Effects of viscosity on (Breaking) Water Waves

19$^{\text{èmes}}$ Journées de l'Hydrodynamique – Nantes 2024

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Assumptions in Water Waves

$$\begin{tikzcd}
 & & & \text{Saint-Venant} \\
\text{Navier-Stokes} \arrow[r, "Re \to +\infty"] & \text{Euler} \arrow[r, "\text{irrotationality } \nabla \times \boldsymbol{u} = 0"] & \text{Bernoulli} \arrow[ur] \arrow[r] \arrow[dr] & \text{K-dV} \\
 & & & \text{Serre-Green-Naghdi}
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$$

Are we sure?

Numerical method

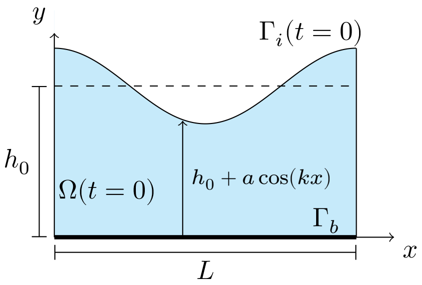
Navier-Stokes
Lagrangian advection
Finite Element Methods

Viscous Water Waves

Nondimensionalization

Nondimensional quantities are defined as follows

$$\boldsymbol{x} \to h_0 \boldsymbol{x}$$
$$\boldsymbol{u} \to \sqrt{g h_0} \cdot \boldsymbol{u}$$
$$p \to \rho g h_0 \cdot p$$



This allows to define the **Reynolds number** Re,

$$Re = \frac{\rho h_0 \sqrt{g h_0}}{\mu}$$

Viscous Water Waves

Navier-Stokes equation

Incompressible, non-dimensional, **Navier-Stokes** equation in $\Omega(t)$:

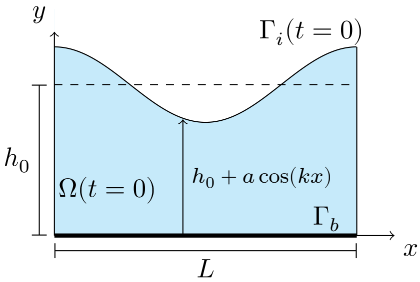
$$\left\{\begin{array}{rcl} \partial_t \boldsymbol{u} + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u} & = & -\nabla p + \frac{1}{Re}\Delta \boldsymbol{u} + \boldsymbol{g} \\ \nabla\cdot\boldsymbol{u} & = & 0 \end{array}\right.$$

Navier boundary conditions on Γ_b,

$$\boldsymbol{u}\cdot\boldsymbol{n} = 0 \quad ; \quad \boldsymbol{t}\cdot\left[\nabla\boldsymbol{u} + (\nabla\boldsymbol{u})^t\right]\cdot\boldsymbol{n} = 0$$

Stress-free boundary condition on $\Gamma_i(t)$,

$$p\boldsymbol{n} - \frac{1}{Re}\cdot\left[\nabla\boldsymbol{u} + (\nabla\boldsymbol{u})^t\right]\cdot\boldsymbol{n} = 0$$



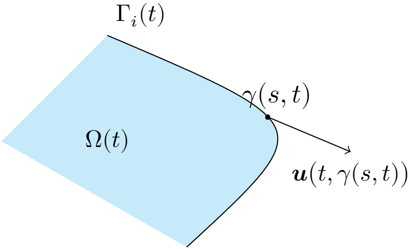
Interface advection

Lagrangian scheme

Interface is a **parametrised curve** $\gamma(s,t) \in \mathbb{R}^2$ whose evolution is given by

$$\frac{\partial \gamma}{\partial t}(s,t) = \boldsymbol{u}\Big(t, \gamma(s,t)\Big)$$

i.e. points on the interface have the same velocity as the fluid particles.



