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# Steenrod operations and algebraic classes

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Based on a relative Wu theorem in étale cohomology, we study the compatibility of Steenrod operations on Chow groups and on étale cohomology. Using the resulting obstructions to algebraicity, we construct new examples of nonalgebraic cohomology classes over various fields ($\mathbb{C}$, $\mathbb{R}$, $\bar{\mathbb{F}}_p$, $\mathbb{F}_q$).

We also use Steenrod operations to study the mod 2 cohomology classes of a compact $\mathcal{C}^\infty$ manifold $M$ that are algebraizable, i.e., algebraic on some real algebraic model of $M$. We give new examples of algebraizable and nonalgebraizable classes, answering questions of Benedetti, Dedò and Kucharz.

## 1. Introduction

The theme of this article is the interaction between Steenrod operations and algebraic cohomology classes of smooth projective varieties.

**1.1.** ***Action of Steenrod operations on algebraic classes.*** Our starting point is a theorem of Kawai [1977] according to which mod $\ell$ Steenrod operations preserve algebraic cohomology classes with $\mathbb{Z}/\ell$ coefficients on smooth projective complex varieties (see also [Fulton 1998, Example 19.1.8]). This implies in particular that mod $\ell$ algebraic classes on smooth projective complex varieties are killed by odd degree Steenrod operations, and hence provides topological obstructions to the algebraicity of such classes. One recovers in this way the first counterexamples to the integral Hodge conjecture, constructed by Atiyah and Hirzebruch [1962].

These arguments were adapted by Colliot-Thélène and Szamuely [2010, théorème 2.1] with help from Totaro to construct counterexamples to the integral Tate conjecture over $\bar{\mathbb{F}}_p$ (whose statement is recalled in Section 1.4). We refer to [Quick 2011; Pirutka and Yagita 2015; Kameko 2015; Antieau 2016] for further examples. In particular, the vanishing of odd degree Steenrod operations on mod $\ell$ algebraic classes over algebraically closed fields of characteristic different from $\ell$, proven in a particular case in [Colliot-Thélène and Szamuely 2010, théorème 2.1(1)], follows in general from Quick's work [2011, Theorem 1.1]. Our first goal is to extend this result to arbitrary, not necessarily algebraically closed, fields.

To do so, we compare Brosnan's [2003] Steenrod operations on Chow groups (see also Section 3.1) and the Steenrod operations in (twisted) étale cohomology (see [Guillou and Weibel 2019] and Sections 2.1–2.2) by means of the mod $\ell$ étale cycle class map cl.

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In the next theorem, we write Sq for the total Steenrod square (when $\ell = 2$) and $P$ for the total Steenrod $\ell$-th power (when $\ell$ is odd). When $\ell = 2$, the cohomology class $\varpi$ is the class of $-1$ in $k^*/(k^*)^2 \simeq H^1_{\text{ét}}(\operatorname{Spec}(k), \mathbb{Z}/2)$. See also Remarks 2.1(ii).

**Theorem 3.4.** *Let $X$ be a smooth quasiprojective variety over a field $k$. Let $\ell$ be a prime number invertible in $k$. For $x \in \mathrm{CH}^c(X)/\ell$, one has*

$$\begin{aligned} \mathrm{Sq}(\mathrm{cl}(x)) &= \sum_{i\geq 0} (1+\varpi)^{c-i} \cdot \mathrm{cl}(\mathrm{Sq}^{2i}(x)) && \text{if } \ell = 2, \\ P(\mathrm{cl}(x)) &= \mathrm{cl}(P(x)) && \text{if } \ell \text{ is odd.} \end{aligned}$$

It follows from Theorem 3.4 that algebraic cohomology classes are preserved by all Steenrod operations as soon as $k$ contains a primitive $\ell^2$-th root of unity, and by Steenrod's $\ell$-th powers when $\ell$ is odd and $k$ is arbitrary (see Proposition 3.7). It is however not true that Steenrod squares always preserve algebraic classes (e.g., if $\ell = 2$ and $k = \mathbb{R}$, see Remark 3.8), and one can view Theorem 3.4 as providing the correct substitute for this incorrect statement.

Theorem 3.4 is proved by comparing two relative Wu theorems (describing the behaviour of Steenrod operations under proper pushforwards): one for Chow groups and another one in étale cohomology. The former is due to Brosnan ([2003], see Proposition 3.1). The latter is due to Scavia and Suzuki [2023, Proposition 3.1] when $\ell = 2$ and $k$ is algebraically closed, and could be deduced in general from the Riemann–Roch theorems of Panin [2004] and Déglise [2018]. We have found it more convenient to give a self-contained proof in Section 2.5 (see Theorem 2.5).

**1.2. *Examples of nonalgebraic cohomology classes.*** Our second goal, following the above-mentioned counterexamples to the integral Hodge and Tate conjectures, is to exploit the obstructions to algebraicity stemming from Theorem 3.4 to give interesting new examples of nonalgebraic cohomology classes.

**1.2.1.** We first apply classical topological obstructions to algebraicity going back to Atiyah and Hirzebruch [1962] (the vanishing of odd degree Steenrod operations) to a variety constructed in [Benoist and Ottem 2021, §5.1]. This yields a 5-dimensional counterexample to the integral Tate conjecture over $\overline{\mathbb{F}}_p$ (see Remark 4.3 for its Hodge counterpart).

**Theorem 4.2.** *Let $p$ be an odd prime number. There exists a smooth projective variety of dimension 5 over $\overline{\mathbb{F}}_p$ on which the 2-adic integral Tate conjecture for codimension 2 cycles fails.*

The counterexamples to the integral Tate conjecture over $\overline{\mathbb{F}}_p$ that have appeared so far in the literature all have dimension at least 7. By [Tate 1966, (5.10); Schoen 1998, Theorem 0.5], it would follow from the (rational) Tate conjecture over $\overline{\mathbb{F}}_p$ that no 3-dimensional counterexample exists. We do not know any 4-dimensional counterexample. In fact, we do not know any counterexample to the integral Tate conjecture for cycles of dimension 2 over $\overline{\mathbb{F}}_p$.

**1.2.2.** We then notice that the stability of algebraic classes by Steenrod operations (over, say, algebraically closed fields) yields obstructions to algebraicity that go beyond the vanishing of odd degree Steenrod operations. For instance, over the complex numbers, it implies that even degree Steenrod operations send mod $\ell$ algebraic classes to reductions mod $\ell$ of integral Hodge classes (and, over $\overline{\mathbb{F}}_p$, to reductions mod $\ell$ of integral Tate classes). This really may obstruct the integral Hodge and Tate conjectures, because there are no reasons why reductions mod $\ell$ of integral Hodge classes (or of integral Tate classes) should be preserved by even degree Steenrod operations. We illustrate these new Hodge-theoretic (or Galois-theoretic) obstructions to the integral Hodge and Tate conjectures by using them to prove the following theorem (see Remark 4.10 for its Hodge counterpart).

**Theorem 4.9.** *Choose $\ell \in \{2, 3, 5\}$ and let $p$ be a prime number distinct from $\ell$. There exists a smooth projective variety $X$ of dimension $2\ell+2$ over $\overline{\mathbb{F}}_p$ with $H^*_{\text{ét}}(X, \mathbb{Z}_\ell)$ torsion-free on which the $\ell$-adic integral Tate conjecture for codimension 2 cycles fails.*

Theorem 4.9 gives the first counterexample to the integral Tate conjecture over $\overline{\mathbb{F}}_p$ with a torsion-free cohomology ring. Indeed, the topological obstructions used in the counterexamples that have previously appeared in the literature force their $\ell$-adic cohomology rings to contain nonzero torsion classes (although the nonalgebraic integral Tate classes of [Pirutka and Yagita 2015] live in torsion-free $\ell$-adic cohomology groups). We also refer to [de Jong and Perry 2022, Theorem 1.10] for more recent examples, valid for all prime numbers $\ell$, and based on entirely different arguments.

Our proof builds upon Pirutka and Yagita [2015]. In this article, the authors use algebraic approximations of classifying spaces of exceptional reductive groups to construct interesting $(2\ell+3)$-dimensional counterexamples to the $\ell$-adic integral Tate conjecture over $\overline{\mathbb{F}}_p$ by the Atiyah–Hirzebruch method, for $\ell \in \{2, 3, 5\}$. We notice that a well-chosen hypersurface in (a blowup of) their variety has a torsion-free cohomology ring, but the integral Tate conjecture may still fail by the obstruction described above.

Totaro [1997] explained how to better understand and generalize the Atiyah–Hirzebruch topological obstructions to the integral Hodge conjecture using complex cobordism. In Section 3.4, we show that our Hodge-theoretic enhancement of these obstructions also has such an interpretation, in terms of the Hodge classes in complex cobordism introduced by Hopkins and Quick [2015].

**1.2.3.** Finally, we give examples of nonalgebraic cohomology classes on varieties defined over nonclosed fields, that become algebraic (and even vanish) over the algebraic closure of the base field. This is particularly interesting because their nonalgebraicity is then a purely arithmetic phenomenon.

Over finite fields, the first examples of such classes were constructed very recently by Scavia and Suzuki [2024, Theorem 1.4]. As they relied on obstructions to algebraicity induced by nonvanishing odd degree Steenrod operations, they had to restrict to finite fields containing a primitive $\ell^2$-th root of unity. Based on the subtler obstructions to algebraicity stemming from Theorem 3.4 (see Proposition 3.12), we are able to remove this hypothesis.

**Theorem 4.12.** *Let $p \neq \ell$ be prime numbers, and let $\mathbb{F}$ be a finite subfield of $\overline{\mathbb{F}}_p$. There exist a smooth projective geometrically connected variety $X$ of dimension $2\ell+3$ over $\mathbb{F}$ and a nonalgebraic class*

$$x \in \mathrm{Ker}\big(H^4_{\text{ét}}(X, \mathbb{Z}_\ell(2)) \to H^4_{\text{ét}}(X_{\overline{\mathbb{F}}_p}, \mathbb{Z}_\ell(2))\big).$$

Scavia and Suzuki's examples are algebraic approximations of classifying spaces of semisimple algebraic groups. To be able to encompass all finite fields, we instead resort to algebraic approximations of classifying spaces of (not necessarily constant) finite étale group schemes.

Over the field $\mathbb{R}$ of real numbers, nonalgebraic classes that are geometrically algebraic are harder to construct on real varieties with no real points (as there are then no obstructions to algebraicity induced by the topology of the set of real points). The example of lowest dimension appearing in the literature is the 7-dimensional anisotropic quadric ([Benoist and Wittenberg 2020a, Example 2.5], see also [Yagita 2010, §5]). Another application of Proposition 3.12 allows us to produce a 4-dimensional example. In the next statement, we set $G := \mathrm{Gal}(\mathbb{C}/\mathbb{R})$ and we let $H^*_G(X(\mathbb{C}), -)$ denote the $G$-equivariant cohomology of the set of complex points of a smooth variety over $\mathbb{R}$.

**Theorem 4.15.** *There exists a smooth projective variety $X$ of dimension 4 over $\mathbb{R}$ such that $X(\mathbb{R}) = \varnothing$, and a nonalgebraic class*

$$x \in \mathrm{Ker}\big(H^4_G(X(\mathbb{C}), \mathbb{Z}(2)) \to H^4(X(\mathbb{C}), \mathbb{Z}(2))\big).$$

Theorem 4.15 gives a new counterexample to the real integral Hodge conjecture introduced in [Benoist and Wittenberg 2020a, Definition 2.2], which is not explained by a failure of the usual (complex) integral Hodge conjecture (see Corollary 4.16). We refer to Remarks 4.17 for a comparison with previously known counterexamples to the real integral Hodge conjecture.

**1.3.** ***Algebraizability of cohomology classes of $\mathcal{C}^\infty$ manifolds.*** In the last section of this article, we consider questions specific to real algebraic geometry, that are thematically related but logically independent of the above.

Let $X$ be a smooth variety over $\mathbb{R}$. Borel and Haefliger ([1961], see also [Benoist and Wittenberg 2020a, §1.6.2]) have constructed a cycle class map

$$\mathrm{cl}_{\mathbb{R}} : \mathrm{CH}^c(X) \to H^c(X(\mathbb{R}), \mathbb{Z}/2),$$

associating with a codimension $c$ integral subvariety $Z \subset X$ the fundamental class of its real locus. Computing or controlling the image $H^*_{\mathrm{alg}}(X(\mathbb{R}), \mathbb{Z}/2)$ of $\mathrm{cl}_{\mathbb{R}}$ is a central topic in real algebraic geometry (see, e.g., [Bochnak et al. 1998, §11.3] or [Bochnak and Kucharz 1998]).

Let $M$ be a compact $\mathcal{C}^\infty$ manifold of dimension $d$. The Nash–Tognoli theorem [Tognoli 1973, corollario p. 176] asserts that there exists a smooth projective variety $X$ of dimension $d$ over $\mathbb{R}$ and a diffeomorphism $\chi : M \xrightarrow{\sim} X(\mathbb{R})$. We call such a pair $(X, \chi)$ an *algebraic model* of $M$. A cohomology class $x \in H^*(M, \mathbb{Z}/2)$ is said to be *algebraizable* if it belongs to $\chi^* H^*_{\mathrm{alg}}(X(\mathbb{R}), \mathbb{Z}/2)$ for some algebraic model $(X, \chi)$ of $M$. The problem of determining which cohomology classes are algebraizable, a cohomological strengthening of the Nash–Tognoli theorem, is still largely open.

It is known since Benedetti and Dedò [1984, Theorem 1] that there exist nonalgebraizable cohomology classes. Teichner [1995, Theorem 1] has found examples in dimension 6 (the smallest dimension possible), and Kucharz [2008, Theorem 1.13] has discovered examples for all even degrees greater or equal to 2. It remained an open problem, raised by Benedetti and Dedò [1984, p. 150] and Kucharz [2005, p. 194], to determine the possible degrees of nonalgebraizable classes, and in particular to construct nonalgebraizable classes of odd degree. We solve this problem entirely.

**Theorem 5.3.** *For all $c \geq 2$, there exists a compact $\mathcal{C}^\infty$ manifold $M$ and a class $x \in H^c(M, \mathbb{Z}/2)$ that is not algebraizable.*

Examples of algebraizable cohomology classes include Stiefel–Whitney classes of topological real vector bundles on $M$ and Poincaré duals of fundamental classes of $\mathcal{C}^\infty$ submanifolds of $M$ (see [Benedetti and Tognoli 1980, Theorem 4.2 and Corollary 4.5] or [Akbulut and King 1981, Theorem 2.10]). Following Kucharz [2005], let us denote by $A(M) \subset H^*(M, \mathbb{Z}/2)$ the subring generated by these classes. It was asked by Benedetti and Dedò [1984, Conjecture 2, p. 150] and again by Kucharz [2005, Conjecture A] whether all algebraizable classes belong to $A(M)$. We give a negative answer to this conjecture.

**Theorem 5.6.** *There exists a compact $\mathcal{C}^\infty$ manifold $M$ and an algebraizable class $x \in H^5(M, \mathbb{Z}/2)$ that does not belong to $A(M)$.*

The proofs of Theorems 5.3 and 5.6 rely on the stability of algebraizable classes by mod 2 Steenrod operations, a result due to Akbulut and King ([1985, Theorem 6.6], see also Corollary 5.2). To prove Theorem 5.3, we construct cohomology classes whose images by a Steenrod operation can be shown to be nonalgebraizable by the method of Kucharz [2008], hence are themselves not algebraizable. As for the proof of Theorem 5.6, it relies on the observation that the subring $A(M)$ of $H^*(M, \mathbb{Z}/2)$ is in general not stable under the action of Steenrod operations.

**1.4.** ***Conventions.*** A variety over a field $k$ is a separated $k$-scheme of finite type.

Let $X$ be a smooth projective variety over the algebraic closure $k$ of a finitely generated field $k_0$. A class in $H^{2c}_{\text{ét}}(X, \mathbb{Z}_\ell(c))$ is said to be a Tate class if it belongs to the subgroup $H^{2c}_{\text{ét}}(X_1 \times_{k_1} k, \mathbb{Z}_\ell(c))^{G_{k_1}}$ of $H^{2c}_{\text{ét}}(X, \mathbb{Z}_\ell(c))$ associated with some model $X_1$ of $X$ over a finite extension $k_1$ of $k_0$ with absolute Galois group $G_{k_1}$. The classes that lie in the image of the cycle class map $\mathrm{CH}^c(X) \to H^{2c}_{\text{ét}}(X, \mathbb{Z}_\ell(c))$ are said to be algebraic, and are automatically Tate classes. In this article, the converse statement that all Tate classes are algebraic is called the integral Tate conjecture over $k$ (see [Colliot-Thélène and Scavia 2023, § 1] for a discussion of variants of this statement).

## 2. A relative Wu theorem in étale cohomology

In topology, the behaviour of Steenrod operations with respect to proper pushforwards is controlled by a Grothendieck–Riemann–Roch-type theorem: the relative Wu theorem of Atiyah and Hirzebruch [1961, Satz 3.2]. In this section, we prove an analogue of this theorem in the étale cohomology of schemes.

We first recall, in Section 2.1 and Section 2.2, a construction of Steenrod operations on the (twisted) étale cohomology of schemes. The characteristic classes appearing in the statement of the relative Wu theorem are defined and studied in Section 2.4, and the relative Wu theorem is proved in Section 2.5.

In this whole section we fix a prime number $\ell$ and we set $S_\ell := \mathrm{Spec}\big(\mathbb{Z}[\frac{1}{\ell}]\big)$.

**2.1.** ***Steenrod operations in étale cohomology.*** Let $\mathcal{A}_\ell$ be the mod $\ell$ Steenrod algebra [Steenrod 1962, Chapter I, §3; Chapter VI, §2]. It is a graded $\mathbb{Z}/\ell$-algebra generated by the Steenrod squares $(\mathrm{Sq}^i)_{i\geq 1}$ which have degree $i$ (when $\ell = 2$), or by Steenrod's $\ell$-th powers $(P^i)_{i\geq 1}$ which have degree $2i(\ell-1)$ and by the Bockstein $\beta$ which has degree 1 (when $\ell$ is odd), subject to the Adem relations. We write $\mathrm{Sq}^0 = P^0 = 1$. If $\ell = 2$, we set $\beta := \mathrm{Sq}^1$. We use the notation $\mathrm{Sq} := \sum_{i\geq 0}\mathrm{Sq}^i$ and $P := \sum_{i\geq 0} P^i$.

The algebra $\mathcal{A}_\ell$ acts functorially on the mod $\ell$ (relative) cohomology of (pairs of) topological spaces [Steenrod 1962, Chapters VII and VIII]. There are several ways to construct analogous Steenrod operations on the mod $\ell$ étale cohomology of schemes [Epstein 1966; May 1970; Jardine 1989; Feng 2020]. We follow Feng [2020, §3] and Scavia and Suzuki [2024, §2] in exploiting étale homotopy theory [Friedlander 1982]. This choice will allow us to use Scavia and Suzuki's analysis of the behaviour of Steenrod operations in Hochschild–Serre spectral sequences in Section 4.4.

