The 2D free-surface Navier-Stokes equations with a Lagrangian advection scheme using FreeFEM

FreeFEM days 2024

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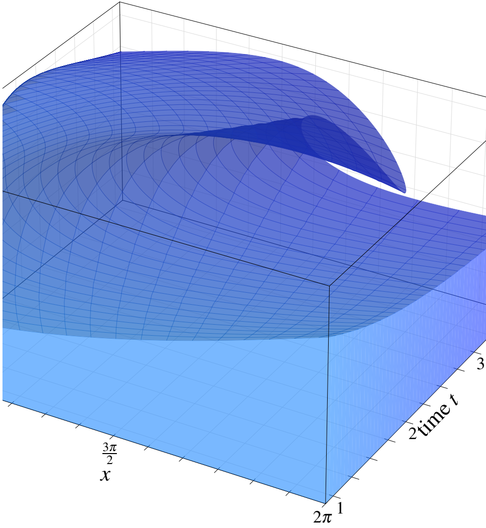
Outline

1. Mathematical modeling
2. FreeFEM to the rescue
3. The $\mathrm{Re} \to +\infty$ limit in (Breaking) Water Waves



Problem formulation

Navier-Stokes

Lagrangian advection

Initial condition

Viscous Water Waves

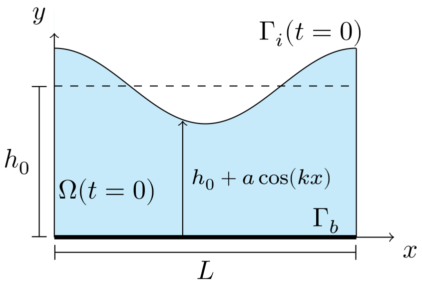
Nondimensionalization

Nondimensional quantities are defined as follows

$$\boldsymbol{x} \to h_0 \boldsymbol{x}$$
$$\boldsymbol{u} \to \sqrt{g h_0} \cdot \boldsymbol{u}$$
$$p \to \rho g h_0 \cdot p$$

This allows to define the **Reynolds number** Re,

$$\mathrm{Re} = \frac{\rho h_0 \sqrt{g h_0}}{\mu}$$



Viscous Water Waves

Navier-Stokes equation

Incompressible, non-dimensional, **Navier-Stokes** equation in $\Omega(t)$:

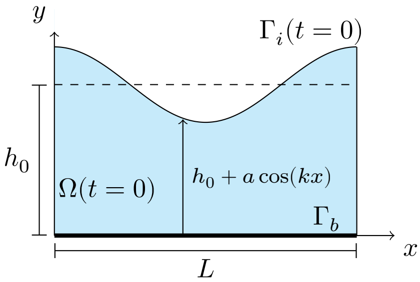
$$\begin{cases} \partial_t \boldsymbol{u} + (\boldsymbol{u} \cdot \nabla)\boldsymbol{u} & = & -\nabla p + \frac{1}{\mathrm{Re}}\Delta \boldsymbol{u} + \boldsymbol{g} \\ \nabla \cdot \boldsymbol{u} & = & 0 \end{cases}$$

Navier boundary conditions on Γ_b,

$$\boldsymbol{u} \cdot \boldsymbol{n} = 0 \quad ; \quad \boldsymbol{t} \cdot \left[\nabla \boldsymbol{u} + (\nabla \boldsymbol{u})^t\right] \cdot \boldsymbol{n} = 0$$

Stress-free boundary condition on $\Gamma_i(t)$,

$$p\boldsymbol{n} - \frac{1}{\mathrm{Re}} \cdot \left[\nabla \boldsymbol{u} + (\nabla \boldsymbol{u})^t\right] \cdot \boldsymbol{n} = 0$$



Weak formulation

Navier-Stokes problem

Velocity u and **pressure** p such that

$$\boxed{\int_{\Omega(t)} \boldsymbol{v} \cdot \partial_t \boldsymbol{u} + \boldsymbol{v} \cdot (\boldsymbol{u} \cdot \nabla)\boldsymbol{u} + \frac{2}{\mathrm{Re}} \mathbb{S}(\boldsymbol{v}) : \mathbb{S}(\boldsymbol{u}) - p \nabla \cdot \boldsymbol{v} + q \nabla \cdot \boldsymbol{u} - \boldsymbol{v} \cdot \boldsymbol{g} = 0}$$

for all $\boldsymbol{v}$ and q.