Initial conditions

Theory of linear waves

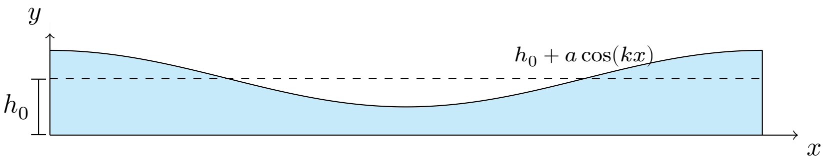
Linear wave of (small) amplitude a,

$$\gamma_0(t,x) = h_0 + a\cos(kx - \omega t) \qquad \text{with} \qquad \omega = \sqrt{gk\tanh(kh_0)}$$

$$\phi_0(t,x,y) = \frac{a\omega}{k}\frac{\cosh(ky)}{\sinh(kh_0)} \cdot \sin(kx - \omega t) + \mathcal{O}(ka)$$

The **small amplitude velocity** is then

$$\boldsymbol{u}_0(t,x,y) = \nabla\phi_0 = \frac{a\omega}{\sinh(kh_0)} \cdot \begin{bmatrix} \cosh(ky)\cos(kx-\omega t) \\ \sinh(ky)\sin(kx-\omega t) \end{bmatrix}$$



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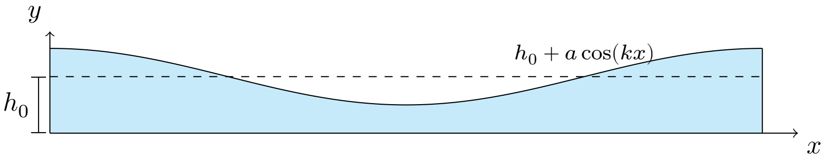
The **finite amplitude velocity** is then

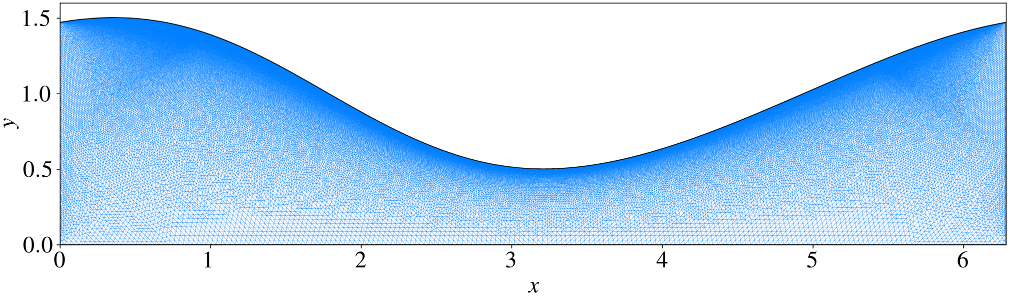
$$\boldsymbol{u}_0(t,x,y) = \nabla\phi_0 = \frac{a\omega}{\sinh(kh_0)}\cdot\begin{bmatrix}\cosh(kh_0)\cos(kx-\omega t)\\ \sinh(kh_0)\sin(kx-\omega t)\end{bmatrix}$$



Finite Elements discretization

We use the `FreeFem` finite elements library Hecht (2012) for

- Mesh generation and handling
- Matrices computations and handling
- Interface with PETSc



$4\,000$ points on the interface, initially $\approx 200\,000$ triangles, $\approx 10^6$ degrees of freedom.

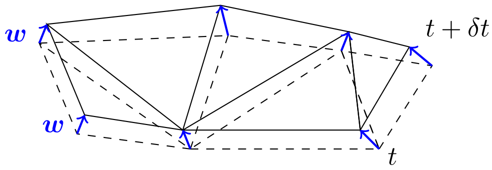
Mesh advection scheme

Let $\boldsymbol{w}$ the **velocity of the mesh**. At each time step, we numerically solve the problem

$$\begin{cases} \Delta \boldsymbol{w} = 0 & \text{in } \Omega(t) \\ \boldsymbol{w} = \boldsymbol{u} & \text{on } \Gamma_i(t) \\ \boldsymbol{w} = 0 & \text{on } \Gamma_b \end{cases}$$

And each point of the mesh is advected with velocity $\boldsymbol{w}$. Points on the interface are thus **purely Lagrangian**!

This is called the **Arbitrary Lagrangian Eulerian** method (ALE).

