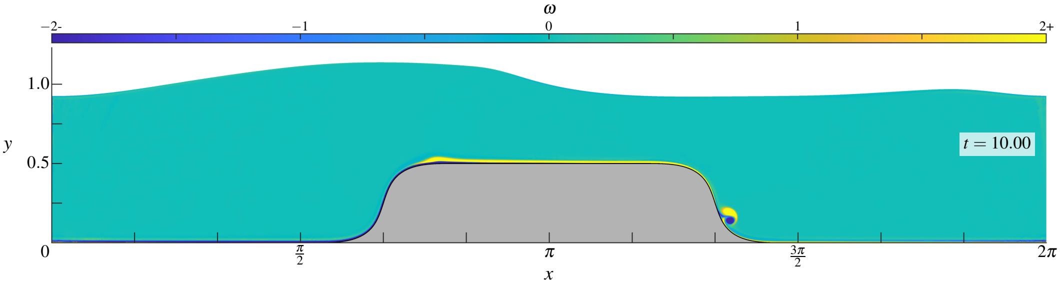


L^2 distance to the $\mathrm{Re} = 10^5$ **solution**

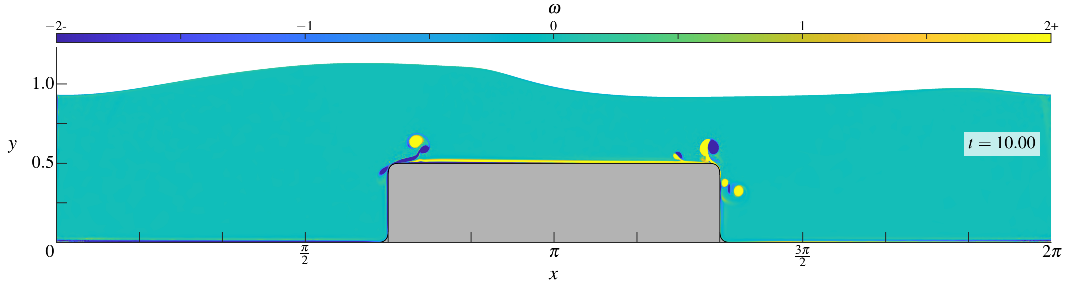
Mollifying the salient edge



Mollifying the salient edge

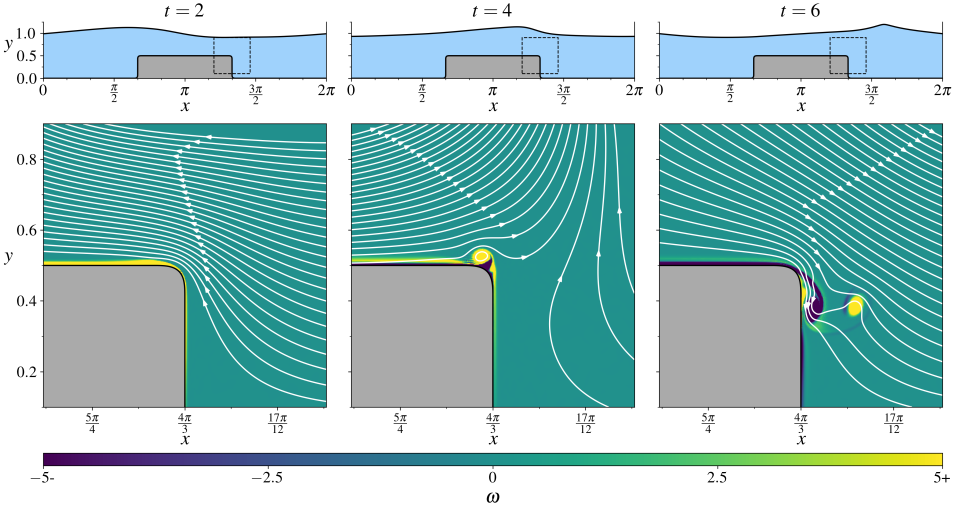


Mollifying the salient edge



Streamlines

$\mathrm{Re} = 10^5$ with mollified edge



Merci pour votre attention !

Plus d'informations : AR & E. Dormy (2024) **Numerical study of a viscous breaking water wave and the limit of vanishing viscosity.** J. Fluid Mech. **984**(R5)

Et allons manger !