$\mathbb{S}(\boldsymbol{u})$ is the symmetric part of the gradient,

$$\mathbb{S}(\boldsymbol{u}) = \frac{1}{2}\left[\nabla \boldsymbol{u} + (\nabla \boldsymbol{u})^t\right]$$

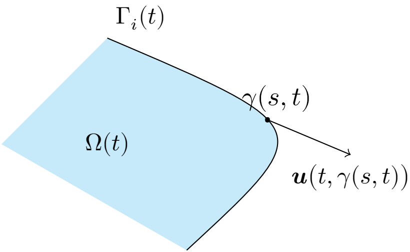
Interface advection

Lagrangian scheme

Interface is a **parametrised curve** $\gamma(s,t) \in \mathbb{R}^2$ whose evolution is given by

$$\frac{\partial \gamma}{\partial t}(s,t) = \boldsymbol{u}\Big(t, \gamma(s,t)\Big)$$

i.e. points on the interface have the same velocity as the fluid particles.



The interface contains all points parametrised by γ,

$$\Gamma_i(t) = \bigcup_s \{\gamma(s,t)\}$$

Initial conditions

$$\boldsymbol{u}_0(t,x,y) = \frac{a\omega}{\sinh(kh_0)} \cdot \begin{bmatrix} \cosh(kh_0)\cos(kx-\omega t) \\ \sinh(kh_0)\sin(kx-\omega t) \end{bmatrix}$$

so the velocity $\boldsymbol{n} \cdot \boldsymbol{u}_0$ in the **normal** direction can be computed from

$$\boldsymbol{n}_0(x) = \frac{1}{\sqrt{1+(\partial_x \gamma_0)}} \begin{bmatrix} -\partial_x \gamma_0 \\ 1 \end{bmatrix} \qquad \text{where} \quad \gamma_0(x) = h_0 + a\cos(kx)$$

The **initial velocity** is then constructed by solving the following Laplace problem

$$\begin{cases} \Delta\phi_0 = 0 & \text{in the domain } \Omega(t) \\ \partial_n \phi_0 = 0 & \text{at the bottom } \Gamma_b \\ \partial_n \phi_0 = \boldsymbol{n}_0 \cdot \boldsymbol{u}_0(y = h_0) & \text{at the surface } \Gamma_i(t=0) \end{cases}$$

and the **initial velocity** is $\boldsymbol{u}(t=0) = \nabla\phi_0$.

Using FreeFEM

Finite elements

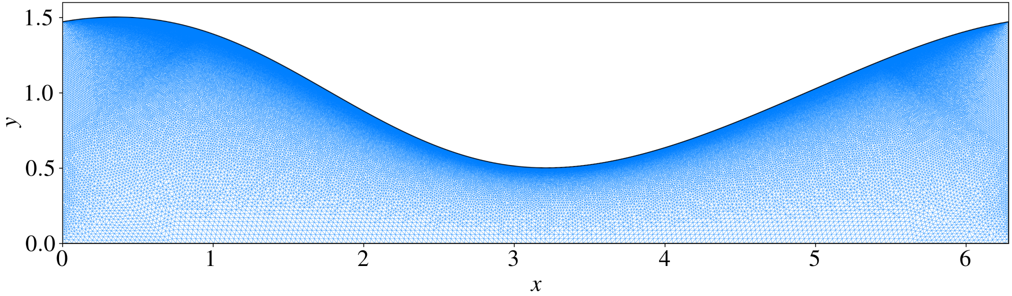
Mixed Lagrangian-Eulerian

Geometric multigrid solver

Finite Elements discretization

We use `FreeFEM` for

- Mesh generation and handling
- Matrices computations and handling
- Interface with PETSc



$4\,000$ points on the interface, initially $\approx 200\,000$ triangles, $\approx 10^6$ degrees of freedom.

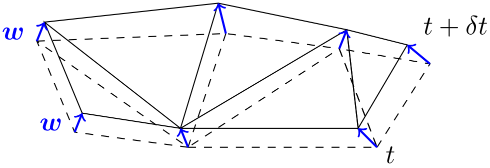
Mesh advection scheme

Let $\boldsymbol{w}$ the **velocity of the mesh**. At each time step, we numerically solve the problem

$$\begin{cases} \Delta \boldsymbol{w} & = & 0 & \text{in } \Omega_t \\ \boldsymbol{w} & = & \boldsymbol{u} & \text{on } \Gamma_{s,t} \\ \boldsymbol{w} & = & 0 & \text{on } \Gamma_b \end{cases}$$

And we use the `movemesh` function to advect the mesh with velocity $\boldsymbol{w}$. Points on the interface are thus **purely Lagrangian**!

This is called the **Arbitrary Lagrangian Eulerian** method (ALE).



Time stepping scheme

Crank-Nicolson second order in time scheme. CFL condition to compute the time step at each iteration.