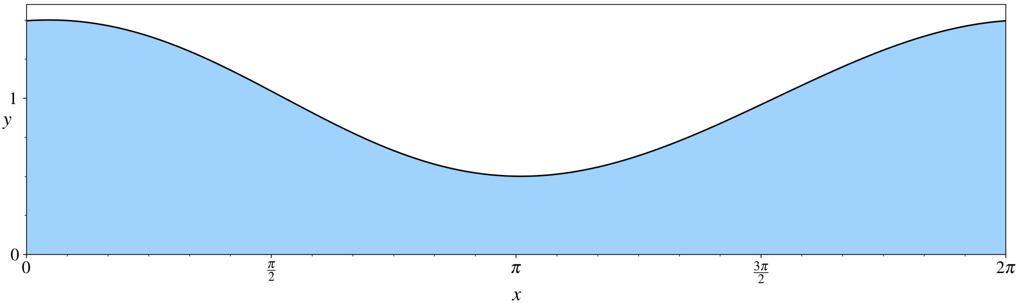


Viscosity and Breaking Waves

Flat Topography

Navier-Stokes $\leftrightarrow$ Euler

$\mathrm{Re} = 10^6$ **result**



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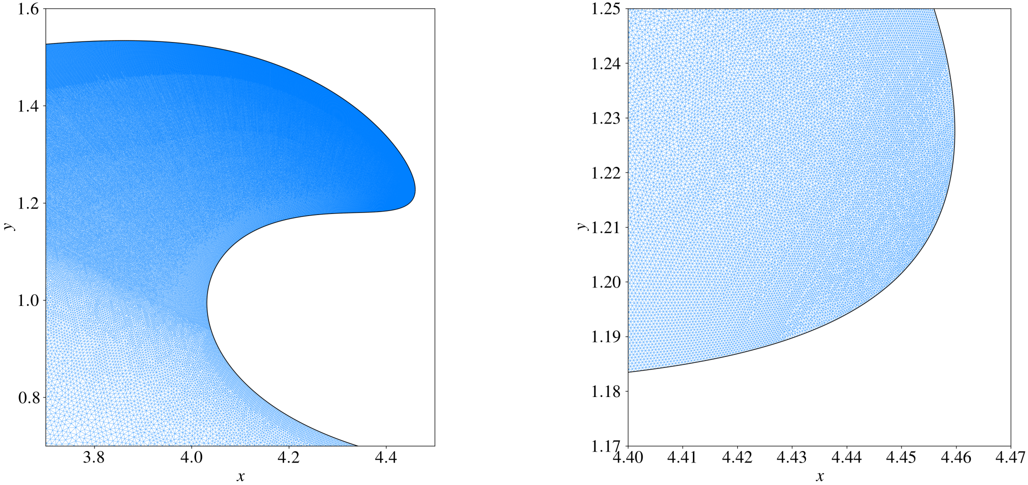
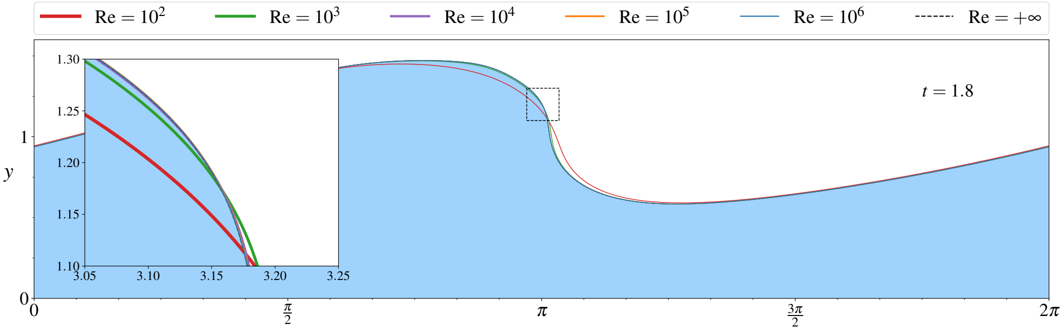
Mesh at $Re = 10^6$



Interface for different values of Re



$Re = +\infty$ simulations (i.e. Euler solution) computed with the numerical methods of Dormy & Lacave (2024).

