STEIN SPACES AND STEIN ALGEBRAS

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ABSTRACT. We prove that the category of Stein spaces and holomorphic maps is anti-equivalent to the category of Stein algebras and $\mathbb{C}$-algebra morphisms. This removes a finite dimensionality hypothesis from a theorem of Forster.

INTRODUCTION

Complex spaces are a generalization of complex manifolds allowing singularities, and as such are the basic objects of study in complex-analytic geometry. Formally, they are defined to be $\mathbb{C}$-ringed spaces that are locally isomorphic to model spaces defined by the vanishing of finitely many holomorphic functions in a domain of $\mathbb{C}^N$ for some $N \geq 0$ (see [GR84, 1, §1.5]). We assume that they are second-countable, but not necessarily reduced or finite-dimensional.

A complex space S is said to be *Stein* if $H^k(S, \mathcal{F}) = 0$ for all coherent sheaves $\mathcal{F}$ on S and all $k > 0$ (see [GR79]). Stein spaces are the complex-analytic analogues of affine algebraic varieties. For instance, the Stein spaces of finite embedding dimension are exactly those complex spaces that may be realized as closed complex subspaces of $\mathbb{C}^N$ for some $N \geq 0$ (see [Nar60, Theorem 6]).

If S is a complex space, the $\mathbb{C}$-algebra $\mathcal{O}(S)$ of holomorphic functions on S carries a canonical Fréchet topology (see [GR79, V, §6]). A topological $\mathbb{C}$-algebra of the form $\mathcal{O}(S)$ for some Stein space S is called a *Stein algebra*.

In algebraic geometry, the anti-equivalence of categories between affine varieties over $\mathbb{C}$ and $\mathbb{C}$-algebras of finite type is a basic tool to study affine algebraic varieties. Our main theorem is a counterpart of this result in complex-analytic geometry.

Theorem 0.1 (Theorem 3.3)**.** *The contravariant functor*

$$\left\{ \begin{array}{c} \textit{Stein spaces} \\ \textit{and holomorphic maps} \end{array} \right\} \to \left\{ \begin{array}{c} \textit{Stein algebras} \\ \textit{and } \mathbb{C}\textit{-algebra morphisms} \end{array} \right\} \tag{0.1}$$

given by $S \mapsto \mathcal{O}(S)$ is an anti-equivalence of categories.

Very significant particular cases of Theorem 0.1 were previously known. First, Forster has shown in [For67, Satz 1] that Theorem 0.1 holds if one replaces the right-hand side of (3.1) by the category of Stein algebras and continuous $\mathbb{C}$-algebra morphisms. From this point of view, our contribution is an automatic continuity result for morphisms of Stein algebras (see Theorem 3.2 below).

Second, Forster has proven this automatic continuity result in restriction to finite-dimensional Stein spaces (see [For66, Theorem 5]). In particular, Theorem 0.1 was already known in restriction to finite-dimensional Stein spaces and their associated Stein algebras. Forster's theorem was later generalized by Markoe [Mar73] and Ephraim [Eph78, Theorem 2.3] who made weaker finite dimensionality assumptions. Our contribution is to remove these finite dimensionality hypotheses altogether. This problem was raised by Forster in [For66, Remark p.162].

Our strategy to prove Theorem 0.1 is to reduce to the finite-dimensional case treated by Forster by means of the next theorem.

Theorem 0.2 (Theorem 2.1)**.** *Let S be a Stein space. Then there exists a holomorphic map $f : S \to \mathbb{C}^2$ all of whose fibers are finite-dimensional.*

Our proof of Theorem 0.2 is an application of Oka theory. It uses in a crucial way new examples of Oka manifolds constructed by Forstnerič and Wold [FW24] (based on and extending earlier work of Kusakabe [Kus21, Kus24]), as well as an extension theorem for holomorphic maps from Stein spaces to Oka manifolds due to Forstnerič [For05, For17].

We note that Theorem 0.2 is optimal in the sense that there may not exist a holomorphic map $f : S \to \mathbb{C}$ with finite-dimensional fibers (see Proposition 2.3). An earlier version of this article, relying on the Oka manifolds constructed by Kusakabe [Kus24, Theorem 1.6], only produced such a map with values in $\mathbb{C}^3$. We are grateful to Franc Forstnerič for drawing our attention to the article [FW24], thereby allowing us to prove Theorem 0.2 in the form stated above.