Friedlander associates with any Noetherian scheme $X$ a pro-simplicial set denoted by $\text{Ét}(X)$ in [Feng 2020]: its étale topological type [Friedlander 1982, Definition 4.4]. Let $f : Y \hookrightarrow X$ be a closed immersion of Noetherian schemes with complement open immersion $j : U \to X$. Let $M(\text{Ét}(j))$ be the mapping cylinder in the sense of [Friedlander 1982, p. 140] of the map $\text{Ét}(j) : \text{Ét}(U) \to \text{Ét}(X)$ induced by $j$. It is a pro-simplicial set endowed with an inclusion $\text{Ét}(U) \hookrightarrow M(\text{Ét}(j))$. For any abelian group $A$, it follows from [Friedlander 1982, Corollary 14.5, Proposition 14.6] that there are natural isomorphisms

$$H^*_{\text{ét},Y}(X, A) \xrightarrow{\sim} H^*(M(\text{Ét}(j)), \text{Ét}(U), A). \tag{2-1}$$

When $A = \mathbb{Z}/\ell$, one can use the geometric realization of simplicial sets to let $\mathcal{A}_\ell$ act on the right-hand side of (2-1). We may then let $\mathcal{A}_\ell$ act on $H^*_{\text{ét},Y}(X, \mathbb{Z}/\ell)$ functorially in $(X, Y)$, using the isomorphism (2-1). By construction, this action has all the properties of the action of $\mathcal{A}_\ell$ on the mod $\ell$ cohomology of topological spaces (see [Steenrod 1962, §I.1 and §VI.1]). We will use them freely in the sequel.

If $X$ is a complex variety, it follows from [Friedlander 1982, Theorem 8.4] that Artin's comparison isomorphism $H^q_{\text{ét}}(X, \mathbb{Z}/\ell) \xrightarrow{\sim} H^q(X(\mathbb{C}), \mathbb{Z}/\ell)$ is $\mathcal{A}_\ell$-equivariant. If $X$ is a variety over $\mathbb{R}$, this statement has a $\mathrm{Gal}(\mathbb{C}/\mathbb{R})$-equivariant analogue: the comparison isomorphism $H^q_{\text{ét}}(X, \mathbb{Z}/\ell) \xrightarrow{\sim} H^q_{\mathrm{Gal}(\mathbb{C}/\mathbb{R})}(X(\mathbb{C}), \mathbb{Z}/\ell)$ of [Scheiderer 1994, Corollary 15.3.1] is $\mathcal{A}_\ell$-equivariant as a consequence of [Cox 1979, Theorem 1.1].

**2.2.** ***Steenrod operations in twisted étale cohomology.*** Guillou and Weibel have explained in [Guillou and Weibel 2019, §§1–3] how to extend the Steenrod operations of Section 2.1 to étale cohomology with twisted coefficients. (They work with the definitions of Epstein [1966] and May [1970].) We give a self-contained treatment adapted to our needs, following their arguments closely.

Consider the morphism of schemes $\pi : S'_\ell \to S_\ell$, where $S'_\ell := \mathrm{Spec}\big(\mathbb{Z}[\frac{1}{\ell}, t]/(\frac{t^\ell - 1}{t-1})\big)$ and $S_\ell := \mathrm{Spec}\big(\mathbb{Z}[\frac{1}{\ell}]\big)$. It is a Galois finite étale cover with Galois group $\Gamma := (\mathbb{Z}/\ell)^*$, where $n \in \Gamma$ sends $t$ to $t^n$. Decomposing $\mathbb{Z}/\ell[\Gamma]$ as a direct sum of one-dimensional $\Gamma$-representations yields an isomorphism

$$\pi_*\mathbb{Z}/\ell = \bigoplus_{r\in\mathbb{Z}/(\ell-1)} \boldsymbol{\mu}_\ell^{\otimes r} \tag{2-2}$$

of sheaves of algebras on the big étale site of $S_\ell$, where $\boldsymbol{\mu}_\ell^{\otimes r}$ is the factor of $\pi_*\mathbb{Z}/\ell$ on which $n \in \Gamma$ acts by multiplication by $n^r$, and where the algebra structure on the right comes from the canonical isomorphism

$$\mathbb{Z}/\ell \xrightarrow{\sim} \boldsymbol{\mu}_\ell^{\otimes \ell-1} \tag{2-3}$$

sending 1 to $\zeta^{\otimes \ell-1}$ for any primitive $\ell$-th root of unity $\zeta$.

Let $f : Y \hookrightarrow X$ be a closed immersion of Noetherian $S_\ell$-schemes, and let $f' : Y' \hookrightarrow X'$ denote its base change by $\pi : S'_\ell \to S_\ell$. As the projection $\pi_X : X' \to X$ is finite, one has $\mathrm{R}^k\pi_{X,*}\mathbb{Z}/\ell = 0$ for $k > 0$. It then follows from (2-2) that one has a $\Gamma$-equivariant decomposition

$$H^*_{\text{ét},Y'}(X', \mathbb{Z}/\ell) = H^*_{\text{ét},Y}(X, \pi_{X,*}\mathbb{Z}/\ell) = \bigoplus_{r \in \mathbb{Z}/(\ell-1)} H^*_{\text{ét},Y}(X, \boldsymbol{\mu}_\ell^{\otimes r}), \tag{2-4}$$

where $n \in \Gamma$ acts by multiplication by $n^r$ on the $r$-th factor of the right-hand side.

The action of $\mathcal{A}_\ell$ on the left side of (2-4) defined in Section 2.1 commutes with the $\Gamma$-action by functoriality, and hence preserves the factors of the right side of (2-4). Let $\mathcal{A}_\ell$ act on the $H^*_{\text{ét}}(X, \boldsymbol{\mu}_\ell^{\otimes r})$ in this way. For all $\alpha \in \mathcal{A}_\ell$, $x \in H^q_{\text{ét},Y}(X, \boldsymbol{\mu}_\ell^{\otimes r})$ and $x' \in H^{q'}{}_{\text{ét}}(X, \boldsymbol{\mu}_\ell^{\otimes r'})$, the standard properties of Steenrod operations imply that $\alpha(x) \in H^{q+\deg(\alpha)}_{\text{ét},Y}(X, \boldsymbol{\mu}_\ell^{\otimes r})$, that the identities

$$\begin{aligned} &\mathrm{Sq}^i(x) = 0 \ \text{ for } q < i \quad \text{and} \quad \mathrm{Sq}^i(x) = x^2 \ \text{ for } q = i \quad \text{if } \ell = 2, \\ &P^i(x) = 0 \ \text{ for } q < 2i \quad \text{and} \quad P^i(x) = x^\ell \ \text{ for } q = 2i \ \text{ if } \ell \text{ is odd}, \end{aligned} \tag{2-5}$$

are verified, and that the Cartan formulas

$$\begin{aligned} \beta(x' \cdot x) &= \beta(x') \cdot x + (-1)^{q'} x' \cdot \beta(x), \\ \mathrm{Sq}(x' \cdot x) &= \mathrm{Sq}(x') \cdot \mathrm{Sq}(x) && \text{if } \ell = 2, \\ P(x' \cdot x) &= P(x') \cdot P(x) && \text{if } \ell \text{ is odd}, \end{aligned} \tag{2-6}$$

hold in $H^*_{\text{ét},Y}(X, \boldsymbol{\mu}_\ell^{\otimes r+r'})$.

**Remarks 2.1.** (i) By construction, the action of $\mathcal{A}_\ell$ on $H^*_{\text{ét},Y}(X, \boldsymbol{\mu}_\ell^{\otimes r})$ only depends on $r \in \mathbb{Z}/(\ell-1)$, if one uses the canonical isomorphism (2-3) to trivialize $\boldsymbol{\mu}_\ell^{\otimes \ell-1}$.

(ii) We will often tacitly identify Tate twists that are congruent modulo $\ell - 1$ in our statements, by means of (2-3). For instance, Theorem 3.4 for $\ell$ odd equates $P^i(\mathrm{cl}(x)) \in H^{2i(\ell-1)+2c}_{\text{ét}}(X, \boldsymbol{\mu}_\ell^{\otimes c})$ and $\mathrm{cl}(P^i(x)) \in H^{2i(\ell-1)+2c}_{\text{ét}}(X, \boldsymbol{\mu}_\ell^{\otimes i(\ell-1)+c})$.

(iii) If an isomorphism $\xi : \mathbb{Z}/\ell \xrightarrow{\sim} \mu_\ell^{\otimes r'}$ over $X$ is fixed, then the induced isomorphisms $H^*_{\text{ét},Y}(X, \boldsymbol{\mu}_\ell^{\otimes r}) \xrightarrow{\sim} H^*_{\text{ét},Y}(X, \boldsymbol{\mu}_\ell^{\otimes r+r'})$ are $\mathcal{A}_\ell$-equivariant by (2-6) applied with $x'$ chosen to be the class $\xi \in H^0_{\text{ét}}(X, \boldsymbol{\mu}_\ell^{\otimes r'})$ of the fixed isomorphism.

(iv) As a consequence of (iii), if $X$ is a variety over a field $k$ in which a primitive $\ell$-th root of unity $\zeta$ has been fixed, one obtains $\mathcal{A}_\ell$-equivariant isomorphisms $H^*_{\text{ét},Y}(X, \boldsymbol{\mu}_\ell^{\otimes r}) \xrightarrow{\sim} H^*_{\text{ét},Y}(X, \boldsymbol{\mu}_\ell^{\otimes s})$ for all $r$ and $s$ (which depend on the choice of $\zeta$).

**2.3.** ***The Bockstein.*** We now proceed to the computation of the Bockstein, following [Guillou and Weibel 2019, Proposition 3.3]. Let $\delta_r : H^q_{\text{ét},Y}(X, \boldsymbol{\mu}_\ell^{\otimes r}) \to H^{q+1}_{\text{ét},Y}(X, \boldsymbol{\mu}_\ell^{\otimes r})$ be the boundary map of the short exact sequence

$$0 \to \boldsymbol{\mu}_\ell^{\otimes r} \to \boldsymbol{\mu}_{\ell^2}^{\otimes r} \to \boldsymbol{\mu}_\ell^{\otimes r} \to 0. \tag{2-7}$$

(In [Guillou and Weibel 2019], such morphisms are called $\beta$, but we reserve this notation for the Bockstein element of $\mathcal{A}_\ell$.) Use a primitive $\ell$-th root of unity $\zeta \in \mathcal{O}(S'_\ell)$ to identify $\boldsymbol{\mu}_\ell$ and $\mathbb{Z}/\ell$ on $S'_\ell$. The short exact sequence (2-7) for $r = 1$ then reads

$$0 \to \mathbb{Z}/\ell \to \boldsymbol{\mu}_{\ell^2} \to \mathbb{Z}/\ell \to 0, \tag{2-8}$$

and we let $\varpi \in H^1_{\text{ét}}(S'_\ell, \mathbb{Z}/\ell)$ be the image of $1 \in H^0_{\text{ét}}(S'_\ell, \mathbb{Z}/\ell)$ by its boundary map. The isomorphism class of the extension (2-8) is independent of the choice of $\zeta$ (if $n \in (\mathbb{Z}/\ell)^*$, multiplication by $n$ on (2-7) induces an isomorphism between short exact sequences (2-8) for choices $\zeta$ and $\zeta^n$ of primitive $\ell$-th roots of unity). It follows that the class $\varpi$ is $\Gamma$-invariant. As the degree of $\pi : S'_\ell \to S_\ell$ is prime to $\ell$, the Hochschild–Serre spectral sequence shows that $\varpi$ comes by pullback from a class that we still denote by $\varpi \in H^1_{\text{ét}}(S_\ell, \mathbb{Z}/\ell)$.

We still call $\varpi \in H^1_{\text{ét}}(X, \mathbb{Z}/\ell)$ the pullback of $\varpi$ to any $S_\ell$-scheme $X$. When $\ell = 2$, the class $\varpi$ is the image of $-1$ by the boundary map of the Kummer short exact sequence. (The class $\varpi$ is called $z$ in [Guillou and Weibel 2019], and $\alpha$ in [Feng 2020] when $\ell = 2$.)

**Lemma 2.2.** *The Bockstein morphism $\beta : H^q_{\text{ét},Y}(X, \boldsymbol{\mu}_\ell^{\otimes r}) \to H^{q+1}_{\text{ét},Y}(X, \boldsymbol{\mu}_\ell^{\otimes r})$ is equal to $x \mapsto \delta_r(x) - r\,\varpi \cdot x$.*

*Proof.* As the degree of the finite étale cover $\pi_X : X' \to X$ is prime to $\ell$, the pullback map $\pi_X^* : H^*_{\text{ét},Y}(X, \boldsymbol{\mu}_\ell^{\otimes r}) \to H^*_{\text{ét},Y'}(X', \boldsymbol{\mu}_\ell^{\otimes r})$ is injective. By functoriality, it thus suffices to prove the lemma on $X'$. Choose an isomorphism $\mathbb{Z}/\ell \xrightarrow{\sim} \boldsymbol{\mu}_\ell$ on $X'$. The morphism $\delta_r$ is then the boundary map of $0 \to \mathbb{Z}/\ell \to \boldsymbol{\mu}_{\ell^2}^{\otimes r} \to \mathbb{Z}/\ell \to 0$, while the morphism $\beta$ is the boundary map of $0 \to \mathbb{Z}/\ell \to \mathbb{Z}/\ell^2 \to \mathbb{Z}/\ell \to 0$ (this is one of the standard properties of Steenrod operations). It now follows from [Greenblatt 1965] that

$$\delta_r(x) = \beta(x) + r\varpi \cdot x.$$

(The reference [Greenblatt 1965] deals with the cohomology of topological spaces. By means of the mapping cone construction, the results proven there are still valid for the relative cohomology of pairs of topological spaces. By making use of the geometric realization functor, they still hold in the relative cohomology of pairs of (pro-)simplicial spaces. One can then apply them in our setting using (2-1).) □

**2.4.** ***Étale Stiefel–Whitney classes.*** If $f : Y \hookrightarrow X$ is a closed immersion of codimension $c$ of regular $S_\ell$-schemes, the Gysin morphism $\mathrm{Cl}_f : \mathbb{Z}/\ell \to \mathrm{R}f^!\boldsymbol{\mu}_\ell^{\otimes c}[2c]$ defined in [Riou 2014, définition 2.3.1] is an isomorphism by Gabber's purity theorem [Riou 2014, théorème 3.1.1]. We may then consider the Thom isomorphisms

$$\phi_f : H^q_{\text{ét}}(Y, \boldsymbol{\mu}_\ell^{\otimes r}) \xrightarrow{\sim} H^{q+2c}_{\text{ét},Y}(X, \boldsymbol{\mu}_\ell^{\otimes r+c}) \tag{2-9}$$

induced by $\mathrm{Cl}_f$, and we let $s_f := \phi_f(1) \in H^{2c}_{\text{ét},Y}(X, \boldsymbol{\mu}_\ell^{\otimes c})$ be the Thom class.

When $X = E$ is a vector bundle on $Y$ with structural morphism $g : E \to Y$ and $f : Y \hookrightarrow E$ is the zero section, one has

$$\phi_f(x) = g^*(x) \cdot s_f \quad \text{for all } x \in H^q_{\text{ét}}(Y, \boldsymbol{\mu}_\ell^{\otimes r}). \tag{2-10}$$

In this setting, we define the étale Stiefel–Whitney classes $w^{\text{ét}}_j(E) \in H^j_{\text{ét}}(Y, \mathbb{Z}/\ell)$ of $E$ by the formulas

$$\begin{aligned} w^{\text{ét}}(E) &:= \phi_f^{-1}(\mathrm{Sq}(s_f)) &&\text{if } \ell = 2,\\ w^{\text{ét}}(E) &:= \phi_f^{-1}(P(s_f)) &&\text{if } \ell \text{ is odd.} \end{aligned}$$

In view of (2-10), they are characterized by the identities

$$\mathrm{Sq}(s_f) = g^* w^{\text{ét}}(E) \cdot s_f \;\text{ if } \ell = 2 \quad\text{and}\quad P(s_f) = g^* w^{\text{ét}}(E) \cdot s_f \;\text{ if } \ell \text{ is odd.} \tag{2-11}$$

When $\ell = 2$, this definition appears in [Urabe 1996, p. 569] for varieties over algebraically closed fields and in [Feng 2020, Definition 5.2] for varieties over arbitrary fields. (These classes are denoted by $w_j(E)$ in [Urabe 1996; Feng 2020], but we want to distinguish them from those defined in [Brosnan 2003], see (3-1).)

The following lemma extends [Urabe 1996, Lemma 2.6(4)] and [Feng 2020, Lemma 5.7]. It allows us to define $w^{\text{ét}}(\kappa) := w^{\text{ét}}(E) \cdot w^{\text{ét}}(F)^{-1}$ for any class $\kappa = [E] - [F] \in K_0(X)$.

**Lemma 2.3.** *If $Y$ is a regular $S_\ell$-scheme and if*

$$0 \to E_1 \to E \to E_2 \to 0 \tag{2-12}$$

*is a short exact sequence of vector bundles on $Y$, one has*

$$w^{\text{ét}}(E) = w^{\text{ét}}(E_1) \cdot w^{\text{ét}}(E_2). \tag{2-13}$$

*Proof.* Let $\sigma_0, \sigma_1 : Y \to \mathbb{A}^1_Y$ be the sections of the projection $\mathbb{A}^1_Y \to Y$ with values 0 and 1. Arguing as in [Urabe 1996, Proof of Lemma 2.7], one constructs a short exact sequence of vector bundles $0 \to F_1 \to F \to F_2 \to 0$ on $\mathbb{A}^1_Y$ whose pullbacks by $\sigma_0$ and $\sigma_1$ are respectively isomorphic to the split exact sequence $0 \to E_1 \to E_1 \oplus E_2 \to E_2 \to 0$ and to (2-12). Since the two pullback morphisms $\sigma_0^*, \sigma_1^* : H^*_{\text{ét}}(\mathbb{A}^1_Y, \mathbb{Z}/\ell) \to H^*_{\text{ét}}(Y, \mathbb{Z}/\ell)$ are isomorphisms by [SGA $4_{\text{III}}$ 1973, exposé XVI, corollaire 2.2], one can assume, to prove (2-13), that the exact sequence (2-12) is split.

In this case, let $f_1 : E_1 \to E$ and $f_2 : E_2 \to E$ be the inclusions of the factors, let $g_1 : E \to E_1$ and $g_2 : E \to E_2$ be the projections and let $h_1 : Y \to E_1$ and $h_2 : Y \to E_2$ be the zero sections. It then follows from [Riou 2014, remarque 2.3.6, proposition 2.3.2] that $s_f = s_{f_1} \cdot s_{f_2} = g_1^* s_{h_1} \cdot g_2^* s_{h_2}$. The multiplicativity (2-13) of $w^{\text{ét}}$ now follows from the Cartan formulas (2-6). □

If $E$ is a vector bundle on an $S_\ell$-scheme $Y$, we let $c^{\text{ét}}_j(E) \in H^{2j}_{\text{ét}}(Y, \boldsymbol{\mu}_\ell^{\otimes j})$ be the mod $\ell$ Chern classes of $E$ (see [Riou 2014, théorème 1.3] for a construction following [Grothendieck 1958]). We also let $c^{\text{ét}}(E) := \sum_{j \geq 0} c^{\text{ét}}_j(E)$ be the total mod $\ell$ Chern class of $E$.

The following lemma was proven in [Feng 2020, Theorem 5.10] when $\ell = 2$ and $Y$ is a variety over a finite field. Recall the definition of $\varpi$ given in Section 2.3.