At each time step, we solve the following problem

$$\int_{\Omega(t)} \boldsymbol{v} \cdot \frac{\boldsymbol{u}^{n+1} - \boldsymbol{u}^n}{\delta t} + \boldsymbol{v} \cdot \left((\boldsymbol{u}^n - \boldsymbol{w}^n) \cdot \nabla \right) \left(\frac{\boldsymbol{u}^{n+1} + \boldsymbol{u}^n}{2} \right) + \frac{2}{\mathrm{Re}} \mathbb{S}(\boldsymbol{v}) : \mathbb{S} \left(\frac{\boldsymbol{u}^{n+1} + \boldsymbol{u}^n}{2} \right)$$
$$+ \left(\frac{p^{n+1} + p^n}{2} \right) \nabla \cdot \boldsymbol{v} + q \nabla \cdot \left(\frac{\boldsymbol{u}^{n+1} + \boldsymbol{u}^n}{2} \right) - \boldsymbol{v} \cdot \boldsymbol{g}$$
$$= 0$$

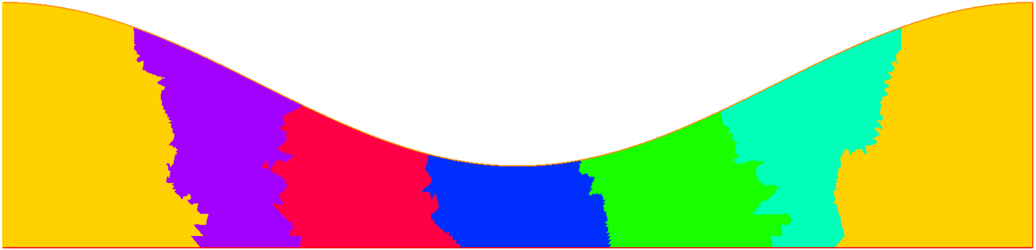
for all $(\boldsymbol{v}, q)$ and then compute $\boldsymbol{w}^{n+1}$ before advecting the mesh.

We use $\mathbb{P}^2$ elements for the velocity and $\mathbb{P}^1$ elements for the pressure.

Domain decomposition

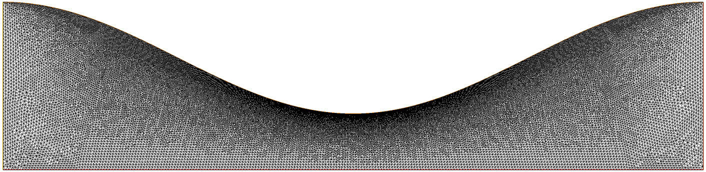
MPI domain decomposition with graph partitioner. **PETSc** matrices and solvers.

Example with **6 domains**:

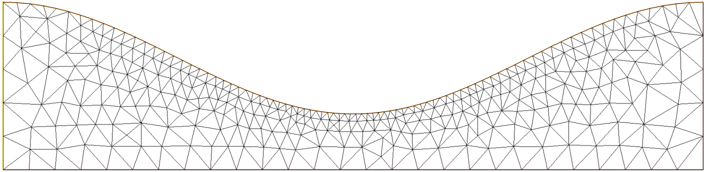


Non-nested Geometric multigrid

We use a **geometric multigrid** solver for fast convergence using a large number of MPI processes.



Level 0 (fine)



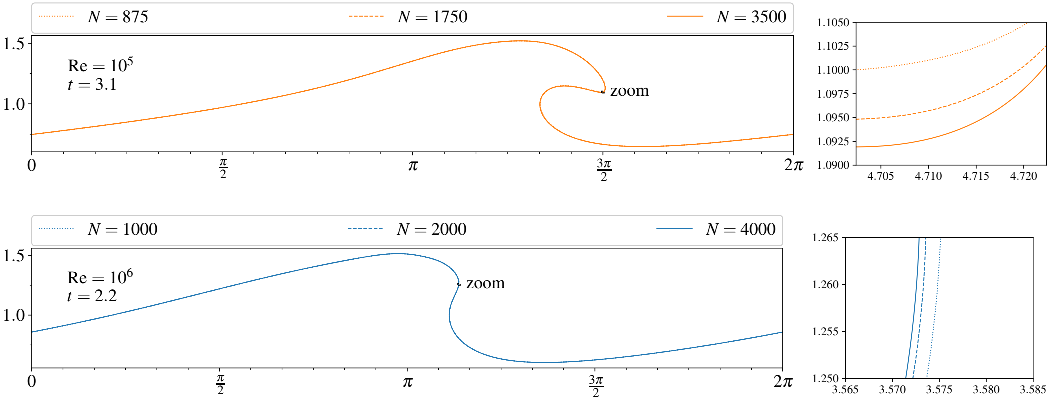
Level 2 (coarse)

Timings and #dofs

Between 1 and 3.5 million unknowns, convergence in $\sim 5 \pm 2$ GMRES iterations to machine precision $\rightarrow$ ~ 12 seconds on 48 MPI cores.