The results of Oka theory that we need are gathered in Section 1. These tools are used to prove Theorem 0.2 in Section 2. In Section 3, we deduce Theorem 0.1 from Theorem 0.2 and from Forster's works [For66, For67].

1. Tools from Oka theory

We recall that a complex manifold Y is said to be *Oka* if for all convex compact subsets $K \subset \mathbb{C}^N$ and all open neighborhoods Ω of K in $\mathbb{C}^N$, any holomorphic map $\Omega \to Y$ can be approximated uniformly on K by holomorphic maps $\mathbb{C}^N \to Y$ (see [For09, Definition 1.2]).

We now introduce the Oka manifolds of interest to us. For $r \in \mathbb{R}$, define

$$Y_r := \{(z_1, z_2) \in \mathbb{C}^2 \mid \mathrm{Im}(z_2) < |z_1|^2 + \mathrm{Re}(z_2)^2 + r\}.$$

The next proposition is a particular case of a theorem of Forstnerič and Wold [FW24, Corollary 1.5] (pointed out in [FW24, (1.2)]).

Proposition 1.1. *For $r \in \mathbb{R}$, the complex manifold Y_r is Oka.*

The following easy lemma implies in particular that Y_r is contractible.

Lemma 1.2. *Fix $r \in \mathbb{R}$. There is a homotopy $(h_t)_{t\in[0,1]} : \mathbb{C}^2 \to \mathbb{C}^2$ inducing strong deformation retractions of both $\mathbb{C}^2$ and Y_r onto $\{(z_1, z_2) \in \mathbb{C}^2 \mid \mathrm{Im}(z_2) \leq r - 1\}$.*

Proof. The homotopy $(h_t)_{t\in[0,1]}$ defined by

$$\begin{aligned} h_t(z_1, z_2) &= (z_1, z_2 - it(\mathrm{Im}(z_2) - r + 1)) && \text{if } \mathrm{Im}(z_2) \geq r - 1 \\ h_t(z_1, z_2) &= (z_1, z_2) && \text{if } \mathrm{Im}(z_2) \leq r - 1 \end{aligned}$$

has the required properties.

We will make use of the Oka property and of the contractibility of Y_r through the next extension result, which is an application of theorems of Forstnerič (see [For05, Theorem 1.1] and the more general [For17, Theorem 5.4.4]).

Proposition 1.3. *Fix $r \in \mathbb{R}$. Let S be a reduced Stein space and let S' be a (possibly nonreduced) closed complex subspace of S. Let $f' : S' \to Y_r$ be a holomorphic map. Then there exists a holomorphic map $f : S \to Y_r$ with $f|_{S'} = f'$.*

Proof. Since S is Stein, the restriction map $\mathcal{O}(S) \to \mathcal{O}(S')$ is onto. It follows that there exists a holomorphic map $f_1 : S \to \mathbb{C}^2$ such that $f_1|_{S'} = f'$.

Define $U := f_1^{-1}(Y_r)$. It is an open neighborhood of S' in S. Let $Z \subset U$ be a closed neighborhood of S' in U. By the Tietze–Urysohn extension theorem, there exists a continuous map $\tau : S \to [0,1]$ which is equal to 0 on Z and to 1 on $S \setminus U$.

Define a continuous map $f_2 : S \to Y_r$ by the formula $f_2(s) = h_{\tau(s)}(f_1(s))$, where $(h_t)_{t\in[0,1]}$ is the homotopy given by Lemma 1.2. Since f_2 is equal to f_1 on U, it is holomorphic in a neighborhood of S' and satisfies $f_2|_{S'} = f'$.

As Y_r is Oka by Proposition 1.1, it now follows from the jet interpolation part of [For17, Theorem 5.4.4] (applied with π equal to be the first projection map $S \times Y \to S$ and with $\mathcal{S}$ equal to the ideal sheaf of S' in S) that f_2 is homotopic to a holomorphic map $f : S \to Y_r$ with $f|_{S'} = f_2|_{S'}$, and hence $f|_{S'} = f'$. This completes the proof of the proposition. □

2. Holomorphic maps with finite-dimensional fibers

The next theorem is the key to our main results.