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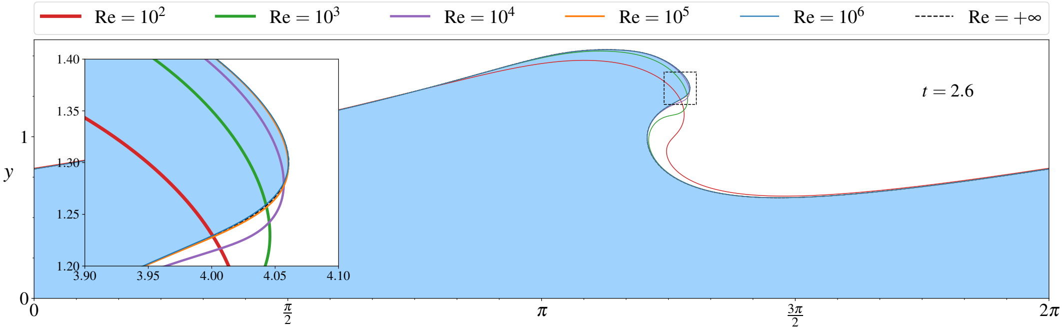

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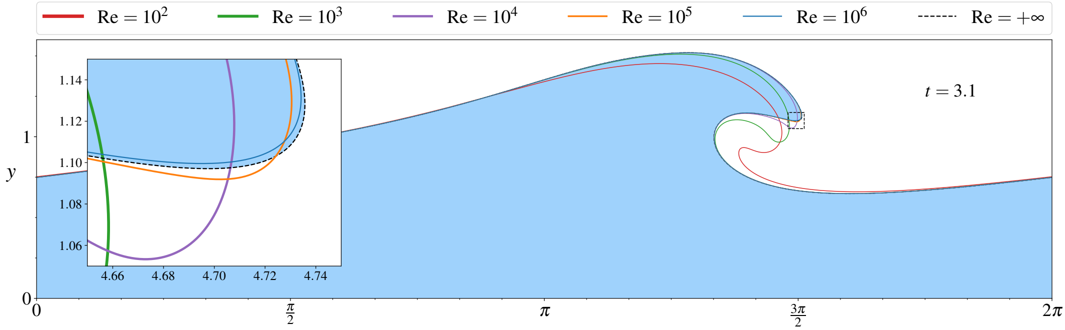

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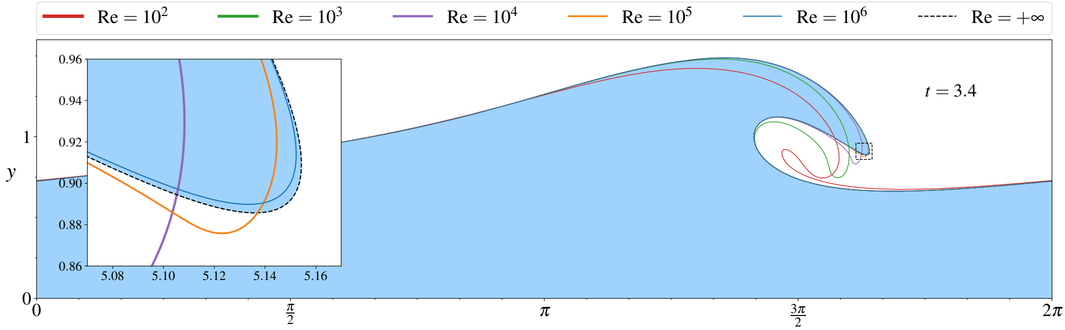

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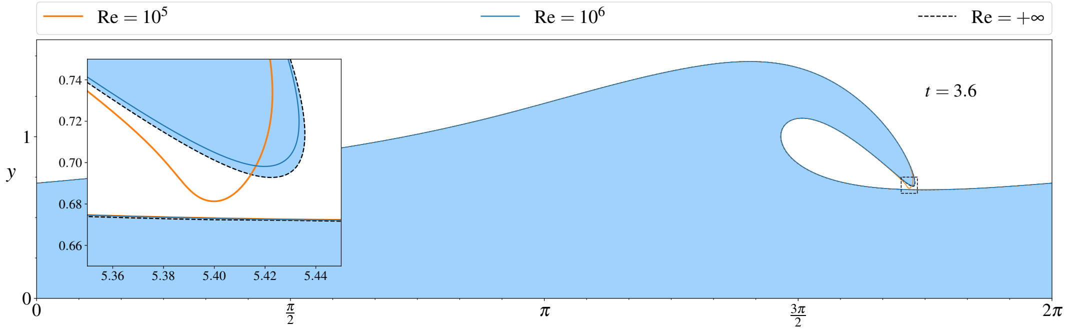

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Where does a fluid dissipates energy?