		# dofs at the **start**			# dofs at the **end**		
Re	N	# triangles	$(\boldsymbol{u}, p)$	$\boldsymbol{w}$	# triangles	$(\boldsymbol{u}, p)$	$\boldsymbol{w}$
10^2	3000	195,314	886,913	198,514	780,572	3,520,574	783,722
10^3	3500	221,748	1,007,116	225,448	615,382	2,675,923	596,294
10^4	3500	221,368	1,005,406	225,068	510,266	2,305,447	513,966
10^5	3500	222,970	1,012,615	226,670	450,648	2,037,166	454,348
10^6	4000	272,948	1,238,766	277,148	498,250	2,252,625	502,450

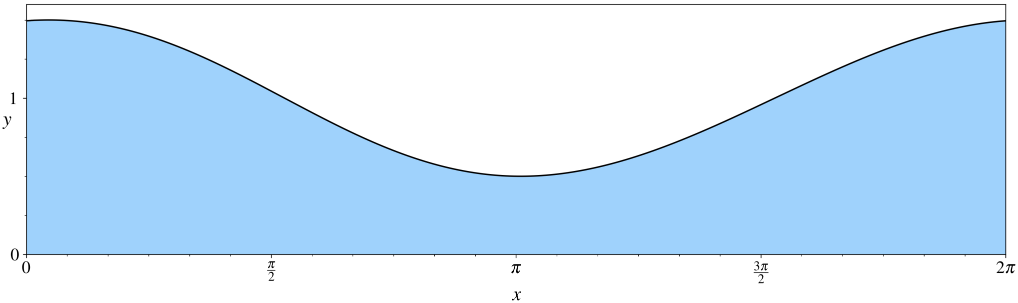
Convergence



The $\mathrm{Re} \to +\infty$ limit

Comparing our results with the inviscid solution

$\mathrm{Re} = 10^6$ result



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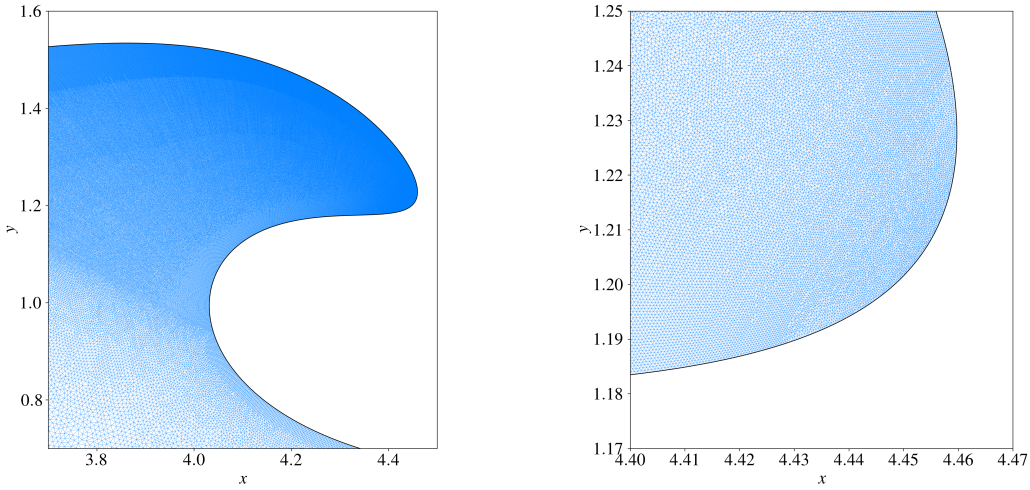
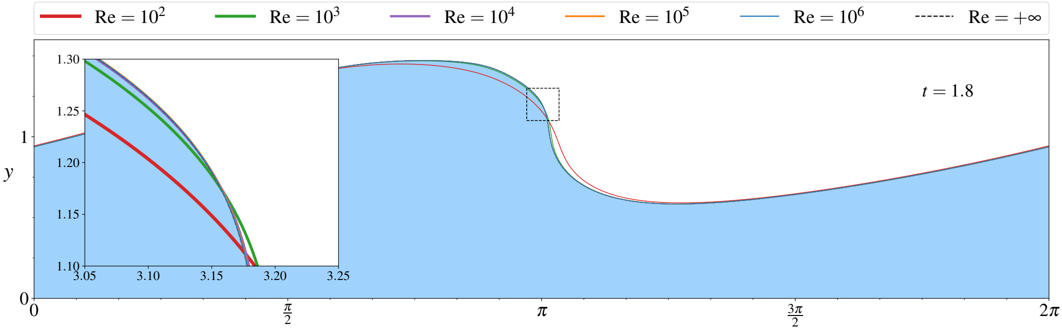
Mesh at $\mathrm{Re} = 10^6$