Theorem 2.1. *Let S be a Stein space. Then there exists a holomorphic map $f : S \to \mathbb{C}^2$ all of whose fibers are finite-dimensional.*

Proof. Let S^{red} be the reduction of S. Since S is Stein, the restriction map $\mathcal{O}(S) \to \mathcal{O}(S^{\mathrm{red}})$ is onto, and we may assume that S is reduced.

Let $(S_k)_{0\leq k<n}$ with $n \in \mathbb{N} \cup \{+\infty\}$ be the irreducible components of S, viewed as reduced closed complex subspaces of S. Let Θ be the collection of all reduced and irreducible closed complex subspaces of S that may be obtained as irreducible components of an intersection of finitely many of the S_k. The set Θ is at most countable, and any compact subset of S meets at most finitely many elements of Θ.

For $d \geq 0$, we let $\Theta_d \subset \Theta$ be the set of all d-dimensional elements of Θ. Let $(Z_{d,j})_{0\leq j<m(d)}$ with $m(d) \in \mathbb{N} \cup \{+\infty\}$ be an enumeration of the elements of Θ_d. We henceforth identify Θ with the set of all pairs (d,j) with $d \geq 0$ and $0 \leq j < m(d)$ and endow it with the lexicographical order. It is a well-ordered set. For all $(d,j) \in \Theta$, we view $W_{d,j} := \cup_{(d',j')\leq(d,j)} Z_{d',j'}$ and $W'_{d,j} := \cup_{(d',j')<(d,j)} Z_{d',j'}$ as reduced closed complex subspaces of S. Finally, for $(d,j) \in \Theta$, we let $r(d,j)$ be the biggest integer $k \geq 1$ such that $Z_{d,j} \subset S_k$.

We will now construct holomorphic functions $f_{d,j} : W_{d,j} \to \mathbb{C}^2$ for all $(d,j) \in \Theta$ with the property that $f_{d,j}|_{W_{d',j'}} = f_{d',j'}$ and $f_{d,j}(Z_{d',j'}) \subset Y_{r(d',j')}$ whenever $(d',j') \leq (d,j)$. The construction is by induction on the pair $(d,j) \in \Theta$ (which is legitimate since Θ is well-ordered).

Assume that the $f_{d',j'}$ for $(d',j') < (d,j)$ have been constructed. Since these maps are compatible, they glue to give rise to a holomorphic map $f'_{d,j} : W'_{d,j} \to \mathbb{C}^2$. Now $W_{d,j} = W'_{d,j} \cup Z_{d,j}$. Define $V_{d,j} := W'_{d,j} \cap Z_{d,j}$. It is a possibly nonreduced closed complex subspace of S. Note that $V_{d,j}$ is set-theoretically a union of some of the $Z_{d',j'}$ with $(d',j') < (d,j)$. If $Z_{d',j'} \subset V_{d,j}$ is one of them, then $Z_{d',j'} \subset Z_{d,j}$ and hence $r(d',j') \geq r(d,j)$. Since $f_{d',j'}(Z_{d',j'}) \subset Y_{r(d',j')} \subset Y_{r(d,j)}$, we deduce that $f'_{d,j}(V_{d,j}) \subset Y_{r(d,j)}$. Proposition 1.3 now implies that the holomorphic map $f'_{d,j}|_{V_{d,j}} : V_{d,j} \to Y_{r(d,j)}$ extends to a holomorphic map $f''_{d,j} : Z_{d,j} \to Y_{r(d,j)}$. Since $f'_{d,j}$ and $f''_{d,j}$ coincide on $V_{d,j} = W'_{d,j} \cap Z_{d,j}$, they glue (by Lemma 2.2 below) to give rise to a holomorphic map $f_{d,j} : W_{d,j} \to \mathbb{C}^2$ with the required properties.