**Lemma 2.4.** *Let $E$ be a vector bundle on a regular $S_\ell$-scheme $Y$. Write formally $c^{\text{ét}}(E)=\prod_i(1+\lambda_i)$, where the $\lambda_i$ are the Chern roots of $E$. Then*

$$w^{\text{ét}}(E)=\prod_i(1+\varpi+\lambda_i) \qquad \text{if } \ell=2, \tag{2-14}$$

$$w^{\text{ét}}(E)=\prod_i(1+\lambda_i^{\ell-1}) \qquad \text{if } \ell \text{ is odd.} \tag{2-15}$$

*Proof.* By Grothendieck's splitting principle, we may assume that $E$ is a successive extension of line bundles. By Lemma 2.3, we may further assume that $E$ is a line bundle. In this case, we let $g:E\to Y$ be the structural morphism and $f:Y\hookrightarrow E$ be the zero section. The image of the Thom class $s_f$ by the morphism $H^2_{\text{ét},Y}(E,\boldsymbol{\mu}_\ell)\to H^2_{\text{ét}}(E,\boldsymbol{\mu}_\ell)$ forgetting the support is equal to $c_1^{\text{ét}}(\mathcal{O}_E(Y))=c_1^{\text{ét}}(g^*E)=g^*c_1^{\text{ét}}(E)$ by [Riou 2014, §2.1]. As a consequence,

$$s_f^2=g^*c_1^{\text{ét}}(E)\cdot s_f \quad \text{in } H^4_{\text{ét},Y}(E,\boldsymbol{\mu}_\ell^{\otimes 2}). \tag{2-16}$$

If $\ell$ is odd, then $P(s_f)=s_f+s_f^\ell$ by (2-5), hence $P(s_f)=(1+g^*c_1^{\text{ét}}(E)^{\ell-1})\cdot s_f$ by (2-16). Using (2-11), we get $w^{\text{ét}}(E)=1+c_1^{\text{ét}}(E)^{\ell-1}$, and (2-15) holds.

If $\ell=2$, one has $\mathrm{Sq}(s_f)=s_f+\mathrm{Sq}^1(s_f)+s_f^2$ by (2-5). As $s_f$ lifts to $H^2_{\text{ét},Y}(E,\boldsymbol{\mu}_4)$ by [Riou 2014, définition 2.3.1], one has $\mathrm{Sq}^1(s_f)=\varpi\cdot s_f$ by Lemma 2.2 and (2-7). Applying (2-16) yields $\mathrm{Sq}(s_f)=(1+\varpi+g^*c_1^{\text{ét}}(E))\cdot s_f$. We deduce from (2-11) that $w^{\text{ét}}(E)=1+\varpi+c_1^{\text{ét}}(E)$, and hence that (2-14) holds. □

**2.5.** ***A relative Wu theorem.*** Let $f:Y\to X$ be a morphism of finite type of regular Noetherian $S_\ell$-schemes admitting ample invertible sheaves, which has virtual relative dimension $-c$ in the sense of [SGA 6 1971, exposé VIII, définition 1.9]. As $f$ may be written as the composition of a regular closed immersion $g:Y\to Z$ and of a smooth morphism $h:Z\to X$ (one may take $Z=\mathbb{P}^n_X$ for $n\gg 0$), one can define its virtual normal bundle $N_f\in K_0(Y)$ to be $N_f:=-(T_f)^\vee$, where $T_f\in K_0(Y)$ is defined in [SGA 6 1971, exposé VIII, corollaire 2.5]. The class $N_f$ has rank $c$. It is equal to $[(I/I^2)^\vee]$ if $f$ is a regular closed immersion defined by an ideal $I\subset\mathcal{O}_X$ and to $-[T_{Y/X}]$ if $f$ is smooth.

One defines as in [Riou 2014, définition 2.5.11] a morphism $\mathrm{Cl}_f:\mathbb{Z}/\ell\to Rf^!\boldsymbol{\mu}_\ell^{\otimes c}[2c]$ which coincides with the one considered in Section 2.4 when $f$ is a closed immersion and which is induced by the trace map when $f$ is smooth [Riou 2014, proposition 2.5.13]. As noted in Remark 2.5.14 of the latter paper, if $f$ is moreover proper, the morphism $\mathrm{Cl}_f$ induces pushforward (or Gysin) morphisms $f_*:H^q_{\text{ét}}(Y,\boldsymbol{\mu}_\ell^{\otimes r})\to H^{q+2c}_{\text{ét}}(X,\boldsymbol{\mu}_\ell^{\otimes r+c})$.

The following theorem may be deduced from the Riemann–Roch theorems of Panin [2004] and Déglise [2018]. The case of Steenrod squares for varieties over algebraically closed fields also appears in [Scavia and Suzuki 2023, Proposition 3.1].

**Theorem 2.5.** *Let $f:Y\to X$ be a proper morphism of regular Noetherian $S_\ell$-schemes admitting ample invertible sheaves. For $x\in H^q_{\text{ét}}(Y,\boldsymbol{\mu}_\ell^{\otimes r})$, one has*

$$\begin{aligned} \mathrm{Sq}(f_*x)&=f_*(\mathrm{Sq}(x)\cdot w^{\text{ét}}(N_f)) && \text{if } \ell=2,\\ P(f_*x)&=f_*(P(x)\cdot w^{\text{ét}}(N_f)) && \text{if } \ell \text{ is odd.}\end{aligned} \tag{2-17}$$

*Proof.* Write $f$ as the composition of a closed immersion $g : Y \hookrightarrow \mathbb{P}^n_X$ and of the natural projection $h : \mathbb{P}^n_X \to X$. Using the covariant functoriality of Gysin morphisms (see [Riou 2014, théorème 2.5.12]), the multiplicativity of $w^{\text{ét}}$ (see Lemma 2.3), the Cartan formulas (2-6), and the additivity of $T_f$ [SGA 6 1971, exposé VIII, corollaire 2.7], we are reduced to proving the theorem for $g$ and $h$ separately. We may thus assume that $f$ is either a closed immersion or a projection $\mathbb{P}^n_X \to X$.

Assume first that $f$ is a closed immersion of codimension $c$. Let $f' : Y \to N_{Y/X}$ and $g' : N_{Y/X} \to Y$ be the zero section and the structural morphism of its normal bundle. Let $\sigma_0, \sigma_1 : X \to \mathbb{A}^1_X$ be the sections of the projection $\mathbb{A}^1_X \to X$ with values 0 and 1. Let $\mathfrak{X} := \mathrm{Bl}_{\sigma_0(Y)}\mathbb{A}^1_X \setminus \mathrm{Bl}_{\sigma_0(Y)}\sigma_0(X)$ be the deformation of $f$ to its normal cone and $h : \mathbb{A}^1_Y \to \mathfrak{X}$ be the strict transform of the natural closed immersion $\mathbb{A}^1_Y \to \mathbb{A}^1_X$. One gets a cartesian diagram

$$\begin{tikzcd} Y \arrow[r] \arrow[d, "f'"] & \mathbb{A}^1_Y \arrow[d, "h"] & Y \arrow[l] \arrow[d, "f"] \\ N_{Y/X} \arrow[r] & \mathfrak{X} & X \arrow[l] \end{tikzcd}$$

in which the left (resp. right) column is cut out in the middle column by the equation $t = 0$ (resp. $t = 1$), where $t$ is the coordinate of the affine line. We deduce from [Riou 2014, proposition 2.3.2] the commutativity of the following induced diagram, where the vertical arrows are Thom isomorphisms (2-9) and where the upper horizontal arrows are isomorphisms by [SGA $4_{\text{III}}$ 1973, exposé XVI, corollaire 2.2]:

$$\begin{tikzcd} H^*(Y, \boldsymbol{\mu}_\ell^{\otimes r}) \arrow[d, "\phi_{f'}", "\wr"'] & H^*(\mathbb{A}^1_Y, \boldsymbol{\mu}_\ell^{\otimes r}) \arrow[l, "\sim"'] \arrow[r, "\sim"] \arrow[d, "\phi_h", "\wr"'] & H^*(Y, \boldsymbol{\mu}_\ell^{\otimes r}) \arrow[d, "\phi_f", "\wr"'] \\ H^{*+2c}_Y(N_{Y/X}, \boldsymbol{\mu}_\ell^{\otimes r+c}) & H^{*+2c}_{\mathbb{A}^1_Y}(\mathfrak{X}, \boldsymbol{\mu}_\ell^{\otimes r+c}) \arrow[l] \arrow[r] & H^{*+2c}_Y(X, \boldsymbol{\mu}_\ell^{\otimes r+c}) \end{tikzcd} \tag{2-18}$$

When $\ell = 2$, one has

$$\mathrm{Sq}(\phi_{f'}(x)) = g'^*\mathrm{Sq}(x) \cdot \mathrm{Sq}(s_{f'}) = g'^*(\mathrm{Sq}(x) \cdot w^{\text{ét}}(N_{f'})) \cdot s_{f'} = \phi_{f'}(\mathrm{Sq}(x) \cdot w^{\text{ét}}(N_{f'}))$$

in $H^*_Y(N_{Y/X}, \boldsymbol{\mu}_2^{\otimes r+c})$ by (2-6), (2-10) and (2-11). In view of (2-18) and since $N_f$ and $N_{f'}$ are both restrictions of $N_h$, we deduce that $\mathrm{Sq}(\phi_f(x)) = \phi_f(\mathrm{Sq}(x) \cdot w^{\text{ét}}(N_f))$ in $H^*_Y(X, \boldsymbol{\mu}_2^{\otimes r+c})$. Forgetting the support yields (2-17). The argument when $\ell$ is odd is identical, replacing 2 by $\ell$ and Sq by $P$.

Assume now that $f : \mathbb{P}^n_X \to X$ is the projection. Suppose first that $\ell = 2$. Let $\lambda := c_1^{\text{ét}}(\mathcal{O}_{\mathbb{P}^n_X}(1)) \in H^2_{\text{ét}}(\mathbb{P}^n_X, \boldsymbol{\mu}_2)$. As $\lambda$ lifts to $H^2_{\text{ét}}(\mathbb{P}^n_X, \boldsymbol{\mu}_4)$ by [Riou 2014, définition 2.3.1], Lemma 2.2 and (2-7) imply that $\mathrm{Sq}^1(\lambda) = \varpi \cdot \lambda$. Taking (2-5) into account, we get

$$\mathrm{Sq}(\lambda) = \lambda + \varpi \cdot \lambda + \lambda^2. \tag{2-19}$$

The Euler exact sequence $0 \to \mathcal{O}_{\mathbb{P}^n_X} \to \mathcal{O}_{\mathbb{P}^n_X}(1)^{\oplus N+1} \to T_{\mathbb{P}^n_X/X} \to 0$ and Lemmas 2.3 and 2.4 imply that $w^{\text{ét}}(T_{\mathbb{P}^n_X/X}) = (1 + \varpi + \lambda)^{n+1}/(1 + \varpi)$, hence that

$$w^{\text{ét}}(N_f) = w^{\text{ét}}(-[T_{\mathbb{P}^n_X/X}]) = (1 + \varpi)/(1 + \varpi + \lambda)^{n+1}. \tag{2-20}$$

As $H^*_{\text{ét}}(\mathbb{P}^n_X, \boldsymbol{\mu}_2^{\otimes *}) = H^*_{\text{ét}}(X, \boldsymbol{\mu}_2^{\otimes *})[\lambda]/\lambda^{n+1}$ by the projective bundle formula (see for instance [Riou 2014, §1]), it suffices to prove the identity (2-17) for $x$ of the form $f^*y \cdot \lambda^m$, where $y \in H^*_{\text{ét}}(X, \boldsymbol{\mu}_2^{\otimes *})$ and $m \in \{0, \dots, n\}$. By the projection formula, the left-hand side of (2-17) is equal to $\mathrm{Sq}(y)$ if $m = n$ and vanishes otherwise. Let us compute the right-hand side of (2-17). One has

$$\mathrm{Sq}(x) = \mathrm{Sq}(f^*y \cdot \lambda^m) = f^*\mathrm{Sq}(y) \cdot \mathrm{Sq}(\lambda)^m = f^*\mathrm{Sq}(y) \cdot \lambda^m (1 + \varpi + \lambda)^m$$

by the Cartan formula (2-6) and (2-19). It follows from (2-20) that

$$\mathrm{Sq}(x) \cdot w^{\text{ét}}(N_f) = f^*\mathrm{Sq}(y) \cdot \lambda^m \eta(\eta + \lambda)^{-k-1} = f^*\mathrm{Sq}(y) \cdot \lambda^m \eta^{m-n} \Big(1 + \frac{\lambda}{\eta}\Big)^{-k-1}, \tag{2-21}$$

where $\eta := 1 + \varpi$ and $k := n - m$. By Lemma 2.6 below applied with $T = \lambda/\eta$, the coefficient of $\lambda^n$ in (2-21) is equal to $f^*\mathrm{Sq}(y)$ if $m = n$ and to 0 otherwise. It then follows from the projection formula that the right-hand side of (2-17) is equal to $\mathrm{Sq}(y)$ if $m = n$ and vanishes otherwise. This concludes the proof when $\ell = 2$.

The argument when $\ell$ is odd is similar, only easier because $\varpi$ is not involved. Equations (2-19) and (2-20) are replaced with the identities $P(\lambda) = \lambda + \lambda^\ell$ and $w^{\text{ét}}(N_f) = 1/(1+\lambda)^{n+1}$. Both the left and the right side of (2-17) for $x = f^*y \cdot \lambda^m$ with $y \in H^{q-2m}_{\text{ét}}(\mathbb{P}^n_X, \boldsymbol{\mu}_\ell^{\otimes r-m})$ are then computed to be equal to $P(y)$ if $m = n$ and to vanish otherwise (using Lemma 2.6 applied with $T = \lambda$). □

**Lemma 2.6.** *The coefficient of $T^k$ in the power series $F(T) = (1+T)^{-k-1} \in \mathbb{Z}[\![T]\!]$ is odd if $k = 0$ and even if $k \geq 1$.*

*Proof.* Since $F^{(k)}(T) = (-1)^k \prod_{i=k+1}^{2k} i \cdot (1+T)^{-2k-1}$, the coefficient of $T^k$ in $F(T)$ is $F^{(k)}(0)/k! = (-1)^k \binom{2k}{k}$. It is equal to 1 if $k = 0$ and it is even if $k \geq 1$. □

**Remark 2.7.** In topology, the Wu theorem describes the interaction of Steenrod operations and Poincaré duality. The same holds for Theorem 2.5, where Poincaré duality appears in the guise of pushforwards. Feng's Wu theorem over finite fields [Feng 2020, Theorem 6.5], however, describes the interaction of Steenrod operations and a subtler duality that combines Poincaré duality and duality in Galois cohomology of finite fields. It is therefore not a particular case of Theorem 2.5.

## 3. Steenrod operations and algebraic classes

We now apply the relative Wu theorem proved in Section 2 to study the compatibility of Brosnan's Steenrod operations on Chow groups (recalled in Section 3.1) and those in étale cohomology (defined in Sections 2.1–2.2). Our main result in this direction is Theorem 3.4. In Section 3.3, we explain how to deduce obstructions to the algebraicity of cohomology classes. We clarify their relation with complex cobordism in Section 3.4.

Throughout this section, we fix a field $k$ and a prime number $\ell$ invertible in $k$.

**3.1.** ***Steenrod operations on Chow groups.*** Let $X$ be a smooth variety over $k$. Brosnan [2003, Definition 8.11 and § 11] has defined an action of $\mathcal{A}_\ell$ on $\mathrm{CH}^*(X)/\ell$ such that all odd degree elements of $\mathcal{A}_\ell$ act by zero, and such that for all $\alpha \in \mathcal{A}_\ell$ of even degree and all $x \in \mathrm{CH}^c(X)/\ell$, one has $\alpha(x) \in \mathrm{CH}^{c+\deg(\alpha)/2}(X)/\ell$.

Let $E$ be a vector bundle on a smooth variety $X$ over $k$. Let $c_j(E) \in \mathrm{CH}^j(X)/\ell$ be the mod $\ell$ Chern classes of $E$ defined as in [Grothendieck 1958] (see also [Fulton 1998, p. 325]), and let $c(E) := \sum_{j\geq 0} c_j(E)$. Write formally $c(E) = \prod_i (1+\lambda_i)$, where the $\lambda_i$ are the Chern roots of $E$. Brosnan [2003, p. 1891] introduces the characteristic classes $w_{2j}(E) \in \mathrm{CH}^j(X)/\ell$ defined by

$$w(E) = \sum_{j\geq 0} w_{2j}(E) := \prod_i (1+\lambda_i^{\ell-1}). \tag{3-1}$$

This definition is extended to $K_0(X)$ by setting $w([E]-[F]) := w(E)\cdot w(F)^{-1}$.

Brosnan uses these classes to describe the behaviour of the Steenrod operations on Chow groups with respect to proper pushforwards.

**Proposition 3.1.** *Let $f : Y \to X$ be a proper morphism between smooth varieties over $k$ with virtual normal bundle $N_f$. For $x \in \mathrm{CH}^*(Y)/\ell$, one has*

$$\begin{aligned} \mathrm{Sq}(f_*x) &= f_*(\mathrm{Sq}(x)\cdot w(N_f)) \quad && \text{if } \ell = 2, \\ P(f_*x) &= f_*(P(x)\cdot w(N_f)) && \text{if } \ell \text{ is odd.} \end{aligned}$$

*Proof.* This follows from [Brosnan 2003, Propositions 9.4(iii) and 10.3]. □

**Remark 3.2.** An earlier construction of Steenrod operations on Chow groups was given by Voevodsky [2003] in the context of motivic cohomology. We do not know a published proof of the fact that his operations are compatible with Brosnan's.

**3.2.** ***Comparing Steenrod operations.*** Combining Proposition 3.1 and Theorem 2.5, we may compare the Steenrod operations on Chow groups and in étale cohomology. To do so, we let $\mathrm{cl} : \mathrm{CH}^*(X)/\ell \to H^{2*}_{\text{ét}}(X, \boldsymbol{\mu}_\ell^{\otimes *})$ denote the mod $\ell$ étale cycle class map of a smooth variety $X$ over $k$.