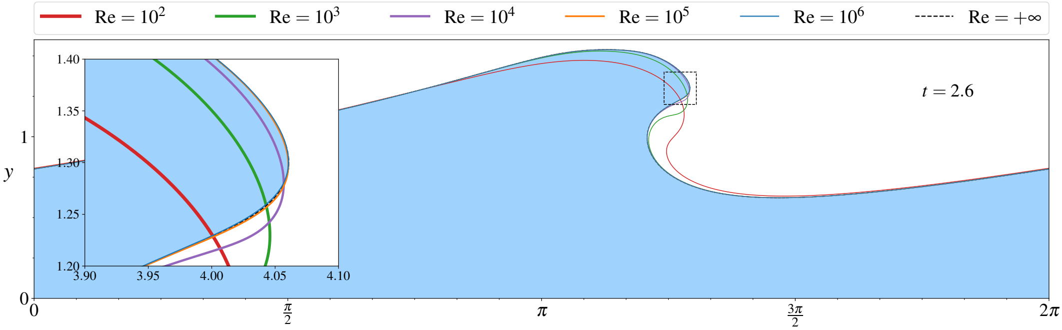


Interface for different values of Re



$Re = +\infty$ simulations (i.e. Euler solution) computed with the numerical methods of Dormy & Lacave (2024).

Interface for different values of Re



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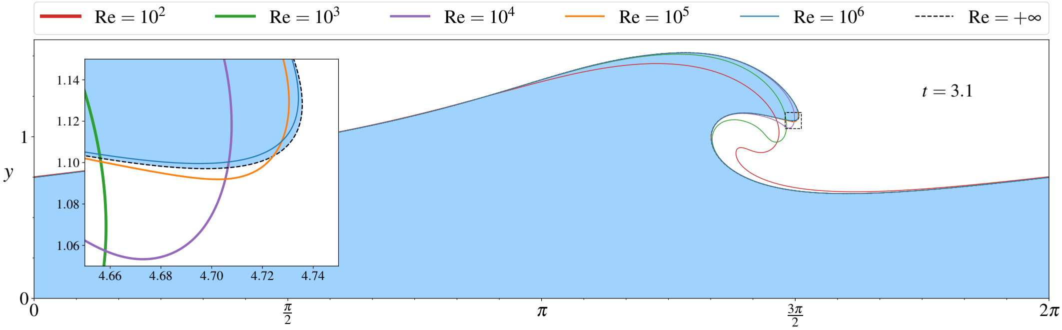

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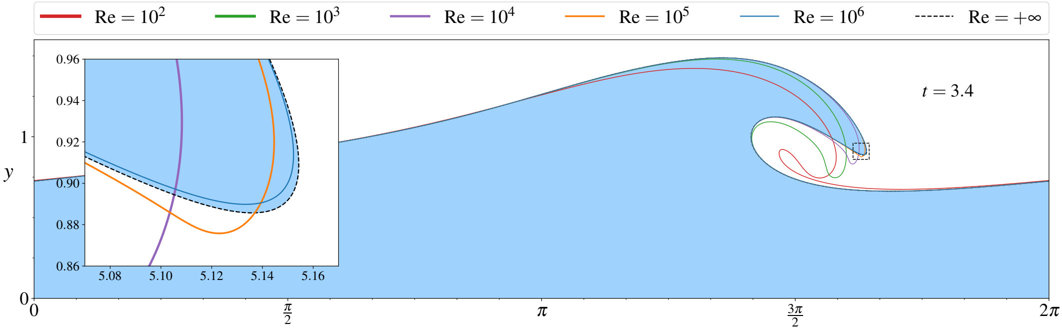

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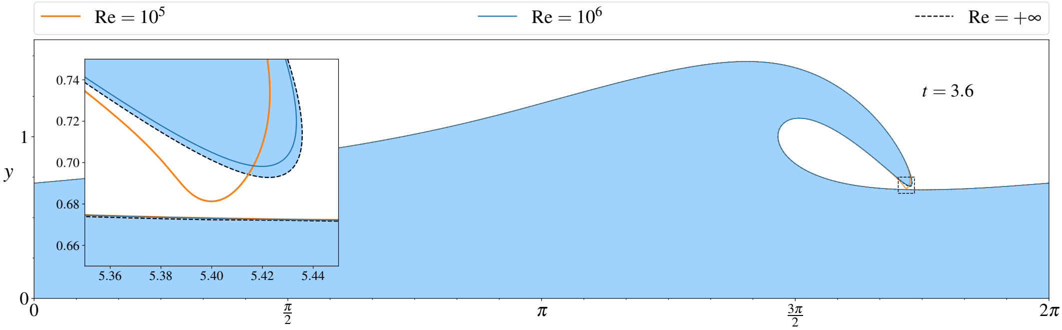


$Re = +\infty$ simulations (i.e. Euler solution) computed with the numerical methods of Dormy & Lacave (2024).