Link with vorticity

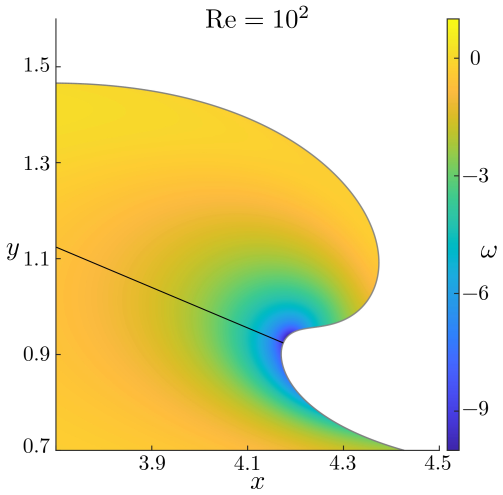
The **kinetic energy equation** can be obtained multiplying Navier-Stokes by $\boldsymbol{u}$,

$$\frac{\mathrm{D}}{\mathrm{D}t}\left(\frac{\boldsymbol{u}^2}{2}\right) = \boldsymbol{g}\cdot\boldsymbol{u} - \boldsymbol{u}\cdot\nabla p - \frac{1}{Re}\left[\nabla\cdot(\boldsymbol{u}\times\omega) + \omega^2\right]$$

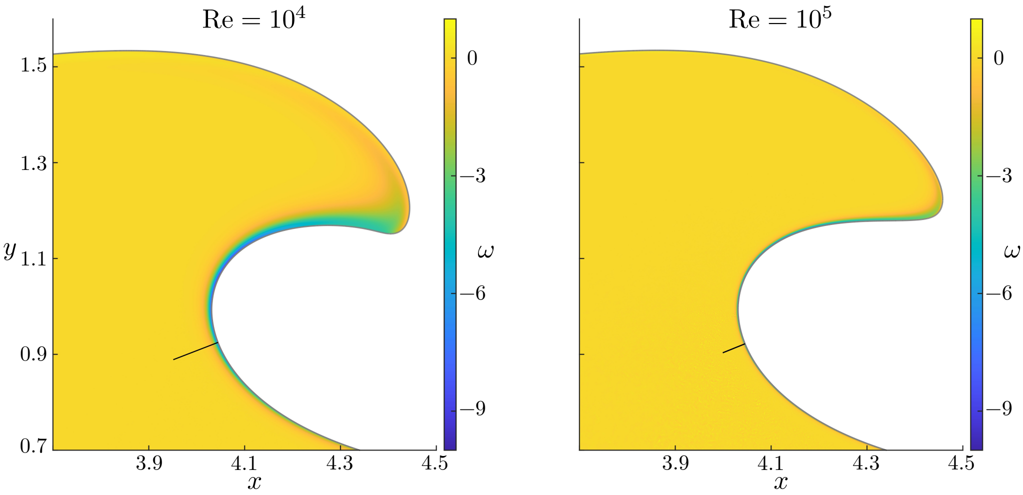
where $\omega = \nabla \times \boldsymbol{u}$ is the **vorticity**.

This shows that the fluid **dissipates energy** where $\omega \neq 0$!