**Lemma 3.3.** *Let $X$ be a smooth variety over $k$ and let $\kappa \in K_0(X)$ be a* K*-theory class of rank $c \in \mathbb{Z}$. Then*

$$\begin{aligned} w^{\text{ét}}(\kappa) &= \sum_{j\geq 0} (1+\varpi)^{c-j}\cdot \mathrm{cl}(w_{2j}(\kappa)) \quad && \text{if } \ell = 2, \\ w^{\text{ét}}(\kappa) &= \mathrm{cl}(w(\kappa)) && \text{if } \ell \text{ is odd.} \end{aligned}$$

*Proof.* It follows from Lemma 2.3 and the definition (3-1) that both sides of these equalities are multiplicative in the class $\kappa$. By the splitting principle, we may thus assume that $\kappa$ is the class of a line bundle $L$. It then follows from Lemma 2.4 and (3-1) that, if $\ell$ is odd, one has

$$w^{\text{ét}}(\kappa) = 1 + c_1^{\text{ét}}(L)^{\ell-1} = \mathrm{cl}(1 + c_1(L)^{\ell-1}) = \mathrm{cl}(w(\kappa)),$$

and that, if $\ell = 2$, one has

$$w^{\text{ét}}(\kappa) = 1 + \varpi + c_1^{\text{ét}}(L) = 1 + \varpi + \mathrm{cl}(c_1(L)) = (1+\varpi)\cdot \mathrm{cl}(w_0(\kappa)) + \mathrm{cl}(w_2(\kappa)).$$ □

**Theorem 3.4.** *Let $X$ be a smooth quasiprojective variety over $k$. For $c \geq 0$ and $x \in \mathrm{CH}^c(X)/\ell$, one has*

$$\mathrm{Sq}(\mathrm{cl}(x)) = \sum_{i \geq 0} (1+\varpi)^{c-i} \cdot \mathrm{cl}(\mathrm{Sq}^{2i}(x)) \quad \text{if } \ell = 2,$$
$$P(\mathrm{cl}(x)) = \mathrm{cl}(P(x)) \quad \text{if } \ell \text{ is odd}.$$

*Proof.* By invariance of étale cohomology under inseparable extensions of the base field [SGA $4_{\text{II}}$ 1972, exposé VIII, corollaire 1.2], we may assume that $k$ is perfect (after replacing it with its perfect closure). We may also assume that $x$ is the class of integral subvariety $g : Z \hookrightarrow X$. As $k$ is perfect, it follows from Gabber's improvement on de Jong's alteration theorem [Illusie and Temkin 2014, Theorem 2.1] that there exists a smooth variety $Y$ over $k$ and a projective morphism $\nu : Y \to Z$ that is generically finite of degree $d$ prime to $\ell$. Set $f := g \circ \nu$. Replacing $x$ by $dx$, which is legitimate because $d \in (\mathbb{Z}/\ell)^*$, we may assume that $x = f_*[Y]$. If $\ell$ is odd, we then compute

$$P(\mathrm{cl}(x)) = P(\mathrm{cl}(f_*[Y])) = P(f_*1) = f_* w^{\text{ét}}(N_f)$$

and

$$\mathrm{cl}(P(x)) = \mathrm{cl}(P(f_*[Y])) = \mathrm{cl}(f_*(P([\widetilde{Y}]) \cdot w(N_f))) = \mathrm{cl}(f_* w(N_f)) = f_*\mathrm{cl}(w(N_f)),$$

where the last equality on the first line is by Theorem 2.5 and the second on the second line is by Proposition 3.1. If $\ell = 2$, the same arguments show that $\mathrm{Sq}(\mathrm{cl}(x)) = f_* w^{\text{ét}}(N_f)$ and $\mathrm{cl}(\mathrm{Sq}(x)) = f_*\mathrm{cl}(w(N_f))$. In both cases, the theorem now follows from Lemma 3.3. □

One can complement Theorem 3.4 with a formula describing the action of the Bockstein on a cycle class.

**Lemma 3.5.** *For a smooth quasiprojective variety $X$ over $k$ and $x \in \mathrm{CH}^c(X)/\ell$,*

$$\beta(\mathrm{cl}(x)) = -c\varpi \cdot \mathrm{cl}(x).$$

*Proof.* This follows from Lemma 2.2 because $\mathrm{cl}(x)$ lifts to $H^{2c}_{\text{ét}}(X, \boldsymbol{\mu}_{\ell^2}^{\otimes c})$. □

**Remark 3.6.** We do not think that Theorem 3.4 was previously known, even in the framework of Voevodsky's motivic Steenrod operations alluded to in Remark 3.2. We note for instance that the class $\varpi$ does not appear in the statement of [Brosnan and Joshua 2015, Theorem 1.1(iii)]. We also note that the claim made in [Guillou and Weibel 2019, Remark 7.3.1] implies that the Steenrod operations in étale cohomology considered in [Guillou and Weibel 2019] preserve algebraic classes, which is not true in general (see Remark 3.8 below).

**3.3. *Obstructions to algebraicity.*** Let $H^{2*}_{\text{ét}}(X, \boldsymbol{\mu}_\ell^{\otimes *})_{\text{alg}} \subset H^{2*}_{\text{ét}}(X, \boldsymbol{\mu}_\ell^{\otimes *})$ be the image of the cycle class map. We obtain stability results of these algebraic classes under Steenrod operations.

**Proposition 3.7.** *Let $X$ be a smooth quasiprojective variety over $k$.*

(i) *If $k$ contains a primitive $\ell^2$-th root of unity, then $H^{2*}_{\text{ét}}(X, \boldsymbol{\mu}_\ell^{\otimes *})_{\text{alg}}$ is $\mathcal{A}_\ell$-stable.*

(ii) *If $\ell$ is odd, then $H^{2*}_{\text{ét}}(X, \boldsymbol{\mu}_\ell^{\otimes *})_{\text{alg}}$ is stable under the action of $P$.*

*Proof.* By construction, $\varpi \in H^1_{\text{ét}}(\operatorname{Spec}(k), \mathbb{Z}/\ell)$ vanishes if $k$ contains a primitive $\ell^2$-th root of unity. Theorem 3.4 and Lemma 3.5 thus imply the proposition. □

**Remark 3.8.** When $\ell = 2$, the hypothesis that $-1$ is a square in Proposition 3.7(i) cannot be removed, even if one restricts to even degree Steenrod squares. For instance, if $k = \mathbb{R}$ and $X = \mathbb{P}^2_{\mathbb{R}}$, then

$$H^*_{\text{ét}}(X, \mathbb{Z}/2) = H^*_{\text{ét}}(\operatorname{Spec}(\mathbb{R}), \mathbb{Z}/2)[\lambda]/\lambda^3 = (\mathbb{Z}/2)[\varpi, \lambda]/\lambda^3,$$

where $\lambda := c_1^{\text{ét}}(\mathcal{O}_X(1))$. The class $\lambda$ is algebraic, hence so is $\lambda^2$. Using (2-5) and (2-6) shows that $\operatorname{Sq}^2(\lambda^2) = \operatorname{Sq}^1(\lambda)^2$. Since $\lambda$ lifts to $H^2_{\text{ét}}(X, \boldsymbol{\mu}_4)$, Lemma 2.2 implies that $\operatorname{Sq}^1(\lambda) = \varpi \cdot \lambda$. It follows that $\operatorname{Sq}^2(\lambda^2) = \varpi^2 \cdot \lambda^2$ is not algebraic. This shows that Steenrod operations do not preserve algebraic classes in general.

In this example, the algebraic class $\lambda^2$ is not sent to an algebraic class by $\operatorname{Sq}^2$, but it is sent to an algebraic class by $\operatorname{Sq}^2 + \varpi^2$. This is a general phenomenon, as we show in Proposition 3.12(i) below.

Traditionally, instances of Proposition 3.7 have been used to prove that some cohomology classes are not algebraic, by showing that they are not killed by odd degree Steenrod operations.

**Corollary 3.9.** *Let $X$ be a smooth quasiprojective variety over $k$. Assume that $k$ contains a primitive $\ell^2$-th root of unity. Then $H^{2c}_{\text{ét}}(X, \boldsymbol{\mu}_\ell^{\otimes c})_{\text{alg}}$ is killed by all odd degree Steenrod operations.*

Even over algebraically closed fields, Proposition 3.7 yields obstructions to the algebraicity of cohomology classes that go beyond the vanishing of odd degree Steenrod operations. Over the complex numbers, that algebraic mod $\ell$ cohomology classes are reductions mod $\ell$ of integral Hodge classes implies the following.

**Corollary 3.10.** *Let $X$ be a smooth projective variety over $\mathbb{C}$. Fix $\alpha \in \mathcal{A}_\ell$ and $x \in H^{2c}(X(\mathbb{C}), \mathbb{Z}/\ell)_{\text{alg}}$.*

(i) *If $\deg(\alpha)$ is odd, then $\alpha(x) = 0$.*

(ii) *If $\deg(\alpha)$ is even, then $\alpha(x)$ is the reduction mod $\ell$ of an integral Hodge class.*

As algebraic classes are reductions mod $\ell$ of Tate classes, we similarly obtain:

**Corollary 3.11.** *Assume that $k$ is the algebraic closure of a finitely generated field. Let $X$ be a smooth projective variety over $k$. Fix $\alpha \in \mathcal{A}_\ell$ and $x \in H^{2c}_{\text{ét}}(X, \boldsymbol{\mu}_\ell^{\otimes c})_{\text{alg}}$.*

(i) *If $\deg(\alpha)$ is odd, then $\alpha(x) = 0$.*

(ii) *If $\deg(\alpha)$ is even, then $\alpha(x)$ is the reduction mod $\ell$ of an integral Tate class.*

Corollaries 3.10(i) and 3.11(i) are the well-known topological obstructions to the integral Hodge and Tate conjectures going back to Atiyah and Hirzebruch [1962]. Corollaries 3.10(ii) and 3.11(ii) yield new Hodge-theoretic (or Galois-theoretic) obstructions to them.

When $k$ does not contain a primitive $\ell^2$-th root of unity, one can still deduce from Theorem 3.4 and Lemma 3.5 stability properties of algebraic cohomology classes by Steenrod operations, and hence obstructions to the algebraicity of cohomology classes that go beyond Proposition 3.7 and Corollary 3.9.

We give an example of such an obstruction, to be used in Section 4.4 and Section 4.5, in the next proposition.

**Proposition 3.12.** *Assume that $\ell = 2$ and let $X$ be a smooth quasiprojective variety over $k$. Then*:

(i) $\mathrm{Sq}^2 + \frac{1}{2}c(c-1)\varpi^2$ *sends* $H^{2c}_{\text{ét}}(X, \boldsymbol{\mu}_2^{\otimes c})_{\text{alg}}$ *to* $H^{2c+2}_{\text{ét}}(X, \boldsymbol{\mu}_2^{\otimes c+1})_{\text{alg}}$.

(ii) $\mathrm{Sq}^3 + (c+1)\varpi \cdot \mathrm{Sq}^2 + \frac{1}{2}c(c-1)\varpi^3$ *kills* $H^{2c}_{\text{ét}}(X, \boldsymbol{\mu}_2^{\otimes c})_{\text{alg}}$.

*Proof.* If $x \in \mathrm{CH}^c(X)/2$, it follows from Theorem 3.4 that

$$\mathrm{cl}(\mathrm{Sq}^2(x)) = \mathrm{Sq}^2(\mathrm{cl}(x)) + \tfrac{1}{2}c(c-1)\varpi^2 \cdot \mathrm{cl}(x).$$

This proves (i). Applying Lemma 3.5, we deduce that $H^{2c}_{\text{ét}}(X, \boldsymbol{\mu}_2^{\otimes c})_{\text{alg}}$ is killed by

$$(\mathrm{Sq}^1 + (c+1)\varpi)\big(\mathrm{Sq}^2 + \tfrac{1}{2}c(c-1)\varpi^2\big).$$

Developing and applying Lemma 3.5 again, we see that this operator equals $\mathrm{Sq}^3 + (c+1)\varpi \cdot \mathrm{Sq}^2 + \frac{1}{2}c(c-1)\varpi^3$, proving (ii). ☐

**3.4. *Relation with complex cobordism.*** Totaro [1997] reinterpreted the Atiyah–Hirzebruch topological obstructions to the integral Hodge conjecture using complex cobordism. His point of view is that algebraic cohomology classes on a smooth projective complex variety $X$ always lie in the image of the natural map $\mu : \mathrm{MU}^*(X) \to H^*(X(\mathbb{C}), \mathbb{Z})$. Consequently, an integral Hodge class not in the image of $\mu$ cannot be algebraic. This fact also explains the topological obstructions to algebraicity given by Corollary 3.10(i) (see [Benoist and Ottem 2021, Proposition 3.6]). We now explain how to similarly understand the obstruction of Corollary 3.10(ii) using complex cobordism.

Let $\mathbf{MU}$ and $\boldsymbol{H}(\mathbb{Z})$ be spectra representing complex cobordism and ordinary cohomology with integral coefficients. The morphism $\mu : \mathrm{MU}^*(X) \to H^*(X(\mathbb{C}), \mathbb{Z})$ is induced by a map of spectra $\mu : \mathbf{MU} \to \boldsymbol{H}(\mathbb{Z})$. Let $\boldsymbol{H}(\pi_*(\mathbf{MU})_{\mathbb{C}})$ denote the Eilenberg–MacLane spectrum of the graded ring $\pi_*(\mathbf{MU})_{\mathbb{C}} := \pi_*(\mathbf{MU}) \otimes_{\mathbb{Z}} \mathbb{C}$. Hopkins and Quick [2015, §5] define a map $\iota : \mathbf{MU} \to \boldsymbol{H}(\pi_*(\mathbf{MU})_{\mathbb{C}})$ inducing the complexification $\pi_*(\mathbf{MU}) \to \pi_*(\mathbf{MU})_{\mathbb{C}}$ on homotopy groups. If $X$ is a smooth projective complex variety of dimension $n$ and $c \in \mathbb{Z}$, the map $\iota$ induces morphisms

$$\iota_* : \mathrm{MU}^{2c}(X) \to \bigoplus_{e=0}^{n-c} H^{2c+2e}(X(\mathbb{C}), \pi_{2e}(\mathbf{MU})_{\mathbb{C}}). \tag{3-2}$$

One recovers the morphism $\mu : \mathrm{MU}^{2c}(X) \to H^{2c}(X(\mathbb{C}), \mathbb{C})$ by projecting (3-2) onto the $e = 0$ summand. Hopkins and Quick [2015, (17)] define $\mathrm{Hdg}^{2c}_{\mathrm{MU}}(X)$ to be the set of $y \in \mathrm{MU}^{2c}(X)$ such that all the components of $\iota_* y$ are Hodge. They prove that any algebraic class $x \in H^{2c}(X(\mathbb{C}), \mathbb{Z})$ belongs to $\mu(\mathrm{Hdg}^{2c}_{\mathrm{MU}}(X))$ (see [Hopkins and Quick 2015, Corollary 7.12]). We claim that this obstruction to algebraicity combining Hodge theory and complex cobordism is finer than Corollary 3.10.

**Proposition 3.13.** *Let $X$ be a smooth projective complex variety and let $\ell$ be a prime number. Fix $\alpha \in \mathcal{A}_\ell$ and let $x$ be a class in the image of the composition*

$$\mathrm{Hdg}^{2c}_{\mathrm{MU}}(X) \xrightarrow{\mu} H^{2c}(X(\mathbb{C}), \mathbb{Z}) \xrightarrow{\rho} H^{2c}(X(\mathbb{C}), \mathbb{Z}/\ell)$$

*of $\mu$ and of the reduction modulo $\ell$ morphism $\rho$.*

(i) *If* $\deg(\alpha)$ *is odd, then* $\alpha(x) = 0$.

(ii) *If* $\deg(\alpha)$ *is even, then* $\alpha(x)$ *is the reduction mod $\ell$ of an integral Hodge class.*

*Proof.* Assertion (i) follows from [Benoist and Ottem 2021, Proposition 3.6] (the argument given there when $\ell = 2$ works as well when $\ell$ is odd). We now prove (ii).

Landweber [1967, Theorem 8.1] has shown the existence of a stable cohomological operation $\tilde{\alpha}$ in complex cobordism such that $(\rho \circ \mu) \circ \tilde{\alpha} = \alpha \circ (\rho \circ \mu)$. Letting $2d$ denote the degree of $\alpha$, we get a diagram of spectra

$$\begin{tikzcd}
\boldsymbol{H}(\pi_*(\mathbf{MU})_{\mathbb{C}}) \arrow[r, "\pi_*(\tilde{\alpha})_{\mathbb{C}}"] & \Sigma^{2d}\boldsymbol{H}(\pi_*(\mathbf{MU})_{\mathbb{C}}) \\
\mathbf{MU} \arrow[u, "\iota"] \arrow[r, "\tilde{\alpha}"] \arrow[d, "\rho\circ\mu"'] & \Sigma^{2d}\mathbf{MU} \arrow[u, "\iota"'] \arrow[d, "\rho\circ\mu"'] \\
\boldsymbol{H}(\mathbb{Z}/\ell) \arrow[r, "\alpha"] & \Sigma^{2d}\boldsymbol{H}(\mathbb{Z}/\ell)
\end{tikzcd} \tag{3-3}$$

where $\boldsymbol{H}(\mathbb{Z}/\ell)$ is the Eilenberg–MacLane spectrum with coefficients $\mathbb{Z}/\ell$. We claim that (3-3) is commutative up to homotopy. The lower square commutes by our choice of $\tilde{\alpha}$. For the upper square, we need to show that $\iota \circ \tilde{\alpha}$ and $\pi_*(\tilde{\alpha})_{\mathbb{C}} \circ \iota$ classify the same element of $H^*(\mathbf{MU}, \pi_{*-2d}(\mathbf{MU})_{\mathbb{C}}) = \mathrm{Hom}(H_*(\mathbf{MU}, \mathbb{Z}), \pi_{*-2d}(\mathbf{MU})_{\mathbb{C}})$. As the Hurewicz morphism $\pi_*(\mathbf{MU}) \to H_*(\mathbf{MU}, \mathbb{Z})$ is injective by [Adams 1974, Part II, Corollary 8.11], it suffices to verify that the maps $\iota \circ \tilde{\alpha}$ and $\pi_*(\tilde{\alpha})_{\mathbb{C}} \circ \iota$ induce the same morphism $\pi_*(\mathbf{MU}) \to \pi_{*-2d}(\mathbf{MU})_{\mathbb{C}}$, which is tautological.

Now let $y \in \mathrm{Hdg}^{2c}_{\mathrm{MU}}(X)$ be such that $x = \rho(\mu(y))$. The upper square of (3-3) shows that $\tilde{\alpha}(y) \in \mathrm{Hdg}^{2c+2d}_{\mathrm{MU}}(X)$. By the lower square of (3-3), the class $\alpha(x)$ is the reduction mod $\ell$ of the integral Hodge class $\mu(\tilde{\alpha}(y))$. □

It should be possible to carry out a similar analysis in the setting of the integral Tate conjecture, by relying on Quick's work [2007; 2011]. We do not do it here.

## 4. Nonalgebraic cohomology classes

We construct new examples of nonalgebraic cohomology classes, over algebraically closed fields (in Section 4.1 and Section 4.3), finite fields (in Section 4.4) and the field $\mathbb{R}$ of real numbers (in Section 4.5). Applications to unramified cohomology are made explicit in Section 4.6. In addition, we review in Section 4.2 the construction of algebraic approximations of classifying spaces that we use in Section 4.3 and Section 4.4.

We denote by $\rho$ all the reduction modulo $\ell$ morphisms: in Betti cohomology, in equivariant Betti cohomology, or in $\ell$-adic étale cohomology, where the prime number $\ell$ is always clear from the context (and equal to 2 in Section 4.1 and Section 4.5).

### 4.1. *A fivefold failing the integral Tate conjecture over $\overline{\mathbb{F}}_p$.*

**Proposition 4.1.** *There exists a smooth projective complex variety $Y$ such that*:

(i) *The variety $Y$ has dimension* 5.

(ii) *There exists $y \in H^4(Y(\mathbb{C}), \mathbb{Z})[2]$ such that* $\mathrm{Sq}^3(\rho(y)) \neq 0$.

(iii) *The variety $Y$ is defined over $\overline{\mathbb{Q}}$, with good reduction at all places not above* 2.