Energy considerations

Link with vorticity

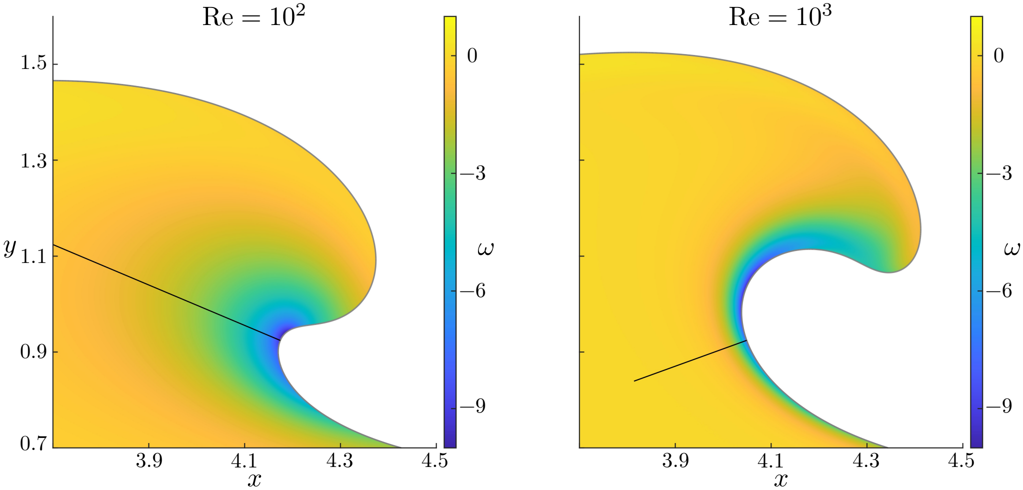
The **kinetic energy equation** can be obtained multiplying Navier-Stokes eq. by $\boldsymbol{u}$,

$$\partial_t\left(\frac{\boldsymbol{u}^2}{2}\right) = \boldsymbol{g}\cdot\boldsymbol{u} - \boldsymbol{u}\cdot\nabla p + \frac{1}{\mathrm{Re}}\left[\nabla\cdot(\boldsymbol{u}^\perp\omega) - \omega^2\right]$$

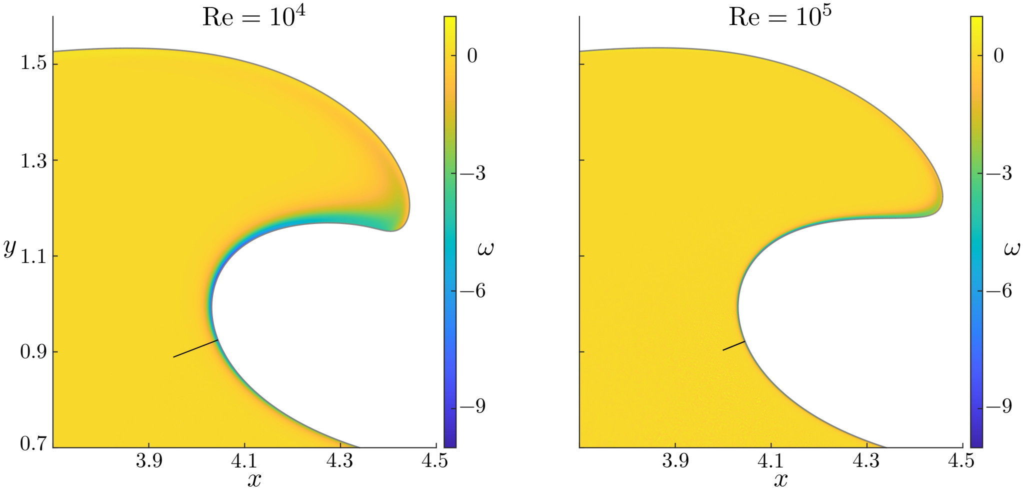
where $\boldsymbol{u}^\perp = [-u_y, u_x]$ and $\omega = \nabla^\perp\cdot\boldsymbol{u}$ is the **vorticity**.

This shows that fluids **dissipates energy** in **the support of** ω!