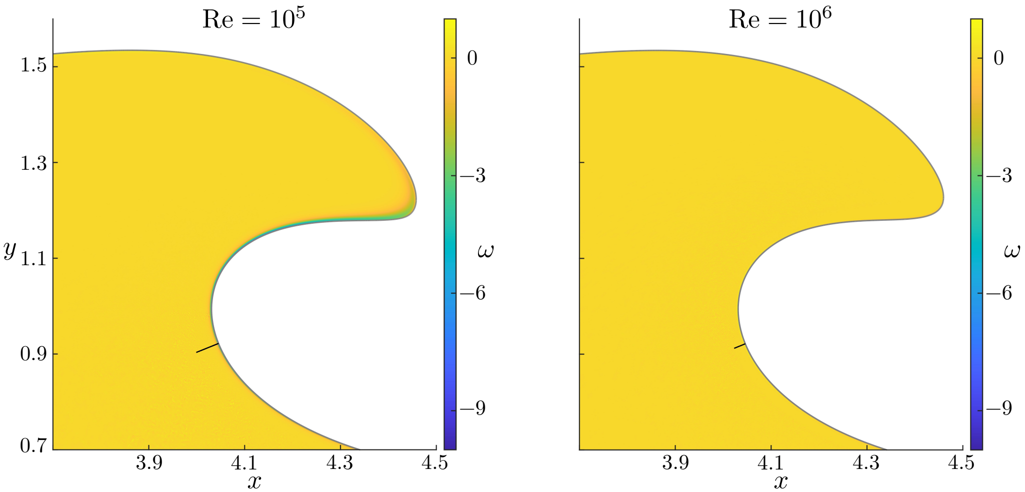
Viscous dissipation



Viscous dissipation



Viscous dissipation



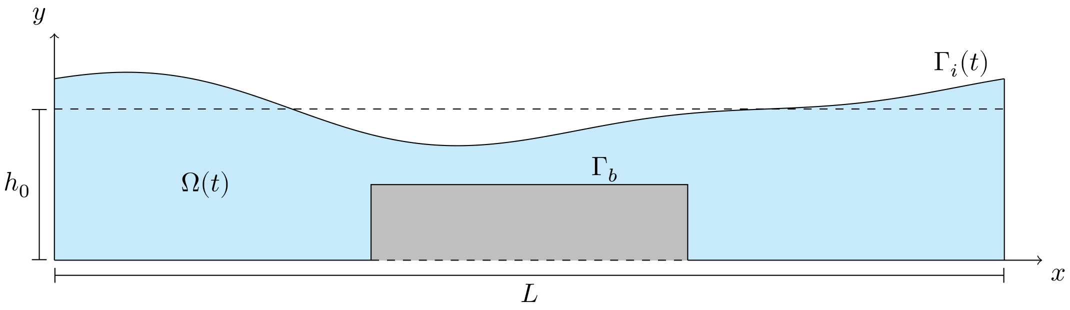
Is irrotationality well motivated?