*Proof.* Let $Z$ be the complex fourfold defined in [Benoist and Ottem 2021, Proposition 5.3]. It carries a class $\sigma \in H^3(Z(\mathbb{C}), \mathbb{Z})[2]$ such that $\rho(\sigma)^2 \neq 0$ in $H^6(Z(\mathbb{C}), \mathbb{Z}/2)$. Let $E$ be a complex elliptic curve. Choose $\tau \in H^1(E(\mathbb{C}), \mathbb{Z})$ not divisible by 2. Set $Y := Z \times E$ and $y := p_1^*\sigma \cdot p_2^*\tau$. Then Cartan's formula and the vanishing of $\mathrm{Sq}^1(\rho(\tau))$ imply

$$\mathrm{Sq}^3(\rho(y)) = p_1^*\mathrm{Sq}^3(\rho(\sigma)) \cdot p_2^*\rho(\tau) = p_1^*\rho(\sigma)^2 \cdot p_2^*\rho(\tau) \neq 0.$$

That $Y$ may be chosen to have good reduction at odd places was explained in [Scavia and Suzuki 2024, §4.1]. To give more details, choose the elliptic curve $E$, as well as all the elliptic curves appearing in the construction of $Z$ given in [Benoist and Ottem 2021, §5.1] to have complex multiplication. Such elliptic curves are defined over a number field and have potentially good reduction everywhere. Combined with the fact that degree 2 finite étale covers of smooth projective varieties specialize in characteristic different from 2 (see [SGA 1 2003, exposé X, théorème 3.8]), this shows that the construction of $Z$, hence also of $Y$, works in mixed characteristic away from characteristic 2. The variety $Y$ is thus defined over $\overline{\mathbb{Q}}$ with good reduction away from 2. □

**Theorem 4.2.** *Let $p$ be an odd prime number. There exists a smooth projective variety $X$ of dimension* 5 *over $\overline{\mathbb{F}}_p$ on which the* 2*-adic integral Tate conjecture for codimension* 2 *cycles fails.*

*Proof.* Let $Y$ and $y$ be as in Proposition 4.1. Let $X$ denote a smooth projective variety over $\overline{\mathbb{F}}_p$ obtained by reduction of $Y$ modulo $p$. Artin's comparison theorem [SGA $4_{\mathrm{III}}$ 1973, exposé XI, théorème 4.4], the invariance of étale cohomology under extensions of algebraically closed fields [SGA $4_{\mathrm{III}}$ 1973, exposé XVI, corollaire 1.6], and the smooth and proper base change theorems [SGA $4_{\mathrm{III}}$ 1973, exposé XVI, corollaire 2.2] yield a commutative diagram

$$\begin{tikzcd} H^*_{\text{ét}}(X, \mathbb{Z}_2) \arrow[r, "\sim"] \arrow[d, "\rho"] & H^*(Y(\mathbb{C}), \mathbb{Z}_2) \arrow[d, "\rho"] \\ H^*_{\text{ét}}(X, \mathbb{Z}/2) \arrow[r, "\sim"] & H^*(Y(\mathbb{C}), \mathbb{Z}/2) \end{tikzcd} \tag{4-1}$$

whose horizontal arrows are isomorphisms, and whose bottom row is $\mathcal{A}_2$-equivariant (by functoriality of Steenrod operations and $\mathcal{A}_2$-equivariance of Artin's comparison isomorphism, see Section 2.1). The class

$x \in H^4_{\text{ét}}(X, \mathbb{Z}_2(2))[2]$ induced by $y$ and a fixed isomorphism $\mathbb{Z}_2 \xrightarrow{\sim} \mathbb{Z}_2(2)$ over $\overline{\mathbb{F}}_p$ is Tate since it is torsion. As $\mathrm{Sq}^3(\rho(x)) \neq 0$ by Proposition 4.1(ii) and (4-1), the class $x$ is not algebraic by Corollary 3.11(i). $\square$

**Remark 4.3.** The same argument shows that the complex variety $Y$ of Proposition 4.1 is a 5-dimensional counterexample to the integral Hodge conjecture: the class $y$ is Hodge because it is torsion, and it is not algebraic by Corollary 3.10(i).

**4.2.** ***Algebraic approximations of classifying spaces.*** Proposition 4.4 below is essentially due to Ekedahl [2009, Theorem 1.3 and §1.1]. Simplifications to the proof were made by Pirutka and Yagita [2015, §4] with further additions by Antieau [2016, §3]. We give a proof for the convenience of the reader, and claim no originality.

Recall that one may let any Lie group $H$ act freely and properly on a contractible space $EH$. The resulting quotient $BH := EH/H$ is the *classifying space* of $H$.

We also recall that a continuous map $f : Y \to X$ between topological spaces is an *$N$-equivalence* if $f_* : \pi_q(Y, y) \to \pi_q(X, f(y))$ is an isomorphism for $q < N$ and a surjection for $q = N$, for all $y \in Y$. By the Hurewicz theorem [Hatcher 2002, Theorem 4.37] and the universal coefficient theorem, the morphism $f^* : H^q(X, A) \to H^q(Y, A)$ is then an isomorphism for $q < N$ and an injection for $q = N$, for all abelian groups $A$.

**Proposition 4.4.** *Let $H$ be a connected reductive complex Lie group. Let $p$ be a prime number, and fix $N \geq 1$. Then there exists a smooth projective complex variety $Y$ of dimension $N$ such that:*

(i) *There exists an $N$-equivalence $Y(\mathbb{C}) \to B(H \times \mathbb{C}^*)$.*

(ii) *The variety $Y$ is defined over $\mathbb{Q}$ and has good reduction at $p$.*

The following lemma is [Ekedahl 2009, Lemma 1.1(ii)–(iv)].

**Lemma 4.5.** *Let $V$ be a $d$-dimensional vector space over an algebraically closed field $K$. For $n \geq d$, let $U_n \subset V^{\oplus n}$ be the open subset of $n$-tuples that span $V$.*

(i) *The complement of $U_n$ in $V^{\oplus n}$ has codimension at least $n + 1 - d$.*

(ii) *The algebraic group $\mathrm{GL}(V)$ acts freely on $U_n$.*

(iii) *The image of $U_n$ in $\mathbb{P}(V^{\oplus n})$ consists of $\mathrm{SL}(V)$-stable points.*

*Proof of Proposition 4.4.* Let $\mathcal{H}$ be a reductive group scheme over $\mathbb{Z}_{(p)}$ (in the sense of [SGA 3$_{\text{III}}$ 2011, exposé XIX, définition 2.7]) with $\mathcal{H}(\mathbb{C}) \simeq H$ (see [SGA 3$_{\text{III}}$ 2011, exposé XXV, corollaire 1.3]). By [SGA 3$_{\text{I}}$ 2011, exposé VI$_B$, proposition 13.2], there exists a closed embedding of group schemes $\mathcal{H} \hookrightarrow \mathrm{GL}_{d-2,\mathbb{Z}_{(p)}}$ for some $d \geq 2$. Composing it with

$$M \mapsto \begin{pmatrix} M & 0 & 0 \\ 0 & \det(M)^{-1} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

yields embeddings $\mathcal{H} \hookrightarrow \mathrm{SL}_{d,\mathbb{Z}_{(p)}}$ and $\mathcal{H} \times \boldsymbol{G}_m \hookrightarrow \mathrm{GL}_{d,\mathbb{Z}_{(p)}}$ (where $\boldsymbol{G}_m$ is embedded by homotheties). Fix $n \gg 0$. Let $\mathcal{H} \times \boldsymbol{G}_m$ act linearly on $\mathbb{A}^{nd}_{\mathbb{Z}_{(p)}} = (\mathbb{A}^d_{\mathbb{Z}_{(p)}})^n$, in a diagonal way using $n$ times the above representation. By Lemma 4.5, there exists an open subset $\mathcal{U} \subset \mathbb{A}^{nd}_{\mathbb{Z}_{(p)}}$ on which $\mathcal{H} \times \boldsymbol{G}_m$ acts freely, whose image $\mathcal{V} \subset \mathbb{P}^{nd-1}_{\mathbb{Z}_{(p)}}$ consists only of $\mathcal{H}$-stable points, and such that the complement of $\mathcal{U}$ in $\mathbb{A}^{nd}_{\mathbb{Z}_{(p)}}$ has codimension at least $N+1$ in the fibres of $\mathbb{A}^{nd}_{\mathbb{Z}_{(p)}} \to \operatorname{Spec}(\mathbb{Z}_{(p)})$ (as $n \gg 0$).

Let $\mathcal{Z} := \mathbb{P}^{nd-1}_{\mathbb{Z}_{(p)}} /\!\!/ \mathcal{H}$ be the GIT quotient (see [Seshadri 1977, Theorem 4]). The complement of the smooth open subset $\mathcal{W} := \mathcal{V}/\mathcal{H} \subset \mathcal{Z}$ has codimension at least $N+1$ in the fibres of $\mathcal{Z} \to \operatorname{Spec}(\mathbb{Z}_{(p)})$. Choosing a projective embedding of the projective $\mathbb{Z}_{(p)}$-scheme $\mathcal{Z}$, one can use Poonen's Bertini theorem [Poonen 2004, Theorem 1.1] to find an $N$-dimensional complete intersection $\mathcal{Y}_{\mathbb{F}_p}$ of $\mathcal{Z}_{\mathbb{F}_p}$ which is smooth and included in $\mathcal{W}_{\mathbb{F}_p}$. Lifting over $\mathbb{Z}_{(p)}$ the equations of the projective hypersurfaces defining $\mathcal{Y}_{\mathbb{F}_p}$ in $\mathcal{Z}_{\mathbb{F}_p}$ yields a complete intersection $\mathcal{Y}$ of $\mathcal{Z}$ which is smooth of relative dimension $N$ over $\mathbb{Z}_{(p)}$, and included in $\mathcal{W}$.

Define $Y := \mathcal{Y}_{\mathbb{C}}$, $V := \mathcal{V}_{\mathbb{C}}$ and $U := \mathcal{U}_{\mathbb{C}}$. The variety $Y$ satisfies (ii) by construction. Applying Hamm's Lefschetz theorem [Hamm 1983, Theorem 2 and the remark below] several times shows that the natural map $Y(\mathbb{C}) \to V(\mathbb{C})/H = U(\mathbb{C})/(H \times \mathbb{C}^*)$ is an $N$-equivalence. In the diagram of fibrations

$$U(\mathbb{C})/(H \times \mathbb{C}^*) \leftarrow (U(\mathbb{C}) \times E(H \times \mathbb{C}^*))/(H \times \mathbb{C}^*) \to B(H \times \mathbb{C}^*),$$

the left-hand side arrow is a weak equivalence because its fibres are isomorphic to $E(H \times \mathbb{C}^*)$ hence contractible, and the right-hand side arrow is an $N$-equivalence as its fibres are isomorphic to $U(\mathbb{C})$ hence are $(N-1)$-connected because the complement of $U(\mathbb{C})$ in $\mathbb{C}^{nd}$ has high codimension when $n \gg 0$. Assertion (i) follows. □

When we apply Proposition 4.4 to the simple simply connected complex Lie group $E_8$ in Section 4.3, we will need the following lemma.

**Lemma 4.6.** *Fix $\ell \in \{2, 3, 5\}$. Then the following assertions hold.*

(i) *There exists a 16-equivalence $f : BE_8 \to K(\mathbb{Z}, 4)$.*

(ii) *The group $H^q(BE_8, \mathbb{Z})$ has no $\ell$-torsion for $q \leq 2\ell + 2$.*

(iii) *There exists $y \in H^4(BE_8, \mathbb{Z})$ such that the classes $\mathrm{Sq}^3(\rho(y))$ (if $\ell = 2$) and $\beta P^1(\rho(y))$ (if $\ell$ is odd) are nonzero.*

*Proof.* As $\pi_1(E_8) = 0$ and $E_8$ is simple, Bott [1956, Theorem A and top of p. 253] has shown that $\pi_2(E_8) = 0$ and $\pi_3(E_8) = \mathbb{Z}$. Moreover, Bott and Samelson [1958, Theorem V, p. 995] have proven that $\pi_q(E_8) = 0$ for $4 \leq q \leq 14$. It follows that $\pi_q(BE_8) = 0$ for $1 \leq q \leq 3$ and $5 \leq q \leq 15$ and that $\pi_4(BE_8) = \mathbb{Z}$. This proves (i).

Assertion (ii) follows from (i) and from the computation of $H_*(K(\mathbb{Z}, 4), \mathbb{Z})$ given in [Cartan 1955, §4]. We now turn to (iii). In view of (i), it suffices to construct such a class $y$ in the cohomology of $K(\mathbb{Z}, 4)$, and by the universal property of $K(\mathbb{Z}, 4)$, it suffices to construct it in the cohomology of a space of our choice. The space $K(\mathbb{Z}/\ell, 3)$ works because the Steenrod operations $\mathrm{Sq}^3\mathrm{Sq}^1$ (when $\ell = 2$) and $\beta P^1 \beta$ (when $\ell$ is odd) are nonzero in general on degree 3 mod $\ell$ cohomology classes. □

In Section 4.4, we will use the following variant of Proposition 4.4.

**Proposition 4.7.** *Let $p$ be a prime number. Let $\mathcal{H}$ be a finite étale group scheme over $\mathbb{Z}_{(p)}$. Fix $N \geq 1$. Then there exist a smooth projective scheme $\mathcal{Y}$ of relative dimension $N$ over $\mathbb{Z}_{(p)}$ and a line bundle $\mathcal{L}$ over $\mathcal{Y}$ with the following properties.*

(i) *There exists an $N$-equivalence $\mathcal{Y}(\mathbb{C}) \to B(\mathcal{H}(\mathbb{C}) \times \mathbb{C}^*)$.*

(ii) *The topological complex line bundle $\mathcal{L}(\mathbb{C}) \to \mathcal{Y}(\mathbb{C})$ is classified by the map $\mathcal{Y}(\mathbb{C}) \to B\mathbb{C}^*$ appearing in* (i).

(iii) *If $N \geq 2$ and $\ell \neq p$ is a prime number, there is a* $\mathrm{Gal}(\overline{\mathbb{F}}_p/\mathbb{F}_p)$*-equivariant map*

$$\mathrm{Hom}(\mathcal{H}(\overline{\mathbb{F}}_p), \mathbb{Z}/\ell) = H^1(\mathcal{H}(\overline{\mathbb{F}}_p), \mathbb{Z}/\ell) \xrightarrow{\sim} H^1_{\text{ét}}(\mathcal{Y}_{\overline{\mathbb{F}}_p}, \mathbb{Z}/\ell). \tag{4-2}$$

*Proof.* Construct $\mathcal{Y}$ exactly as in the proof of Proposition 4.4. (As the reference [Seshadri 1977] assumes that $\mathcal{H}$ has connected geometric fibres, one has to replace it with [Raynaud 1967, théorème 1(iv)].) Assertion (i) is verified there.

By construction, there exists an $\mathcal{H}$-torsor $\widetilde{\mathcal{Y}} \to \mathcal{Y}$ whose total space is a relative complete intersection in $\mathbb{P}^m_{\mathbb{Z}_{(p)}}$ for some $m$. The tautological bundle $\mathcal{O}_{\widetilde{\mathcal{Y}}}(1)$ is naturally $\mathcal{H}$-linearized, and hence descends to a line bundle $\mathcal{L}$ on $\mathcal{Y}$. Inspecting the proof of Proposition 4.4 shows that (ii) holds.

As $\widetilde{\mathcal{Y}}_{\overline{\mathbb{F}}_p}$ is simply connected by [SGA 2 2005, exposé XII, corollaire 3.5], the Hochschild–Serre spectral sequence for $\widetilde{\mathcal{Y}}_{\overline{\mathbb{F}}_p} \to \mathcal{Y}_{\overline{\mathbb{F}}_p}$ yields the isomorphism (4-2). It is canonical, hence $\mathrm{Gal}(\overline{\mathbb{F}}_p/\mathbb{F}_p)$-equivariant. ☐

**4.3.** ***A torsion-free counterexample to the integral Tate conjecture over $\overline{\mathbb{F}}_p$.*** The proof of Proposition 4.8 below is to be compared with [Schoen 1998]. In this article, Schoen overcomes the difficulty of constructing hyperplane sections of smooth projective varieties over $\overline{\mathbb{F}}_p$ whose vanishing cohomology contains interesting Tate classes. In contrast, the main ingredient of the proof of Proposition 4.8 is the construction of a hyperplane section whose vanishing cohomology contains no nonzero Tate classes.

**Proposition 4.8.** *Choose $\ell \in \{2, 3, 5\}$ and let $p$ be a prime number distinct from $\ell$. There exist a smooth projective variety $X$ of dimension $2\ell + 2$ over $\overline{\mathbb{F}}_p$ and a class $x \in H^4_{\text{ét}}(X, \mathbb{Z}_\ell(2))$ such that the following assertions hold.*

(i) *The ring $H^*_{\text{ét}}(X, \mathbb{Z}_\ell)$ is torsion-free.*

(ii) *A multiple of the class $x$ is algebraic.*

(iii) *Define $\alpha := \mathrm{Sq}^2$ if $\ell = 2$ and $\alpha := P^1$ if $\ell$ is odd. Then the class $\alpha(\rho(x))$ is not the reduction modulo $\ell$ of an integral Tate class.*

*Proof.* We split the proof in several steps.

**Step 1** (An algebraic approximation of the classifying space of $E_8$). Let $Y$ be a variety obtained by applying Proposition 4.4 with $H = E_8$ and $N = 2\ell + 3$. Let $Z$ be a smooth projective variety over a finite field $\mathbb{F}$ obtained by reducing $Y$ modulo $p$. After maybe enlarging $\mathbb{F}$, we may assume that $Z(\mathbb{F}) \neq \varnothing$. Define $\nu : W \to Z$ to be the identity if $\ell \in \{2, 5\}$ and the blowup of an $\mathbb{F}$-point of $Z$ if $\ell = 3$. Set $\overline{Z} := Z_{\overline{\mathbb{F}}_p}$ and $\overline{W} := W_{\overline{\mathbb{F}}_p}$.

Proposition 4.4(i), Artin's comparison theorem [SGA $4_{\mathrm{III}}$ 1973, exposé XI, théorème 4.4], the invariance of étale cohomology under extensions of algebraically closed fields [SGA $4_{\mathrm{III}}$ 1973, exposé XVI, corollaire 1.6] and the smooth and proper base change theorems [SGA $4_{\mathrm{III}}$ 1973, exposé XVI, corollaire 2.2] yield a commutative diagram

$$\begin{tikzcd} H^*(BE_8,\mathbb{Z}_\ell)[t]=H^*(B(E_8\times\mathbb{C}^*),\mathbb{Z}_\ell) \arrow[r] \arrow[d, "\rho"] & H^*_{\text{ét}}(\bar{Z},\mathbb{Z}_\ell) \arrow[r, "\nu^*"] \arrow[d, "\rho"] & H^*_{\text{ét}}(\overline{W},\mathbb{Z}_\ell) \arrow[d, "\rho"] \\ H^*(BE_8,\mathbb{Z}/\ell)[t]=H^*(B(E_8\times\mathbb{C}^*),\mathbb{Z}/\ell) \arrow[r] & H^*_{\text{ét}}(\bar{Z},\mathbb{Z}/\ell) \arrow[r, "\nu^*"] & H^*_{\text{ét}}(\overline{W},\mathbb{Z}/\ell) \end{tikzcd} \tag{4-3}$$

whose left horizontal arrows are isomorphisms in degree at most $2\ell+2$ and injective in degree $2\ell+3$, whose right horizontal arrows are injective by computation of the cohomology of a blowup, whose bottom row is $\mathcal{A}_\ell$-equivariant, and where $t\in H^2(B\mathbb{C}^*,\mathbb{Z})$ is a generator of $H^*(B\mathbb{C}^*,\mathbb{Z})=H^*(\mathbb{P}^\infty(\mathbb{C}),\mathbb{Z})=\mathbb{Z}[t]$.

Since $H^{2\ell+2}_{\text{ét}}(\bar{Z},\mathbb{Q}_\ell)=\mathbb{Q}_\ell^{\oplus s}$ with $s=2$ if $\ell=2$, $s=3$ if $\ell=3$ and $s=4$ if $\ell=5$, by (4-3) and Lemma 4.6, we deduce from the computation of the cohomology of a blowup that $H^{2\ell+2}_{\text{ét}}(\overline{W},\mathbb{Q}_\ell)$ always has even rank.