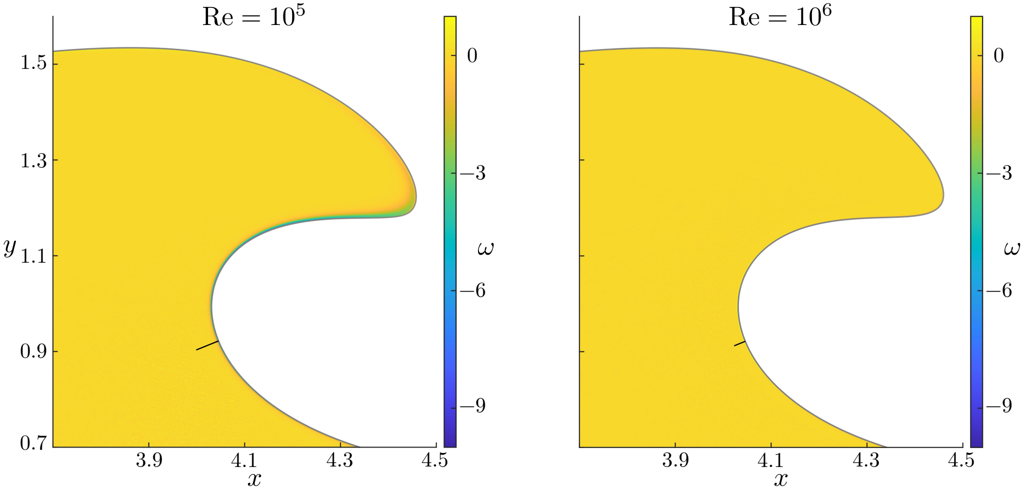
Viscous dissipation



Viscous dissipation



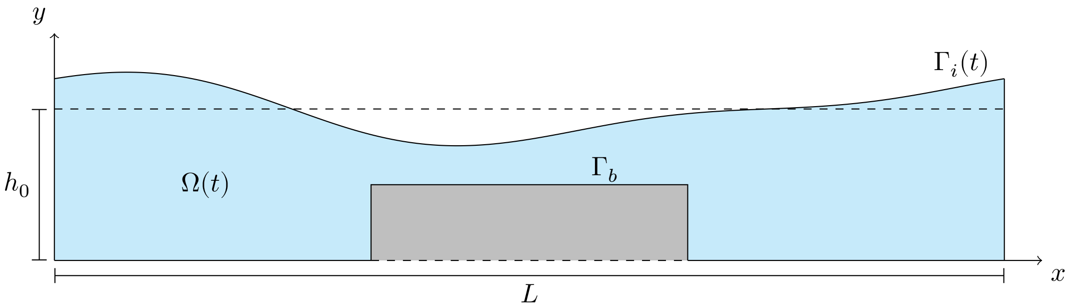
Viscous dissipation



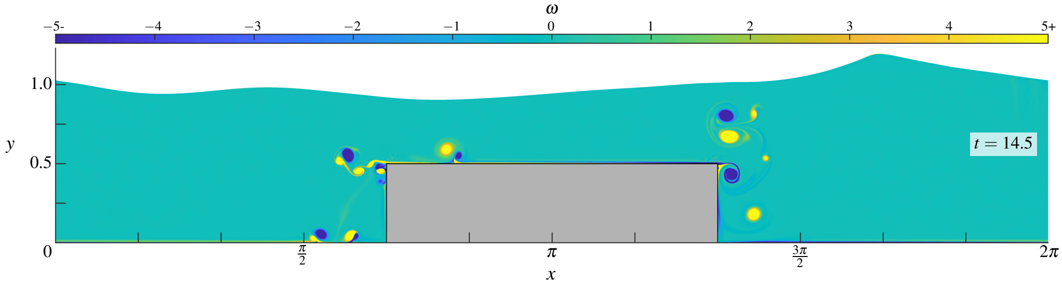
Rectangular step

No slip / **Dirichlet** condition on the bottom,

$$\boldsymbol{u} = 0 \qquad \text{on } \Gamma_b$$



Vortices at $Re = 10^5$



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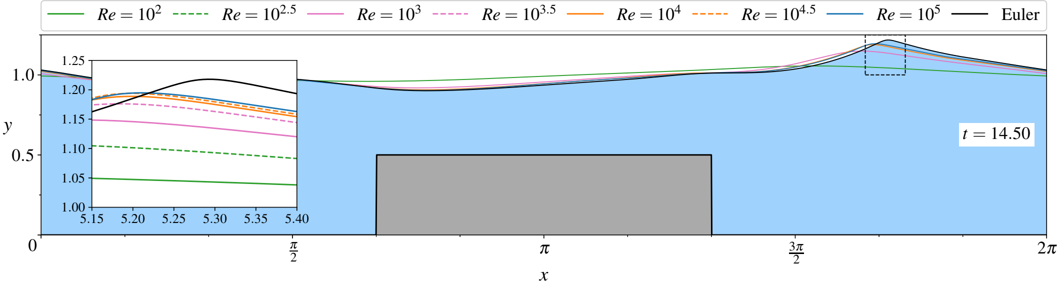
Comparing the interfaces