Non-flat topography
Navier-Stokes $\not\leftrightarrow$ Euler

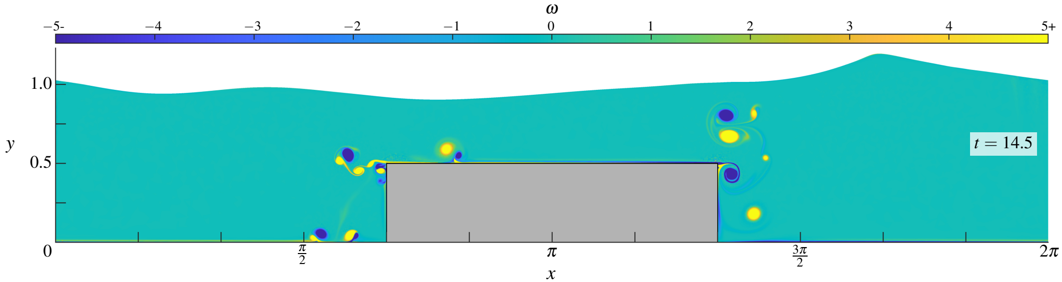
Rectangular step

No slip condition on the bottom,

$$\boldsymbol{u} = 0 \qquad \text{on } \Gamma_b$$



Vortices at $Re = 10^5$



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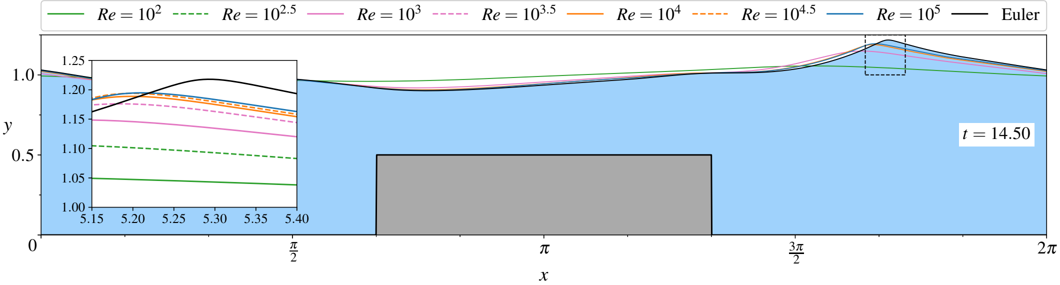
Comparing the interfaces



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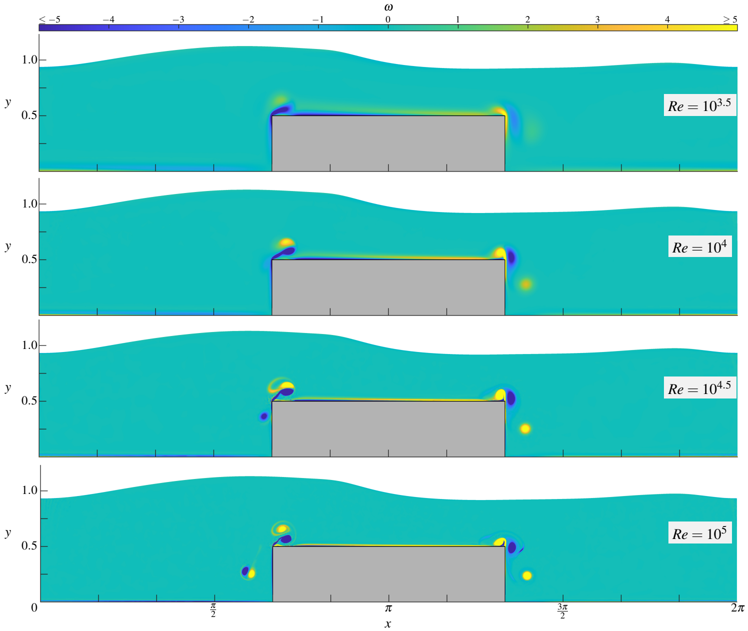
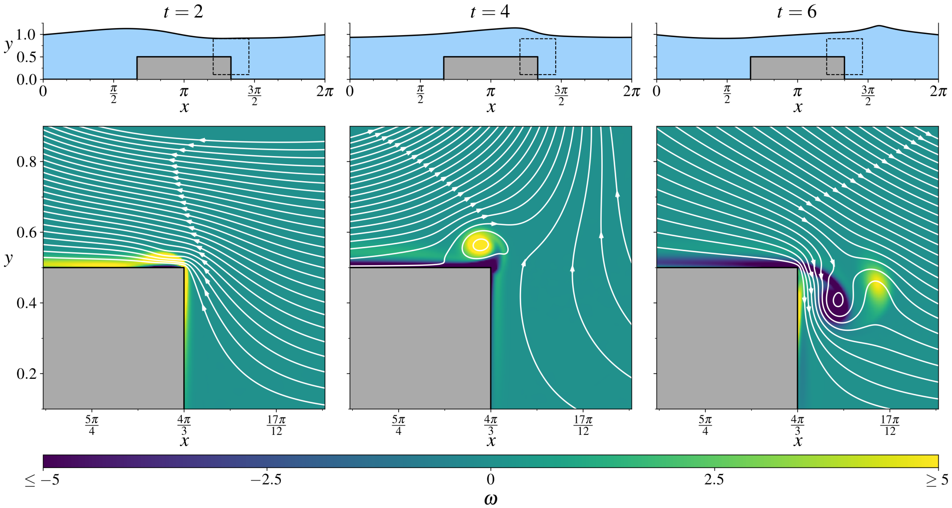
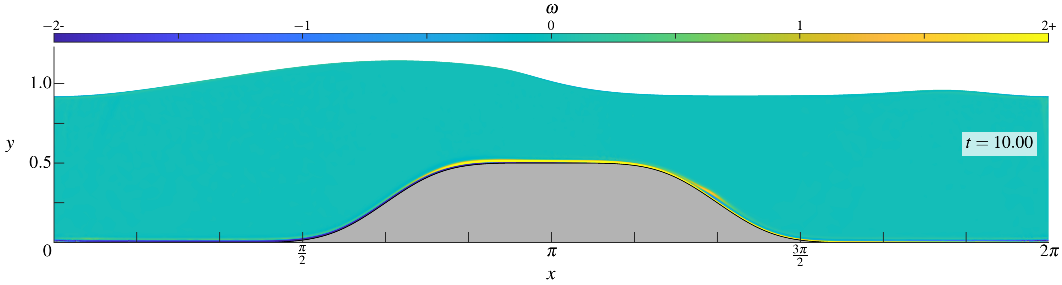