**Step 2** (A Lefschetz pencil). Let $\mathcal{L}$ be a very ample line bundle on $W$ and fix $m\geq 3$. After maybe enlarging the finite field $\mathbb{F}$, we may choose a Lefschetz pencil $f:\widetilde{W}\to\mathbb{P}^1_{\mathbb{F}}$ on $W$ associated with $\mathcal{L}^{\otimes 2m}$ (see [SGA $7_{\mathrm{II}}$ 1973, exposé XVII, théorème 2.5]). Let $U\subset\mathbb{P}^1_{\mathbb{F}}$ be the smooth locus of the pencil, and let $\bar{\eta}$ be a geometric generic point of $\mathbb{P}^1_{\mathbb{F}}$.

By the hard Lefschetz theorem [Deligne 1980, corollaire 4.3.9], one has a decomposition

$$H^{2\ell+2}_{\text{ét}}(\widetilde{W}_{\bar{\eta}},\mathbb{Q}_\ell(\ell+1))=H^{2\ell+2}_{\text{ét}}(\overline{W},\mathbb{Q}_\ell(\ell+1))\oplus H^{2\ell+2}_{\text{ét}}(\widetilde{W}_{\bar{\eta}},\mathbb{Q}_\ell(\ell+1))_{\text{van}} \tag{4-4}$$

which is orthogonal with respect to the cup-product pairing. Let $O$ be the orthogonal group of the vanishing cohomology $H^{2\ell+2}_{\text{ét}}(\widetilde{W}_{\bar{\eta}},\mathbb{Q}_\ell(\ell+1))_{\text{van}}$ endowed with its cup-product pairing, and let $O^+\subset O$ be the special orthogonal subgroup. We view $O$ and $O^+$ as $\ell$-adic Lie groups. By [Schoen 1998, Proposition 1.1(iii)], the image of the geometric monodromy representation $\pi_1(U_{\overline{\mathbb{F}}_p},\bar{\eta})\to O$ is infinite (in [loc. cit.], the proof is written for a Lefschetz pencil on a threefold, but the argument works as well in our situation). It follows from [Deligne 1980, théorème 4.4.1] that this image is open for the $\ell$-adic topology. So is, a fortiori, the image $I\subset O$ of the arithmetic monodromy representation $\pi_1(U,\bar{\eta})\to O$. We deduce that $I$ contains a nonempty open subset $\Omega$ of $O^+$.

**Step 3** (Applying the Chebotarev density theorem). By [Schoen 1999, proof of Lemma 9.2.1(i)], the dimension of $H^{2\ell+2}_{\text{ét}}(\widetilde{W}_{\bar{\eta}},\mathbb{Q}_\ell(\ell+1))$ is even. Given that $H^{2\ell+2}_{\text{ét}}(\overline{W},\mathbb{Q}_\ell(\ell+1))$ has even rank, which we had checked at the end of Step 1, it follows from (4-4) that the dimension of $H^{2\ell+2}_{\text{ét}}(\widetilde{W}_{\bar{\eta}},\mathbb{Q}_\ell(\ell+1))_{\text{van}}$ is also even, say, equal to $2d$. Let $n_0\in\mathbb{N}$ be such that $\varphi(n)>2d$ for all $n\geq n_0$, where $\varphi$ is Euler's totient function. Since there exists an element of $\mathrm{SO}_{2d}(\overline{\mathbb{Q}}_\ell)\simeq\mathrm{SO}_{2d}(\mathbb{C})$ none of whose eigenvalues are roots of unity, a nonempty Zariski-open subset of $\mathrm{SO}_{2d}(\overline{\mathbb{Q}}_\ell)$ consists of matrices with no roots of unity of order less than $n_0$ as eigenvalues. It follows that there exists a nonempty open subset $\Omega' \subset \Omega$ for the $\ell$-adic topology containing only matrices none of whose eigenvalues are roots of unity of order less than $n_0$.

Chebotarev's theorem now shows the existence of a closed point $u \in U$ such that the image in $O$ of the Frobenius $F_u$ at $u$ belongs to $\Omega'$. Denoting by $X := (\widetilde{W}_u)_{\overline{\mathbb{F}}_p}$ the geometric fibre of $f$ above $u$, this implies that $F_u$ acts on the vanishing cohomology $H^{2\ell+2}_{\text{ét}}(X, \mathbb{Q}_\ell(\ell+1))_{\text{van}}$ with no roots of unity of order less than $n_0$ as eigenvalues. By [Deligne 1974, théorème 1.6], the characteristic polynomials of $F_u$ acting on $H^{2\ell+2}_{\text{ét}}(\overline{W}, \mathbb{Q}_\ell(\ell+1))$ and $H^{2\ell+2}_{\text{ét}}(X, \mathbb{Q}_\ell(\ell+1))$ have rational coefficients. By hard Lefschetz [Deligne 1980, corollaire 4.3.9], so is the characteristic polynomial of $F_u$ acting on $H^{2\ell+2}_{\text{ét}}(X, \mathbb{Q}_\ell(\ell+1))_{\text{van}}$. By our choice of $n_0$, this characteristic polynomial, which has degree $2d$, cannot annihilate any root of unity of order at least $n_0$. Consequently, none of the eigenvalues of $F_u$ acting on $H^{2\ell+2}_{\text{ét}}(X, \mathbb{Q}_\ell(\ell+1))_{\text{van}}$ are roots of unity. In other words, $H^{2\ell+2}_{\text{ét}}(X, \mathbb{Q}_\ell(\ell+1))_{\text{van}}$ contains no nonzero Tate classes.

**Step 4** (The cohomology ring $H^*_{\text{ét}}(X, \mathbb{Z}_\ell)$ is torsion-free). By the weak Lefschetz theorem, the restriction map $H^q_{\text{ét}}(\overline{W}, \mathbb{Z}_\ell) \to H^q_{\text{ét}}(X, \mathbb{Z}_\ell)$ is an isomorphism for $q \leq 2\ell+1$ and is injective with torsion-free cokernel for $q = 2\ell+2$ (see [Deligne 1980, (4.1.6)]). In view of (4-3) and Lemma 4.6, this implies that $H^q_{\text{ét}}(X, \mathbb{Z}_\ell)$ is torsion-free for $q \leq 2\ell+2$. Let us show by descending induction on $q$ that $H^q_{\text{ét}}(X, \mathbb{Z}_\ell)$ is also torsion-free for $q \geq 2\ell+3$. The assertion holds for $q \geq 4\ell+5$ by cohomological dimension [SGA $4_{\text{III}}$ 1973, exposé X, corollaire 4.3]. Fix $2\ell+3 \leq q \leq 4\ell+4$. Since both $H^{q+1}_{\text{ét}}(X, \mathbb{Z}_\ell)$ and $H^{4\ell+5-q}_{\text{ét}}(X, \mathbb{Z}_\ell)$ are torsion-free, the long exact sequences of cohomology associated with $0 \to \mathbb{Z}_\ell \xrightarrow{\ell^m} \mathbb{Z}_\ell \to \mathbb{Z}/\ell^m \to 0$ yield isomorphisms

$$H^q_{\text{ét}}(X, \mathbb{Z}_\ell)/\ell^m \xrightarrow{\sim} H^q_{\text{ét}}(X, \mathbb{Z}/\ell^m) \quad \text{and} \quad H^{4\ell+4-q}_{\text{ét}}(X, \mathbb{Z}_\ell)/\ell^m \xrightarrow{\sim} H^{4\ell+4-q}_{\text{ét}}(X, \mathbb{Z}/\ell^m)$$

for all $m \in \mathbb{N}$. As $H^q_{\text{ét}}(X, \mathbb{Z}/\ell^m)$ and $H^{4\ell+4-q}_{\text{ét}}(X, \mathbb{Z}/\ell^m)$ have the same cardinality by Poincaré duality (see [SGA $4_{\text{III}}$ 1973, exposé XVIII, (3.2.6.2)]), and as the cardinality of $H^{4\ell+4-q}_{\text{ét}}(X, \mathbb{Z}_\ell)/\ell^m$ is a linear function of $\ell^m$ because $H^{4\ell+4-q}_{\text{ét}}(X, \mathbb{Z}_\ell)$ is torsion-free, we deduce that the cardinality of $H^q_{\text{ét}}(X, \mathbb{Z}_\ell)/\ell^m$ is a linear function of $\ell^m$. It follows that $H^q_{\text{ét}}(X, \mathbb{Z}_\ell)$ is torsion-free. We have thus proven assertion (i).

**Step 5** (The cohomology class $x$). Let $y \in H^4(BE_8, \mathbb{Z})$ be the class given by Lemma 4.6, let $z \in H^4_{\text{ét}}(\overline{W}, \mathbb{Z}_\ell)$ be the class it induces via (4-3), and let $x \in H^4_{\text{ét}}(X, \mathbb{Z}_\ell(2))$ be the class obtained from $z|_X \in H^4_{\text{ét}}(X, \mathbb{Z}_\ell)$ by choosing an isomorphism $\mathbb{Z}_\ell \xrightarrow{\sim} \mathbb{Z}_\ell(2)$ over $\overline{\mathbb{F}}_p$.

By Edidin and Graham [1997, Theorem 1(c)] (see also [Totaro 2014, Theorem 2.14]), the cycle class map $\mathrm{CH}^2(B(E_8 \times \mathbb{C}^*)) \otimes_{\mathbb{Z}} \mathbb{Q} \to H^4(B(E_8 \times \mathbb{C}^*), \mathbb{Q})$ is surjective. Consequently, a multiple of the class $y$, hence also of $x$, is algebraic, proving (ii).

Assume by contradiction that $\alpha(\rho(x)) \in H^{2\ell+2}_{\text{ét}}(X, \boldsymbol{\mu}_\ell^{\otimes 2}) = H^{2\ell+2}_{\text{ét}}(X, \boldsymbol{\mu}_\ell^{\otimes \ell+1})$ is the reduction modulo $\ell$ of an integral Tate class $w \in H^{2\ell+2}_{\text{ét}}(X, \mathbb{Z}_\ell(\ell+1))$. By the weak Lefschetz theorem (see [Deligne 1980, (4.1.6)]), there is an exact sequence

$$0 \to H^{2\ell+2}_{\text{ét}}(\overline{W}, \mathbb{Z}_\ell(\ell+1)) \to H^{2\ell+2}_{\text{ét}}(X, \mathbb{Z}_\ell(\ell+1)) \to K \to 0 \tag{4-5}$$

with $K$ torsion-free. Since $K \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell = H^{2\ell+2}_{\text{ét}}(X, \mathbb{Q}_\ell(\ell+1))_{\text{van}}$ contains no nonzero Tate class by Step 3, the image of $w$ in $K$ is torsion, hence zero because $K$ is torsion-free. By (4-5), we deduce the existence of a Tate class $v \in H^{2\ell+2}_{\text{ét}}(\overline{W}, \mathbb{Z}_\ell(\ell+1))$ with $\alpha(\rho(x)) = \rho(v|_X)$. As the restriction map $H^{2\ell+2}_{\text{ét}}(\overline{W}, \boldsymbol{\mu}_\ell^{\otimes \ell+1}) \to H^{2\ell+2}_{\text{ét}}(X, \boldsymbol{\mu}_\ell^{\otimes \ell+1})$ is injective, again by weak Lefschetz [Deligne 1980, (4.1.6)], one has $\alpha(\rho(z)) = \rho(v)$ in $H^{2\ell+2}_{\text{ét}}(\overline{W}, \boldsymbol{\mu}_\ell^{\otimes \ell+1})$. Applying $\beta$ to this identity shows that $\beta\alpha(\rho(z)) = 0$, because $\beta$ kills the reductions mod $\ell$ of integral classes. This is a contradiction because $\beta\alpha(\rho(z))$ is nonzero by (4-3) and Lemma 4.6. We have finally proved (iii). □

**Theorem 4.9.** *Choose $\ell \in \{2, 3, 5\}$ and let $p$ be a prime number distinct from $\ell$. There exists a smooth projective variety $X$ of dimension $2\ell+2$ over $\overline{\mathbb{F}}_p$ with $H^*_{\text{ét}}(X, \mathbb{Z}_\ell)$ torsion-free on which the $\ell$-adic integral Tate conjecture for codimension 2 cycles fails.*

*Proof.* Let $X$ and $x$ be as in Proposition 4.8. The class $x$ is Tate because a multiple of it is algebraic by Proposition 4.8(ii). It cannot be algebraic itself by Corollary 3.11(ii) and Proposition 4.8(iii). □

**Remark 4.10.** Fix $\ell \in \{2, 3, 5\}$. The proofs of Proposition 4.8 and Theorem 4.9 may be adapted to show that, over the complex numbers, a $(2\ell+2)$-dimensional complete intersection of very general sufficiently ample hypersurfaces in a smooth projective variety obtained by applying Proposition 4.4 to $H = E_8$ fails the integral Hodge conjecture for codimension 2 cycles (but their cohomology ring has no $\ell$-torsion, and in fact no torsion at all when $\ell = 2$).

The main modifications to implement are the following. First, one has to use Corollary 3.10(ii) instead of Corollary 3.11(ii). More importantly, instead of verifying that the vanishing cohomology contains no nonzero Tate classes as in Steps 2 and 3 of the proof of Proposition 4.8, one needs to verify that it contains no nonzero Hodge classes. This is a consequence of a Noether–Lefschetz theorem (see [Voisin 2002, théorème 18.28]).

**4.4.** ***Geometrically trivial nonalgebraic classes over all finite fields.*** In this paragraph, we work over a finite field $\mathbb{F}$ of characteristic $p$, with algebraic closure $\overline{\mathbb{F}}_p$. If $X$ is a variety over $\mathbb{F}$, we set $\overline{X} := X_{\overline{\mathbb{F}}_p}$. Let $G := \mathrm{Gal}(\overline{\mathbb{F}}_p/\mathbb{F})$ be the absolute Galois group of $\mathbb{F}$, and $F \in G$ be the geometric Frobenius (see [Deligne 1974, (1.15.1)]). Recall that $G$ has cohomological dimension 1 and that, if $M$ is a finite $G$-module, one has a natural isomorphism

$$M/(F - \mathrm{Id}) \xrightarrow{\sim} H^1(G, M). \tag{4-6}$$

**Proposition 4.11.** *Let $p \neq \ell$ be prime numbers, and let $\mathbb{F}$ be a finite field of characteristic $p$. There exist a smooth projective geometrically connected variety $X$ of dimension $2\ell+3$ over $\mathbb{F}$ and a class $z \in H^2_{\text{ét}}(\overline{X}, \boldsymbol{\mu}_\ell^{\otimes 2})$ such that the class $\mathrm{Sq}^3\mathrm{Sq}^1(z)$ (if $\ell = 2$), or $\beta P^1\beta(z)$ (if $\ell$ is odd), does not belong to the image of $F - \mathrm{Id}$.*

*Proof.* Consider the étale abelian sheaf $(\boldsymbol{\mu}_\ell)^2$ over $\mathbb{Z}_{(p)}$ and view it as a finite étale group scheme $\mathcal{H}$ over $\mathbb{Z}_{(p)}$. Let $\mathcal{Y}$ be a smooth projective scheme over $\mathbb{Z}_{(p)}$ obtained by applying Proposition 4.7 to $\mathcal{H}$, with $N = 2\ell+3$. Define $X := \mathcal{Y}_{\mathbb{F}}$.

Proposition 4.7(i), Artin's comparison theorem [SGA $4_{\text{III}}$ 1973, exposé XI, théorème 4.4], the invariance of étale cohomology under extensions of algebraically closed fields [SGA $4_{\text{III}}$ 1973, exposé XVI, corollaire 1.6] and the smooth and proper base change theorems [SGA $4_{\text{III}}$ 1973, exposé XVI, corollaire 2.2] yield a morphism

$$\Phi : H^*(B((\mathbb{Z}/\ell)^2 \times \mathbb{C}^*), \mathbb{Z}/\ell) \to H^*_{\text{ét}}(\overline{X}, \mathbb{Z}/\ell)$$

which is an isomorphism in degree at most $2\ell+2$, and which is $\mathcal{A}_\ell$-equivariant. That $\Phi$ is an isomorphism in degree 0 implies that $X$ is geometrically connected.

It is well known that $H^*(B\mathbb{C}^*, \mathbb{Z}/\ell) = (\mathbb{Z}/\ell)[t]$ with $t \in H^2(B\mathbb{C}^*, \mathbb{Z}/\ell)$, that $H^*(B\mathbb{Z}/2, \mathbb{Z}/2) = (\mathbb{Z}/2)[x]$ with $x \in H^1(B\mathbb{Z}/2, \mathbb{Z}/2)$, and that, when $\ell$ is odd, $H^*(B\mathbb{Z}/\ell, \mathbb{Z}/\ell) = (\mathbb{Z}/\ell[x, y])/\langle x^2\rangle$ with $x \in H^1(B\mathbb{Z}/\ell, \mathbb{Z}/\ell)$ and $y = \beta(x)$. It then follows from the Künneth formula that

$$H^*(B((\mathbb{Z}/2)^2 \times \mathbb{C}^*), \mathbb{Z}/2) = (\mathbb{Z}/2)[x_1, x_2, t],$$

and that, when $\ell$ is odd, one has

$$H^*(B((\mathbb{Z}/\ell)^2 \times \mathbb{C}^*), \mathbb{Z}/\ell) = (\mathbb{Z}/\ell[x_1, y_1, x_2, y_2, t])/\langle x_1^2, x_2^2\rangle.$$

We now compute Galois actions. Let $|\mathbb{F}| = p^a$ be the cardinality of $\mathbb{F}$. By $G$-equivariance of (4-2) and our choice of $\mathcal{H}$, one has $F(\Phi(x_i)) = p^a\Phi(x_i)$ in $H^1_{\text{ét}}(\overline{X}, \mathbb{Z}/\ell)$ for $i \in \{1, 2\}$. When $\ell$ is odd, one also has $F(\Phi(y_i)) = p^a\Phi(y_i)$ in $H^2_{\text{ét}}(\overline{X}, \mathbb{Z}/\ell)$ for $i \in \{1, 2\}$ by $G$-equivariance of the Bockstein. In addition, since $\Phi(t) \in H^2_{\text{ét}}(\overline{X}, \mathbb{Z}/\ell)$ is the first Chern class of a line bundle which is defined over $\mathbb{F}$, by Proposition 4.7(ii), one has $F(\Phi(t)) = p^a\Phi(t)$ in $H^2_{\text{ét}}(\overline{X}, \mathbb{Z}/\ell)$.