As the $(f_{d,j})_{(d,j)\in\Theta}$ are compatible, they induce a holomorphic map $f : S \to \mathbb{C}^2$. Let us verify that this map has the required property. One has $f(S_k) \subset Y_k$ for all $0 \leq k < n$ (as S_k is one of the $Z_{d,j}$). Since the $(Y_k)_{k\geq 0}$ form a decreasing family of subsets of $\mathbb{C}^2$ with empty intersection, we deduce that any point of $\mathbb{C}^2$ belongs to at most finitely many of the $f(S_k)$. In other words, any fiber of f intersects at most finitely many of the S_k. It follows that all the fibers of f are finite-dimensional. □

Lemma 2.2. *Let S be a complex space. Let S_1 and S_2 be closed complex subspaces of S. Set $T := S_1 \cap S_2$. The following diagram of sheaves on S is exact:*

$$\mathcal{O}_S \xrightarrow{f\mapsto(f|_{S_1},f|_{S_2})} \mathcal{O}_{S_1} \oplus \mathcal{O}_{S_2} \xrightarrow{(g,h)\mapsto g|_T - h|_T} \mathcal{O}_T \to 0. \tag{2.1}$$

If moreover S is reduced and $S = S_1 \cup S_2$, then the left arrow of (2.1) is injective.

Proof. Fix $s \in S$. Write $A = \mathcal{O}_{S,s}$ and let I_1 (resp. I_2) be the ideal of A consisting of germs of functions vanishing on S_1 (resp. S_2). Then the exactness of (2.1) at s results from the exactness of $A \to A/I_1 \oplus A/I_2 \to A/\langle I_1, I_2\rangle \to 0$, which is valid for any two ideals I_1 and I_2 of a commutative ring A.

If $S = S_1 \cup S_2$, then a holomorphic function in the kernel of the left arrow of (2.1) vanishes at all points and hence vanishes if S is reduced. □

The next proposition shows the optimality of Theorem 2.1.

Proposition 2.3. *There exists a Stein space S such that all holomorphic maps $f : S \to \mathbb{C}$ admit an infinite-dimensional fiber.*

Proof. For $n \geq 1$, set $S_n := \mathbb{C}^n$. Define $T_n := \{(z_1, \ldots, z_n) \in S_n \mid z_n = 0\}$ and $T'_n := \{(z_1, \ldots, z_{n+1}) \in S_{n+1} \mid z_n = 0 \text{ and } z_{n+1} = 1\}$. Let $\varphi_n : T_n \xrightarrow{\sim} T'_n$ be the isomorphism given by $\varphi_n(z_1, \ldots, z_{n-1}, 0) = (z_1, \ldots, z_{n-1}, 0, 1)$. Let S be the complex space obtained from $\sqcup_{n\geq 1} S_n$ by gluing S_n and S_{n+1} transversally along T_n and T'_n by means of φ_n (for all $n \geq 1$). The complex space S is Stein because so is its normalization $\sqcup_{n\geq 1} S_n$ (see [Nar62, Theorem 1]).

Let $f : S \to \mathbb{C}$ be a holomorphic map. Assume first that $f|_{S_n}$ is constant for all $n \gg 0$. As the subset $S_n \cap S_{n+1}$ of S is nonempty, the value taken by $f|_{S_n}$ does not depend on $n \gg 0$. It follows that f has a (single) infinite-dimensional fiber.

Assume now that the set $\Sigma := \{n \in \mathbb{N}_{\geq 1} \mid f|_{S_n} \text{ is not constant}\}$ is infinite. For all $n \in \Sigma$, the map $f|_{S_n} : S_n \to \mathbb{C}$ omits at most one value, by Picard's little theorem. We deduce that at most one complex number is not the image of $f|_{S_n}$ for all but finitely many $n \in \Sigma$. Consequently, all complex numbers except possibly one are in the image of infinitely of the $f|_{S_n}$. As the nonempty fibers of $f|_{S_n}$ have dimension $\geq n-1$, we deduce that all the fibers of f except possibly one are infinite-dimensional. □

3. Morphisms of Stein algebras

Theorem 3.1. *Let S be a Stein space. Let $\chi : \mathcal{O}(S) \to \mathbb{C}$ be a $\mathbb{C}$-algebra morphism. Then χ is continuous and there exists $s \in S$ such that $\chi(f) = f(s)$ for all $f \in \mathcal{O}(S)$.*

Proof. Let $f : S \to \mathbb{C}^2$ be as in Theorem 2.1. Let $(f_i)_{1\leq i\leq 2}$ be the components of f. Set $\lambda_i := \chi(f_i) \in \mathbb{C}$. Let $T \subset S$ be the closed complex subspace defined by the equations $\{f_i = \lambda_i\}_{1\leq i\leq 2}$. Let $r_{S,T} : \mathcal{O}(S) \to \mathcal{O}(T)$ be the restriction map, which is continuous by [GR79, V, §6.4 Theorem 6]. By [Eph78, Lemma 1.7], there exists a morphism of $\mathbb{C}$-algebras $\chi_T : \mathcal{O}(T) \to \mathbb{C}$ such that $\chi = \chi_T \circ r_{S,T}$.