ω
≤ −5
−4
−3
−2
−1
0
1
2
3
4
≥ 5
1.0
0.5
y
$Re = 10^{3.5}$
$Re = 10^{4}$
$Re = 10^{4.5}$
$Re = 10^{5}$
0
$\frac{\pi}{2}$
π
$\frac{3\pi}{2}$
2π
x

Streamlines

Re $= 10^4$



What about a smooth edge?



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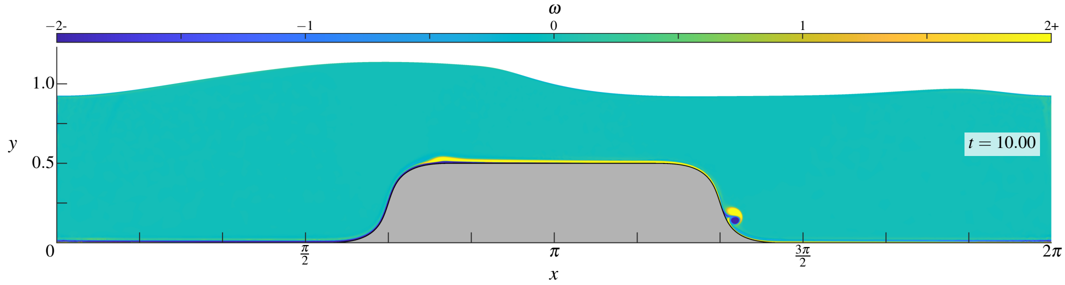

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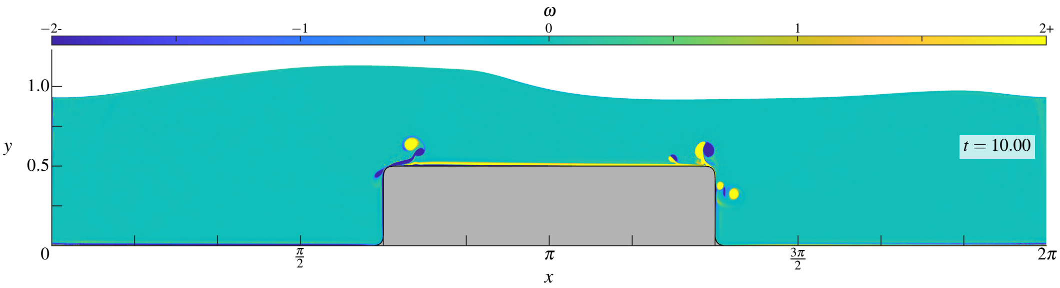

Conclusion

Can we assume that the flow is inviscid and irrotational?

		Inviscid & **Irrotational**
Shallow Water	with **flat** topography	✓
	with **curved** topography	be cautious!
Deep Water		not discussed

Merci !

Viscosity and Breaking Waves: A. R. & E. Dormy (2024) **Numerical study of a viscous breaking water wave and the limit of vanishing viscosity**, J. Fluid Mech. (Rapids) **984**, R5

Irrotationality: A. R. & E. Dormy (2024) **Irrotationality of Water Waves and Topography**, Submitted. `arXiv:2411.09291 [physics.flu-dyn]`