Define $w := \Phi(x_1x_2)$. When $\ell = 2$, one has

$$\mathrm{Sq}^3\mathrm{Sq}^1(w) = \Phi(\mathrm{Sq}^3\mathrm{Sq}^1(x_1x_2)) = \Phi(\mathrm{Sq}^3(x_1^2x_2 + x_1x_2^2)) = \Phi(x_1^4x_2^2 + x_1^2x_2^4).$$

When $\ell$ is odd, one computes similarly that

$$\beta P^1\beta(w) = \Phi(\beta P^1\beta(x_1x_2)) = \Phi(\beta P^1(y_1x_2 - x_1y_2)) = \Phi(\beta(y_1^\ell x_2 - x_1y_2^\ell)) = \Phi(y_1^\ell y_2 - y_1y_2^\ell).$$

Choose a class $\xi \in H^0_{\text{ét}}(\overline{X}, \boldsymbol{\mu}_\ell^{\otimes 2})$ and define $z := \xi \cdot w \in H^2_{\text{ét}}(\overline{X}, \boldsymbol{\mu}_\ell^{\otimes 2})$. The action of $F$ on the group $H^{2\ell+2}_{\text{ét}}(\overline{X}, \boldsymbol{\mu}_\ell^{\otimes 2})$ is diagonal in a basis formed by multiples by $\xi$ of appropriate monomials in $\Phi(t)$, $\Phi(x_1)$, $\Phi(x_2)$ (and, when $\ell$ is odd, $\Phi(y_1)$ and $\Phi(y_2)$). Paying attention to the Tate twist, we see that the elements $\xi \cdot \Phi(x_1^4x_2^2)$ (when $\ell = 2$) and $\xi \cdot \Phi(y_1^\ell y_2)$ (when $\ell$ is odd) of this group are $F$-stable. Taking Remarks 2.1(iv) into account, we deduce that $\mathrm{Sq}^3\mathrm{Sq}^1(z) = \xi \cdot \Phi(x_1^4x_2^2 + x_1^2x_2^4)$ (when $\ell = 2$) and $\beta P^1\beta(z) = \xi \cdot \Phi(y_1^\ell y_2 - y_1y_2^\ell)$ (when $\ell$ is odd), are not in the image of $F - \mathrm{Id}$. □

The following theorem was proven by Scavia and Suzuki [2024, Theorem 1.3] when $\mathbb{F}$ contains a primitive $\ell^2$-th root of unity.

**Theorem 4.12.** *Let $p \neq \ell$ be prime numbers, and let $\mathbb{F}$ be a finite field of characteristic $p$. There exist a smooth projective geometrically connected variety $X$ of dimension $2\ell+3$ over $\mathbb{F}$ and a nonalgebraic class*

$$x \in \mathrm{Ker}\big(H^4_{\text{ét}}(X, \mathbb{Z}_\ell(2)) \to H^4_{\text{ét}}(\overline{X}, \mathbb{Z}_\ell(2))\big).$$

*Proof.* Let $X$ and $z$ be as in Proposition 4.11. The inverse limit, when $m$ varies, of the short exact sequences

$$0 \to H^1(G, H^{q-1}_{\text{ét}}(\overline{X}, \boldsymbol{\mu}_{\ell^m}^{\otimes r})) \to H^q_{\text{ét}}(X, \boldsymbol{\mu}_{\ell^m}^{\otimes r}) \to H^q_{\text{ét}}(\overline{X}, \boldsymbol{\mu}_{\ell^m}^{\otimes r})^G \to 0 \tag{4-7}$$

stemming from Hochschild–Serre spectral sequences yield short exact sequences

$$0 \to H^1\big(G, H^{q-1}_{\text{ét}}(\overline{X}, \mathbb{Z}_\ell(r))\big) \to H^q_{\text{ét}}(X, \mathbb{Z}_\ell(r)) \to H^q_{\text{ét}}(\overline{X}, \mathbb{Z}_\ell(r))^G \to 0 \tag{4-8}$$

because the groups $H^q_{\text{ét}}(\overline{X}, \boldsymbol{\mu}_{\ell^m}^{\otimes r})$ are finite and the $G$-cohomology groups of a finite $G$-module are finite.

Denote by $\psi_X : H^{q-1}_{\text{ét}}(\overline{X}, \boldsymbol{\mu}_{\ell^m}^{\otimes r}) \to H^q_{\text{ét}}(X, \boldsymbol{\mu}_{\ell^m}^{\otimes r})$ the compositions

$$H^{q-1}_{\text{ét}}(\overline{X}, \boldsymbol{\mu}_{\ell^m}^{\otimes r}) \to H^{q-1}_{\text{ét}}(\overline{X}, \boldsymbol{\mu}_{\ell^m}^{\otimes r})/(F - \mathrm{Id}) \to H^1(G, H^{q-1}_{\text{ét}}(\overline{X}, \boldsymbol{\mu}_{\ell^m}^{\otimes r})) \to H^q_{\text{ét}}(X, \boldsymbol{\mu}_{\ell^m}^{\otimes r})$$

of the quotient map, of the isomorphism (4-6), and of the left arrow of (4-7). Consider the element $y := \psi_X(\beta(z))$ of $H^4_{\text{ét}}(X, \boldsymbol{\mu}_\ell^{\otimes 2})$. As $\beta$ is the Bockstein, $\beta(z)$ lifts compatibly to $H^3_{\text{ét}}(\overline{X}, \boldsymbol{\mu}_{\ell^m}^{\otimes 2})$ for all $m$. Applying $\psi_X$ to these compatible lifts, we deduce from (4-8) that $y$ lifts to a class $x \in H^1\big(G, H^3_{\text{ét}}(\overline{X}, \mathbb{Z}_\ell(2))\big) \subset H^4_{\text{ét}}(X, \mathbb{Z}_\ell(2))$. Moreover, it follows from (4-8) that $x \in \mathrm{Ker}\big(H^4_{\text{ét}}(X, \mathbb{Z}_\ell(2)) \to H^4_{\text{ét}}(\overline{X}, \mathbb{Z}_\ell(2))\big)$.

It remains to prove that $x$ is not algebraic. We will prove the stronger statement that $y$ is not algebraic. Recall from Section 2.2 the definition of the finite étale cover $\pi : S'_\ell \to S_\ell$ with Galois group $\Gamma := (\mathbb{Z}/\ell)^*$, set $X' := X \times_{S_\ell} S'_\ell$ and $\overline{X}' := \overline{X} \times_{S_\ell} S'_\ell$, and define $\psi_{X'}$ in the same way as $\psi_X$. For $\alpha = \beta P^1$ (if $\ell$ is odd), or $\alpha \in \{\mathrm{Sq}^2, \mathrm{Sq}^3\}$ (if $\ell = 2$), the diagram

$$\begin{tikzcd}
H^3_{\text{ét}}(\overline{X}', \mathbb{Z}/\ell) \arrow[r, "\alpha"] \arrow[d, "\psi_{X'}"] & H^{3+\deg(\alpha)}_{\text{ét}}(\overline{X}', \mathbb{Z}/\ell) \arrow[d, "\psi_{X'}"] \\
H^4_{\text{ét}}(X', \mathbb{Z}/\ell) \arrow[r, "\alpha"] & H^{4+\deg(\alpha)}_{\text{ét}}(X', \mathbb{Z}/\ell)
\end{tikzcd}$$

commutes by [Scavia and Suzuki 2024, Corollary 3.5]. Retaining the appropriate isotypic components for the action of $\Gamma$, we deduce the commutativity of the following diagram:

$$\begin{tikzcd}
H^3_{\text{ét}}(\overline{X}, \boldsymbol{\mu}_\ell^{\otimes r}) \arrow[r, "\alpha"] \arrow[d, "\psi_X"] & H^{3+\deg(\alpha)}_{\text{ét}}(\overline{X}, \boldsymbol{\mu}_\ell^{\otimes r}) \arrow[d, "\psi_X"] \\
H^4_{\text{ét}}(X, \boldsymbol{\mu}_\ell^{\otimes r}) \arrow[r, "\alpha"] & H^{4+\deg(\alpha)}_{\text{ét}}(X, \boldsymbol{\mu}_\ell^{\otimes r})
\end{tikzcd} \tag{4-9}$$

Assume first that $\ell$ is odd. As $\beta P^1\beta(z)$ does not belong to the image of $F - \mathrm{Id}$ by Proposition 4.11, one has $\psi_X(\beta P^1\beta(z)) \neq 0$, hence $\beta P^1(y) \neq 0$ by the commutativity of (4-9) for $\alpha = \beta P^1$. It follows from Proposition 3.7(ii) and Lemma 3.5 that $\beta P^1$ kills $H^4_{\text{ét}}(X, \boldsymbol{\mu}_\ell^{\otimes 2})_{\text{alg}}$. Hence $y$ cannot be algebraic.

Assume now that $\ell = 2$. As $\mathrm{Sq}^3\mathrm{Sq}^1(z)$ does not belong to the image of $F - \mathrm{Id}$ by Proposition 4.11, one has $\psi_X(\mathrm{Sq}^3\mathrm{Sq}^1(z)) \neq 0$, hence $\mathrm{Sq}^3(y) \neq 0$ by the commutativity of (4-9) for $\alpha = \mathrm{Sq}^3$. The commutativity of (4-9) for $\alpha = \mathrm{Sq}^2$ shows that $\mathrm{Sq}^2(y) \in H^1(G, H^5_{\text{ét}}(\overline{X}, \boldsymbol{\mu}_2^{\otimes 2})) \subset H^6_{\text{ét}}(X, \boldsymbol{\mu}_2^{\otimes 2})$. In view of (4-7), one thus has

$$\mathrm{Sq}^2(y) \in F^1 H^6_{\text{ét}}(X, \boldsymbol{\mu}_2^{\otimes 2}) = \mathrm{Ker}(H^6_{\text{ét}}(X, \boldsymbol{\mu}_2^{\otimes 2}) \to H^6_{\text{ét}}(\overline{X}, \boldsymbol{\mu}_2^{\otimes 2})),$$

where $F^\bullet H^q_{\text{ét}}(X, \boldsymbol{\mu}_2^{\otimes r})$ denotes the abutment filtration of the Hochschild–Serre spectral sequence. Since $\varpi \in F^1 H^1_{\text{ét}}(X, \mathbb{Z}/2) = \mathrm{Ker}(H^1_{\text{ét}}(X, \mathbb{Z}/2) \to H^1_{\text{ét}}(\overline{X}, \mathbb{Z}/2))$, the multiplicativity properties of Hochschild–Serre spectral sequences imply that $\varpi \cdot \mathrm{Sq}^2(y) \in F^2 H^7_{\text{ét}}(X, \boldsymbol{\mu}_2^{\otimes 2})$. This group vanishes because $G$ has cohomological dimension 1, so that $\varpi \cdot \mathrm{Sq}^2(y) = 0$. It follows that $(\mathrm{Sq}^3 + \varpi \cdot \mathrm{Sq}^2)(y) \neq 0$. The class $y$ is not algebraic because $\mathrm{Sq}^3 + \varpi \cdot \mathrm{Sq}^2$ kills $H^4_{\text{ét}}(X, \boldsymbol{\mu}_2^{\otimes 2})_{\text{alg}}$ by Proposition 3.12(ii). □

**4.5.** ***A fourfold failing the integral Hodge conjecture over*** $\mathbb{R}$ ***but not over*** $\mathbb{C}$. In this section, we work over $\mathbb{R}$. Let $G := \mathrm{Gal}(\mathbb{C}/\mathbb{R}) \simeq \mathbb{Z}/2$ be generated by the complex conjugation $\sigma$. Let $\mathbb{Z}(k)$ be the $G$-module with underlying abelian group $\mathbb{Z}$, on which $\sigma$ acts by $(-1)^k$. Let $\omega \in H^1(G, \mathbb{Z}(1)) = H^1_G(\mathrm{pt}, \mathbb{Z}(1)) \simeq \mathbb{Z}/2$ and $\varpi \in H^1(G, \mathbb{Z}/2) = H^1_G(\mathrm{pt}, \mathbb{Z}/2) \simeq \mathbb{Z}/2$, and these two elements be the generators. One has $\rho(\omega) = \varpi$. We still denote by $\omega \in H^1_G(X(\mathbb{C}), \mathbb{Z}(1))$ and $\varpi \in H^1_G(X(\mathbb{C}), \mathbb{Z}/2)$ their pullbacks to the $G$-equivariant Betti cohomology of any variety $X$ over $\mathbb{R}$. The class $\varpi \in H^1_G(X(\mathbb{C}), \mathbb{Z}/2)$ corresponds to the one defined in Section 2.3 through the isomorphism $H^*_{\text{ét}}(X, \mathbb{Z}/2) \xrightarrow{\sim} H^*_G(X(\mathbb{C}), \mathbb{Z}/2)$ between étale cohomology and equivariant Betti cohomology [Scheiderer 1994, Corollary 15.3.1]. If $X$ is smooth, Krasnov ([1991, §2.1], see also [Benoist and Wittenberg 2020a, §1.6.1]) has defined, for all $c \geq 0$, a cycle class map

$$\mathrm{cl} : \mathrm{CH}^c(X) \to H^{2c}_G(X(\mathbb{C}), \mathbb{Z}(c)). \tag{4-10}$$

The classes in the image of (4-10) are said to be algebraic.

Let $E$ be the real elliptic curve with set of complex points $E(\mathbb{C}) = \mathbb{C}/(\mathbb{Z} \oplus i\mathbb{Z})$, on which $\sigma \in G$ acts through the usual complex conjugation of $\mathbb{C}$.

**Proposition 4.13.** *There exist $a \in H^1_G(E(\mathbb{C}), \mathbb{Z})$ and $b \in H^1_G(E(\mathbb{C}), \mathbb{Z}(1))$ such that $a \cdot b \in H^2_G(E(\mathbb{C}), \mathbb{Z}(1))$ is the cycle class of a real point $P \in E(\mathbb{R})$.*

*Proof.* One has $H^1(E(\mathbb{C}), \mathbb{Z}) \simeq \mathbb{Z} \oplus \mathbb{Z}(1)$ as $G$-modules, where the first factor is generated by the class $a' \in H^1(E(\mathbb{C}), \mathbb{Z})$ of the loop $\mathbb{R}/\mathbb{Z} \hookrightarrow \mathbb{C}/(\mathbb{Z} \oplus i\mathbb{Z})$ and the second factor by the class $b' \in H^1(E(\mathbb{C}), \mathbb{Z})$ of the loop $i\mathbb{R}/i\mathbb{Z} \hookrightarrow \mathbb{C}/(\mathbb{Z} \oplus i\mathbb{Z})$. We note that $a' \cdot b' \in H^2(E(\mathbb{C}), \mathbb{Z})$ has degree 1.

By work of Krasnov, the Hochschild–Serre spectral sequences

$$E_2^{s,t} = H^s\big(G, H^t(E(\mathbb{C}), \mathbb{Z}(k))\big) \Longrightarrow H^{s+t}_G(E(\mathbb{C}), \mathbb{Z}(k))$$

degenerate (combine [Krasnov 1983, §5.1, Proposition 3.6, Corollary 3.4] and [Krasnov 1998, Remark 1.2]). It follows that there exist $a \in H^1_G(E(\mathbb{C}), \mathbb{Z})$ and $b \in H^1_G(E(\mathbb{C}), \mathbb{Z}(1))$ lifting $a'$ and $b'$. By Krasnov's real Lefschetz (1, 1) theorem (see for instance [van Hamel 2000, Chapter IV, Theorem 4.1]), the class $a \cdot b \in H^2_G(E(\mathbb{C}), \mathbb{Z}(1))$ is the cycle class of a divisor $D$ on $E$. As $a' \cdot b'$ has degree 1, the divisor $D$ has degree 1. The Riemann–Roch theorem applied on the elliptic curve $E$ now shows that $D$ is effective, hence rationally equivalent to a point $P \in E(\mathbb{R})$. □

Let $Q := \{\sum_{i=0}^4 X_i^2 = 0\} \subset \mathbb{P}^4_{\mathbb{R}}$ be the 3-dimensional anisotropic quadric. We consider the smooth projective real algebraic fourfold $X := E \times_{\mathbb{R}} Q^3$.

**Proposition 4.14.** *There is a class* $x \in \mathrm{Ker}\big(H^4_G(X(\mathbb{C}), \mathbb{Z}(2)) \to H^4(X(\mathbb{C}), \mathbb{Z}(2))\big)$ *such that* $\mathrm{Sq}^3(\rho(x)) + \varpi \cdot \mathrm{Sq}^2(\rho(x)) + \varpi^3 \cdot \rho(x) \neq 0$ *in* $H^7_G(X(\mathbb{C}), \mathbb{Z}/2)$.

*Proof.* Let $a$, $b$ and $P$ be as in Proposition 4.13. Let $f : X \to E$ and $g : X \to Q$ be the projections. Set $x := f^*b \cdot \omega^3 \in H^4_G(X(\mathbb{C}), \mathbb{Z}(4))$. View it as an element of $H^4_G(X(\mathbb{C}), \mathbb{Z}(2))$ using an isomorphism of $G$-modules $\mathbb{Z}(4) \simeq \mathbb{Z}(2)$. The image of $x$ in $H^4(X(\mathbb{C}), \mathbb{Z}(2))$ vanishes because so does the image of $\omega$ in $H^1(X(\mathbb{C}), \mathbb{Z}(1))$.

As the elements $\varpi$ and $f^*\rho(b)$ of $H^1_G(X(\mathbb{C}), \mathbb{Z}/2)$ lift to $H^1_G(X(\mathbb{C}), \mathbb{Z}(1))$ and $\mathrm{Sq}^1$ is the Bockstein, one has $\mathrm{Sq}^1(\varpi) = \varpi^2$ and $\mathrm{Sq}^1(f^*\rho(b)) = f^*\rho(b) \cdot \varpi$ (see for instance [Benoist 2019, §2.2]). Since $\rho(x) = f^*\rho(b) \cdot \varpi^3$ in $H^4_G(X(\mathbb{C}), \mathbb{Z}/2)$, the Cartan formula implies that

$$\mathrm{Sq}^3(\rho(x)) + \varpi \cdot \mathrm{Sq}^2(\rho(x)) + \varpi^3 \cdot \rho(x) = f^*\rho(b) \cdot \varpi^6$$

in $H^7_G(X(\mathbb{C}), \mathbb{Z}/2)$. This class is nonzero because the element

$$g_*(f^*\rho(a) \cdot f^*\rho(b) \cdot \varpi^6) = g_*(f^*\rho(\mathrm{cl}(P)) \cdot \varpi^6) = \rho(g_* f^*\mathrm{cl}(P)) \cdot \varpi^6 = \varpi^6$$

of $H^6_G(Q(\mathbb{C}), \mathbb{Z}/2)$ is nonzero by [Benoist and Wittenberg 2024, Proposition 6.1]. □

We now reach the goal of this section.

**Theorem 4.15.** *There exists a smooth projective variety $X$ of dimension 4 over $\mathbb{R}$ such that $X(\mathbb{R}) = \varnothing$, and a nonalgebraic class*

$$x \in \mathrm{Ker}\big(H^4_G(X(\mathbb{C}), \mathbb{Z}(2)) \to H^4(X(\mathbb{C}), \mathbb{Z}(2))\big).$$

*Proof.* Let $X$ be as above and let $x \in H^4_G(X(\mathbb{C}), \mathbb{Z}(2))$ be as in Proposition 4.14. One has $X(\mathbb{R}) = \varnothing$ because $Q^3$ is anisotropic. The inverse image $y \in H^4_{\text{ét}}(X, \boldsymbol{\mu}_2^{\otimes 2})$ of $\rho(x)$ by the comparison isomorphism $H^4_{\text{ét}}(X, \boldsymbol{\mu}_2^{\otimes 2}) \xrightarrow{\sim} H^4_G(X(\mathbb{C}), \mathbb{Z}/2(2))$ of [Scheiderer 1994, Corollary 15.3.1] (which is $\mathcal{A}_2$-equivariant, see Section 2.1) is not algebraic by Propositions 3.12(ii) and 4.14. As a consequence, neither is $x$. (This last assertion follows from the compatibility of cycle class maps in étale and equivariant Betti cohomology. This result holds true, but we know no published reference. Alternatively, one can use the fact that the degree $2c$ classes that are algebraic are exactly those of coniveau at least $c$, both in étale and equivariant Betti cohomology, by purity — for which see [SGA $4_{\text{III}}$ 1973, exposé XIX, théorèmes 3.2 et 3.4] and [Benoist and Wittenberg 2020a, (1.21)].) □

If $X$ is a smooth projective variety over $\mathbb{R}$, the real integral Hodge conjecture for codimension $c$ cycles on $X$, defined in [Benoist and Wittenberg 2020a, Definition 2.2], is the statement that all the classes of $H^{2c}_G(X(\mathbb{C}), \mathbb{Z}(c))$ whose images in $H^{2c}(X(\mathbb{C}), \mathbb{Z})$ are Hodge, and that satisfy a topological condition described in [Benoist and Wittenberg 2020a, Definition 1.19] (and which is always satisfied if $X(\mathbb{R}) = \varnothing$), are algebraic.