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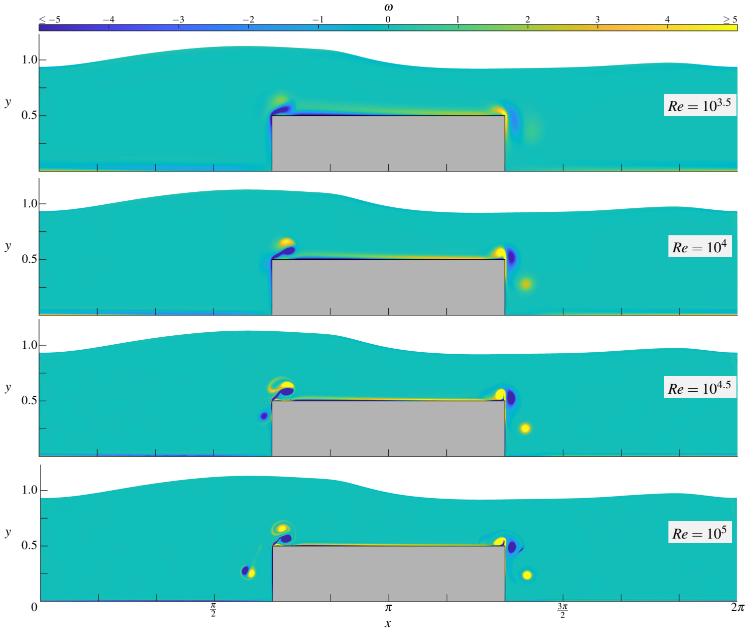
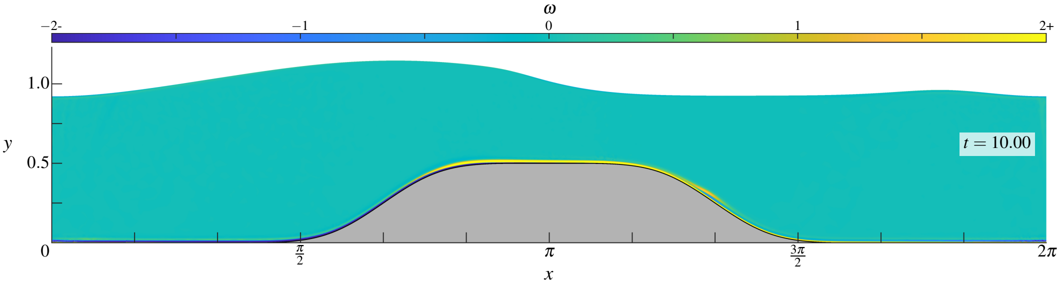

ω
≤ −5
−4
−3
−2
−1
0
1
2
3
4
≥ 5
1.0
y
0.5
Re = 10^3.5
Re = 10^4
Re = 10^4.5
Re = 10^5
0
π/2
π
3π/2
2π
x

What about a smooth edge?



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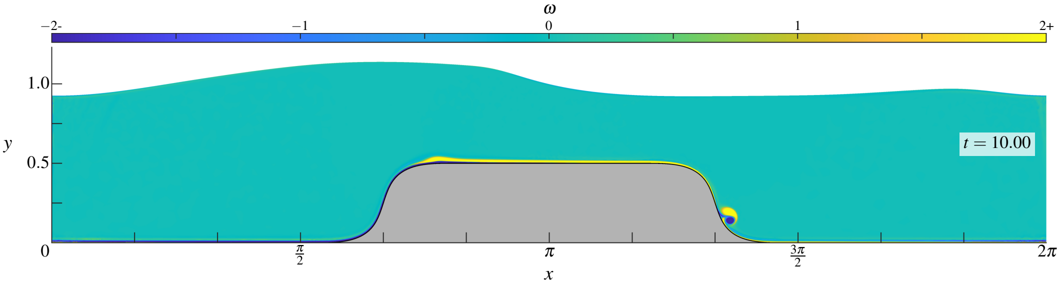

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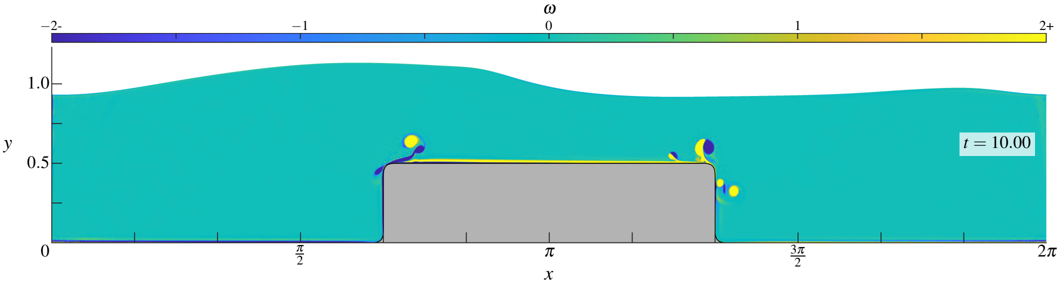

Thank you!

Viscosity and Breaking Waves: A. R. & E. Dormy (2024) **Numerical study of a viscous breaking water wave and the limit of vanishing viscosity**, J. Fluid Mech. (Rapids) **984**, R5

Irrotationality: A. R. & E. Dormy (2024) **Irrotationality of Water Waves and Topography**, Submitted. `arXiv:2411.09291 [physics.flu-dyn]`