Our choice of f implies that T is a finite-dimensional Stein space. It therefore follows from Forster's theorem [For66, Theorem 5] that χ_T is continuous and hence that so is χ. Another theorem of Forster [For67, Satz 1] then implies that there exists $s \in S$ such that $\chi(f) = f(s)$ for all $f \in \mathcal{O}(S)$. ☐

Theorem 3.2. *Any $\mathbb{C}$-algebra morphism between Stein algebras is continuous.*

Proof. Let S and S' be two Stein spaces, and let $\xi : \mathcal{O}(S') \to \mathcal{O}(S)$ be a $\mathbb{C}$-algebra morphism. Fix a finitely generated maximal ideal $\mathfrak{m} \subset \mathcal{O}(S)$. There exists $s \in S$ such that $\mathfrak{m} = \{f \in \mathcal{O}(S) \mid f(s) = 0\}$ (see e.g. [GR79, V, §7.1, statement above Theorem 1]). Evaluation at s therefore induces an isomorphism $\mathcal{O}(S)/\mathfrak{m} \xrightarrow{\sim} \mathbb{C}$. We let $\chi : \mathcal{O}(S) \to \mathbb{C}$ be the induced map.

Apply Theorem 3.3 to the $\mathbb{C}$-algebra morphism $\chi \circ \xi : \mathcal{O}(S') \to \mathbb{C}$. We deduce the existence of $s' \in S'$ such that $\chi \circ \xi(f) = f(s')$ for all $f \in \mathcal{O}(S')$. It then follows that $\xi^{-1}(\mathfrak{m}) = \{f \in \mathcal{O}(S') \mid f(s') = 0\}$. This maximal ideal is closed (by continuity of the evaluation map $f \mapsto f(s')$), and hence finitely generated by [For67, Theorem 2].

Since $\mathfrak{m}$ was arbitrary, the continuity of ξ is now an application of the criterion given in [For67, Theorem 3]. ☐

Theorem 3.3. *The contravariant functor*

$$\left\{ \begin{array}{c} \textit{Stein spaces} \\ \textit{and holomorphic maps} \end{array} \right\} \to \left\{ \begin{array}{c} \textit{Stein algebras} \\ \textit{and } \mathbb{C}\textit{-algebra morphisms} \end{array} \right\} \tag{3.1}$$

given by $S \mapsto \mathcal{O}(S)$ is an anti-equivalence of categories.

Proof. Since $\mathbb{C}$-algebra morphisms of Stein algebras are automatically continuous by Theorem 3.2, the theorem is equivalent to [For67, Satz 1]. ☐

We finally record the following consequence of Theorem 3.3 for later use in [Ben24]. If S is a Stein space, we let $\lambda_S : S \to \mathrm{Spec}(\mathcal{O}(S))$ be the unique morphism of locally ringed spaces such that $\lambda_S^* : \mathcal{O}(S) \to \mathcal{O}(S)$ is the identity (see [SP, Lemma 01I1]).

Proposition 3.4. *Let X be a complex space and let S be a Stein space. The map*

$$\left\{ \begin{array}{c} \textit{holomorphic maps} \\ X \to S \end{array} \right\} \to \left\{ \begin{array}{c} \textit{morphisms of } \mathbb{C}\textit{-locally ringed spaces} \\ X \to \mathrm{Spec}(\mathcal{O}(S)) \end{array} \right\} \tag{3.2}$$

given by $f \mapsto \lambda_S \circ f$ is a bijection.

Proof. As the statement is local on X, we may assume that X is Stein. In this case, the proposition follows from Theorem 3.3 since the global sections functor induces a bijection between the set of morphisms of $\mathbb{C}$-locally ringed spaces $X \to \mathrm{Spec}(\mathcal{O}(S))$ and the set of $\mathbb{C}$-algebra morphisms $\mathcal{O}(S) \to \mathcal{O}(X)$ (see [SP, Lemma 01I1]). ☐

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