**Corollary 4.16.** *There exists a smooth projective variety $X$ of dimension 4 over $\mathbb{R}$ such that $X$ fails the real integral Hodge conjecture but $X_{\mathbb{C}}$ satisfies the usual (complex) integral Hodge conjecture.*

*Proof.* Let $X$ and $x$ be as in Theorem 4.15. The class $x \in H^4_G(X(\mathbb{C}), \mathbb{Z}(2))$ is Hodge because it vanishes in $H^4(X(\mathbb{C}), \mathbb{Z}(2))$. It satisfies the topological condition appearing in the definition of the real integral Hodge conjecture because $X(\mathbb{R}) = \varnothing$. As $x$ is not algebraic, the variety $X$ does not satisfy the real integral Hodge conjecture.

Finally, as $H^*(Q^3(\mathbb{C}), \mathbb{Z})$ is algebraic and torsion-free and $H^{2*}(E(\mathbb{C}), \mathbb{Z})$ is algebraic, the Künneth formula implies that $H^{2*}(X(\mathbb{C}), \mathbb{Z})$ is algebraic, hence that $X_{\mathbb{C}}$ satisfies the integral Hodge conjecture. □

**Remarks 4.17.** (i) Three-dimensional counterexamples to the real integral Hodge conjecture are constructed in [Benoist and Wittenberg 2020a, Examples 4.7 and 4.8]. This is the smallest dimension possible by [Benoist and Wittenberg 2020a, Propositions 2.8 and 2.10].

(ii) The counterexamples to the real integral Hodge conjecture mentioned in (i) are explained by a failure of the complex integral Hodge conjecture. A 7-dimensional counterexample to the real integral Hodge conjecture for which the complex integral Hodge conjecture holds is described in [Benoist and Wittenberg 2020a, Example 2.5]. The fourfold of Theorem 4.15 is the first counterexample of this kind in dimension lesser or equal to 6. We do not know if some exist in dimension 3 (a question closely related to [Benoist and Wittenberg 2020a, Question 4.9]).

(iii) Over the nonarchimedean real closed field $R := \bigcup_n \mathbb{R}((t^{1/n}))$, there exist smooth projective threefolds failing the (analogue in this context of the) real integral Hodge conjecture, but for which the (analogue of the) complex integral Hodge conjecture holds, see [Benoist and Wittenberg 2020b, Examples 9.5, 9.6, 9.7, 9.10 and 9.22].

**4.6.** ***Applications to unramified cohomology.*** Let $X$ be a smooth variety over a field $k$. Let $\ell$ be a prime number invertible in $k$, and let

$$\mathrm{cl} : \mathrm{CH}^2(X) \otimes_{\mathbb{Z}} \mathbb{Z}_\ell \to H^4_{\mathrm{cont}}(X, \mathbb{Z}_\ell(2))$$

be Jannsen's cycle class map to continuous étale cohomology (see [Jannsen 1988]). (For the concrete fields considered in this article, such as algebraically closed fields, real closed fields, or finite fields, the continuous étale cohomology group $H^4_{\mathrm{cont}}(X, \mathbb{Z}_\ell(2))$ coincides with the $\ell$-adic cohomology group $H^4_{\text{ét}}(X, \mathbb{Z}_\ell(2)) := \varprojlim_m H^4_{\text{ét}}(X, \boldsymbol{\mu}_{\ell^m}^{\otimes 2})$; see [Jannsen 1988, Remark 3.5(c)].)

As was discovered by Colliot-Thélène and Voisin over the complex numbers [2012, théorème 3.7], and extended to arbitrary fields by Kahn [2012, théorème 1.1], the torsion subgroup $T$ of the cokernel of cl is a quotient of the third unramified cohomology group $H^3_{\mathrm{nr}}(X, \mathbb{Q}/\mathbb{Z}(2))$. It follows from the proofs of Theorems 4.2, 4.9, 4.12 and 4.15 that the smooth projective varieties $X$ considered there satisfy $T \neq 0$. As a consequence, one has $H^3_{\mathrm{nr}}(X, \mathbb{Q}/\mathbb{Z}(2)) \neq 0$ for any of these varieties.

## 5. Algebraizability of cohomology classes of $\mathcal{C}^\infty$ manifolds

We study the algebraizability of cohomology classes of compact $\mathcal{C}^\infty$ manifolds. We refer to Section 1.3 for context and for the relevant definitions.

**5.1.** ***Steenrod operations and algebraizable classes.*** Our main results (Theorems 5.3 and 5.6 below) both rely on the fact that Steenrod operations preserve algebraizable classes. This statement follows from the next proposition, which is due to Akbulut and King [1985, Theorem 6.6] at least when $X$ is projective. For the convenience of the reader, we give a short proof based on the relative Wu theorem of Atiyah and Hirzebruch [1961, Satz 3.2]. Our argument is also closely related to [Borel and Haefliger 1961, Proposition 5.19].

**Proposition 5.1.** *Let $X$ be a smooth variety over $\mathbb{R}$. Then $H^*_{\mathrm{alg}}(X(\mathbb{R}), \mathbb{Z}/2)$ is stable under mod 2 Steenrod operations.*

*Proof.* Let $g : Z \hookrightarrow X$ be an integral subvariety of codimension $c$ of $X$ and let $\nu : Y \to Z$ be a resolution of singularities of $Z$ (see [Hironaka 1964]). Set $f := g \circ \nu$ and denote by $f(\mathbb{R}) : Y(\mathbb{R}) \to X(\mathbb{R})$ the map induced between real point sets. Let $w(N_{f(\mathbb{R})})$ be the total Stiefel–Whitney class of the virtual normal bundle $N_{f(\mathbb{R})} := f^*T_{X(\mathbb{R})} - T_{Y(\mathbb{R})}$ of $f(\mathbb{R})$. The relative Wu theorem of Atiyah and Hirzebruch [1961, Satz 3.2] applied to the map $f(\mathbb{R})$ and to the class $1 \in H^0(Y(\mathbb{R}), \mathbb{Z}/2)$ (exactly as in [Benoist and Wittenberg 2020a, (1.60) and Remark 1.6(i)]) implies that

$$\mathrm{Sq}(f(\mathbb{R})_*1) = f(\mathbb{R})_* w(N_{f(\mathbb{R})}). \tag{5-1}$$

One has $w(T_{X(\mathbb{R})}) \in H^*_{\mathrm{alg}}(X(\mathbb{R}), \mathbb{Z}/2)$ and $w(T_{Y(\mathbb{R})}) \in H^*_{\mathrm{alg}}(Y(\mathbb{R}), \mathbb{Z}/2)$ (see [Bochnak and Kucharz 1998, Theorem 1.5] when $X$ and $Y$ are projective; in general, one may combine [Kahn 1987, Theorem 4] and [Benoist and Wittenberg 2020a, Theorem 1.18]). As algebraic classes are stable under pullbacks, proper pushforwards and cup-product (see [Bochnak and Kucharz 1998, Theorems 1.3 and 1.4] under projectivity hypotheses and [Benoist and Wittenberg 2020a, § 1.6.2] in general), the class (5-1) is algebraic. Since $H^*_{\mathrm{alg}}(X(\mathbb{R}), \mathbb{Z}/2)$ is generated by classes of the form $\mathrm{cl}_{\mathbb{R}}(Z) = f(\mathbb{R})_*1$, the proof is now complete. □

Proposition 5.1 implies at once the following corollary.

**Corollary 5.2.** *Let $M$ be a compact $\mathcal{C}^\infty$ manifold. The subset of $H^*(M, \mathbb{Z}/2)$ consisting of algebraizable classes is stable under mod 2 Steenrod operations.*

**5.2.** ***Examples of nonalgebraizable classes.*** Here is our first main result.

**Theorem 5.3.** *For all $c \geq 2$, there exists a compact $\mathcal{C}^\infty$ manifold $M$ and a class $x \in H^c(M, \mathbb{Z}/2)$ that is not algebraizable.*

*Proof.* Grant and Szücs [2013, Proof of Theorem 1.1, p. 334] have constructed a compact $\mathcal{C}^\infty$ manifold $M$ and a class $x \in H^c(M, \mathbb{Z}/2)$ such that $x^2$ is not in the image of the reduction modulo 2 map $H^{2c}(M, \mathbb{Z}) \to H^{2c}(M, \mathbb{Z}/2)$ (if $c$ is even), or such that $(\mathrm{Sq}^2\mathrm{Sq}^1x)^2$ is not in the image of the reduction modulo 2 map $H^{2c+6}(M, \mathbb{Z}) \to H^{2c+6}(M, \mathbb{Z}/2)$ (if $c$ is odd).

Akbulut and King [1993, Theorem A(b)] (see also [Krasnov 1994, Remark 4.8]) have shown that squares of algebraizable classes are reductions modulo 2 of integral classes. We deduce that $x$ is not algebraizable (if $c$ is even) and that $\mathrm{Sq}^2\mathrm{Sq}^1 x$ is not algebraizable (if $c$ is odd). Corollary 5.2 now shows that $x$ is not algebraizable even when $c$ is odd. □

**Remark 5.4.** When $c$ is even, the proof of Theorem 5.3 is exactly the one given in [Kucharz 2008, Theorem 1.13].

**5.3. *Examples of algebraizable classes.*** Before stating our second main result, we recall the properties of Thom spaces that will be used in its proof. Let $\mathrm{BO}(n)$ be the classifying space for $O(n)$, constructed as the increasing union of the Grassmannians $\mathrm{Grass}(n, \mathbb{R}^{n+m})$ of $n$-planes in $\mathbb{R}^{n+m}$ (for $m \geq 0$). Let $E(n)$ be the tautological rank $n$ topological real vector bundle on $\mathrm{BO}(n)$ ($\mathrm{BO}(n)$ and $E(n)$ are denoted by $G_n$ and $\gamma^n$ in [Milnor and Stasheff 1974, p. 63]). One has

$$H^*(\mathrm{BO}(n), \mathbb{Z}/2) = (\mathbb{Z}/2)[w_1, \dots, w_n], \tag{5-2}$$

where the $w_i$ are the Stiefel–Whitney classes of $E(n)$ (see [Milnor and Stasheff 1974, Theorem 7.1]).

Let $\mathrm{MO}(n)$ be the Thom space of $E(n)$ (defined in [Milnor and Stasheff 1974, p. 205]). Denoting by $s_n \in H^n(\mathrm{MO}(n), \mathbb{Z}/2)$ the Thom class of $E(n)$, the Thom isomorphism theorem [Milnor and Stasheff 1974, Theorem 10.2] shows that cup-product by $s_n$ induces an isomorphism $H^*(\mathrm{BO}(n), \mathbb{Z}/2) \xrightarrow{\sim} \widetilde{H}^{*+n}(\mathrm{MO}(n), \mathbb{Z}/2)$. By Thom's definition of Stiefel–Whitney classes (see [Milnor and Stasheff 1974, p. 91]), one has

$$\mathrm{Sq}^i(s_n) = w_i \cdot s_n. \tag{5-3}$$

In particular, one computes that

$$s_n^2 = \mathrm{Sq}^n(s_n) = w_n \cdot s_n. \tag{5-4}$$

Moreover, the Wu formula [Milnor and Stasheff 1974, Problem 8-B] shows that

$$\mathrm{Sq}^i(w_j) = \sum_{t=0}^{i} \binom{j+t-i-1}{t} w_{i-t} w_{j+t}, \tag{5-5}$$

where it is understood that $w_j = 0$ in $H^*(\mathrm{BO}(n), \mathbb{Z}/2)$ for $j > n$. These pieces of information and Cartan's formula entirely determine $H^*(\mathrm{MO}(n), \mathbb{Z}/2)$ as an algebra endowed with an action of the mod 2 Steenrod algebra.

Let $M$ be a compact $\mathcal{C}^\infty$ manifold and let $x \in H^n(M, \mathbb{Z}/2)$ be a cohomology class. It follows from the Thom–Pontryagin construction [Thom 1954, théorème II.1] that $x$ is Poincaré dual to the fundamental class of a $\mathcal{C}^\infty$ submanifold of codimension $n$ in $M$ if and only if there exists a continuous map $f : M \to \mathrm{MO}(n)$ such that $x = f^* s_n$.

We also include a proof of the following well-known lemma for lack of a suitable reference.

**Lemma 5.5.** *Let $K$ be a finite CW-complex. For all $N \geq 0$, there exists a compact $\mathcal{C}^\infty$ manifold $M$ and an $N$-equivalence $f : M \to K$.*

*Proof.* By [Hatcher 2002, Theorem 2C.5], we may assume that $K$ is a finite simplicial complex. Viewing $K$ as a subcomplex of a simplex of high dimension, we may embed it piecewise linearly in $\mathbb{R}^d$ for $d \gg 0$. Let $\Theta \subset \mathbb{R}^d$ be a neighbourhood of $K$ that is a smooth regular neighbourhood as in [Hirsch 1962, Theorem 1(a) and (a′)], and set $M := \partial\Theta$. It follows from [Hudson and Zeeman 1964, Corollary 1, p. 725] that the inclusion $M \hookrightarrow \Theta \setminus K$ is a weak homotopy equivalence, the inclusion $\Theta \setminus K \hookrightarrow \Theta$ is an $N$-equivalence because $K$ has high codimension in $\Theta$ when $d \gg 0$, and the inclusion $K \hookrightarrow \Theta$ is a homotopy equivalence by choice of $\Theta$. These facts prove the lemma. □

We now reach the goal of this section.

**Theorem 5.6.** *There exists a compact $\mathcal{C}^\infty$ manifold $M$ and an algebraizable class $x \in H^5(M, \mathbb{Z}/2)$ that does not belong to the subring $A(M)$ of $H^*(M, \mathbb{Z}/2)$ generated by Stiefel–Whitney classes of vector bundles on $M$ and Poincaré duals of fundamental classes of $\mathcal{C}^\infty$ submanifolds of $M$.*

*Proof.* Let $K$ be the Thom space of the tautological rank $n$ vector bundle on $\mathrm{Grass}(3, \mathbb{R}^{3+m})$. Comparing [Milnor and Stasheff 1974, Theorem 7.1 and Problem 7-B] and applying the Thom isomorphism theorem [Milnor and Stasheff 1974, Theorem 10.2] shows that the restriction map $H^*(\mathrm{MO}(3), \mathbb{Z}/2) \to H^*(K, \mathbb{Z}/2)$ is an isomorphism in degree at most 18 if $m \gg 0$. As $K$ is a finite CW-complex, it follows from Lemma 5.5 that there exists a compact $\mathcal{C}^\infty$ manifold $M$ and a continuous map $f : M \to \mathrm{MO}(3)$ such that the pullback map $f^* : H^*(\mathrm{MO}(3), \mathbb{Z}/2) \to H^*(M, \mathbb{Z}/2)$ is an isomorphism in degree at most 18.

By the Thom–Pontryagin construction, the class $f^*s_3$ is Poincaré dual to the fundamental class of a $\mathcal{C}^\infty$ submanifold of $M$, hence is algebraizable by [Benedetti and Tognoli 1980, Corollary 4.5] or [Akbulut and King 1981, Theorem 1.1]. It follows Corollary 5.2 that the class $x := f^*(w_2 \cdot s_3) \in H^5(M, \mathbb{Z}/2)$, which is equal to $\mathrm{Sq}^2(f^*s_3)$ by (5-3), is algebraizable.

Assume for contradiction that $x \in A(M)$. As $H^q(M, \mathbb{Z}/2) = 0$ for $q \in \{1, 2\}$ and $H^5(M, \mathbb{Z}/2)$ is generated by $x = f^*(w_2 \cdot s_3)$ and by $y := f^*((w_1^2 + w_2) \cdot s_3)$, we deduce that there exists $z \in \{x, y\}$ that is either the fifth Stiefel–Whitney class of a vector bundle $E$ on $M$, or Poincaré dual to the fundamental class of a $\mathcal{C}^\infty$ submanifold of codimension 5 in $M$. In the first case, one has

$$\mathrm{Sq}^1(z) = \mathrm{Sq}^1(w_5(E)) = w_1(E) \cdot w_5(E) = 0$$

in $H^6(M, \mathbb{Z}/2)$, where the second equality follows from (5-5) and the third one holds since $H^1(M, \mathbb{Z}/2) = 0$. One then computes (using the Cartan formula, (5-3) and (5-5)) that $\mathrm{Sq}^1(x) = f^*(w_3 \cdot s_3)$ and $\mathrm{Sq}^1(y) = f^*((w_1^3 + w_3) \cdot s_3)$. This is a contradiction since both classes are nonzero by the computation of $H^*(\mathrm{MO}(3), \mathbb{Z}/2)$ recalled above and by the injectivity of $f^*$ in degree 6.

In the second case, the Thom–Pontryagin construction shows the existence of a continuous map $g : M \to \mathrm{MO}(5)$ with $z = g^*s_5$. Using the Cartan formula, (5-3) and (5-5), one computes that

$$\mathrm{Sq}^2\mathrm{Sq}^1(s_5) \cdot s_5^2 = \mathrm{Sq}^2(w_1 \cdot s_5) \cdot s_5^2 = w_1^3 \cdot s_5^3 + w_1 w_2 \cdot s_5^3 = \mathrm{Sq}^1(s_5)^3 + \mathrm{Sq}^1(s_5) \cdot \mathrm{Sq}^2(s_5) \cdot s_5.$$

Pulling back this equality by the map $g$, we deduce that

$$\mathrm{Sq}^2\mathrm{Sq}^1(z) \cdot z^2 + \mathrm{Sq}^1(z)^3 + \mathrm{Sq}^1(z) \cdot \mathrm{Sq}^2(z) \cdot z = 0 \tag{5-6}$$

in $H^{18}(M,\mathbb{Z}/2)$. Using the Cartan formula, (5-3), (5-4) and (5-5), one shows that

$$\begin{aligned}\mathrm{Sq}^2\mathrm{Sq}^1(x)\cdot x^2+\mathrm{Sq}^1(x)^3+\mathrm{Sq}^1(x)\cdot\mathrm{Sq}^2(x)\cdot x&\\ &=f^*(w_1^2w_3s_3\cdot(w_2s_3)^2+(w_3s_3)^3+w_3s_3\cdot(w_1^2w_2s_3+w_1w_3s_3)\cdot w_2s_3)\\ &=f^*((w_1w_2w_3^4+w_3^5)\cdot s_3),\end{aligned}$$

and a similar lengthier computation shows that

$$\mathrm{Sq}^2\mathrm{Sq}^1(y)\cdot y^2+\mathrm{Sq}^1(y)^3+\mathrm{Sq}^1(y)\cdot\mathrm{Sq}^2(y)\cdot y=f^*((w_1^3w_2^3w_3^2+w_1^2w_2^2w_3^3+w_1w_2w_3^4+w_3^5)\cdot s_3).$$

As both these classes are nonzero by the computation of $H^*(\mathrm{MO}(3),\mathbb{Z}/2)$ recalled above and by the injectivity of $f^*$ in degree 18, this contradicts (5-6). □


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