On Hilbert's 17th problem in low degree

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A polynomial $f \in \mathbb{R}[X_1, \ldots, X_n]$ is said to be ≥ 0 if $f(x_1, \ldots, x_n) \geq 0$ for every $(x_1, \ldots, x_n) \in \mathbb{R}^n$. The topic of Hilbert's 17th problem is to try to explain the positivity of f by writing it as a sum of squares. It was understood by Hilbert that there is no hope to write f as a sum of squares of polynomials in general, and that one should consider sums of squares of rational functions instead. This question was settled by Artin [1]:

Theorem 1 (Artin)**.** *A polynomial $f \in \mathbb{R}[X_1, \ldots, X_n]$ that is ≥ 0 is a sum of squares in $\mathbb{R}(X_1, \ldots, X_n)$.*

This result was later improved by Pfister [8], who realized that the number of squares needed only depends on the number of variables:

Theorem 2 (Pfister)**.** *A polynomial $f \in \mathbb{R}[X_1, \ldots, X_n]$ that is ≥ 0 is a sum of 2^n squares in $\mathbb{R}(X_1, \ldots, X_n)$.*

Proving that Pfister's result is optimal (i.e. showing that it is not possible to improve on the bound 2^n) may be the most important related open problem [9, §4 Problem 1]. If $n = 1$, it is obviously optimal, because $X_1^2 + 1$ is not a square.

In two variables, it is also known that Pfister's result is optimal: Cassels, Ellison and Pfister [4] have shown that the polynomial $1 + X_1^2X_2^4 + X_1^4X_2^2 - 3X_1^2X_2^2$ is ≥ 0, but not a sum of 3 squares in $\mathbb{R}(X_1, X_2)$.

When $n \geq 3$, this question is completely open.

We explore another direction: is it possible to improve on Pfister's result, when the degree d of f is low ? Two results were previously known. One is very easy: if $f \in \mathbb{R}[X_1, \ldots, X_n]$ is ≥ 0 of degree 2, diagonalization of quadratic forms shows that it is a sum of $n + 1$ squares. The other is due to Hilbert [7]: a degree 4 polynomial in $\mathbb{R}[X_1, X_2]$ that is ≥ 0 is a sum of 3 squares.

Our main result [2, Theorem 0.1] generalizes this last theorem in more variables:

Theorem 3. *Let $n \geq 2$. A polynomial $f \in \mathbb{R}[X_1, \ldots, X_n]$ of degree $d \leq 2n$ that is ≥ 0 is a sum of $2^n - 1$ squares in $\mathbb{R}(X_1, \ldots, X_n)$, with possible exceptions if $n \geq 7$ is odd and $d = 2n$.*

The particular case with 3 variables is new when $d = 4$ or $d = 6$:

Corollary 1. *A polynomial $f \in \mathbb{R}[X_1, X_2, X_3]$ that is ≥ 0 and of degree ≤ 6 is a sum of 7 squares in $\mathbb{R}(X_1, X_2, X_3)$.*

Another reason why Theorem 3 is interesting lies in the expectation that the bound $d \leq 2n$ on the degree is the best possible, and that from $d \geq 2n + 2$ on, there should exist polynomials achieving Pfister's bound.

The geometric proof of Theorem 3 uses an algebraic variety X naturally associated to f, extending to a higher number of variables arguments that have been used by Colliot-Thélène [5] when $n = 2$. Let $F \in \mathbb{R}[X_0, \dots, X_n]$ be the homogenization of f, and introduce $X := \{Y^2 + F(X_0, \dots, X_n) = 0\}$: a real algebraic variety that is a double cover of $\mathbb{P}^n_{\mathbb{R}}$ ramified over the hypersurface $\{F = 0\}$.

The first step of the proof is to reformulate Theorem 3 into a geometric statement about X. Using the work of Pfister on multiplicative quadratic forms and Voevodsky's proof of the Milnor conjecture, one proves that f being a sum of $2^n - 1$ squares is equivalent to the cohomology class $\{-1\}^n \in H^n(X, \mathbb{Z}/2\mathbb{Z})$ being of coniveau 1, that is vanishing on a non-empty Zariski open subset of X.

A key idea is to work with cohomology with integral coefficients (say, 2-adic cohomology) instead of the mod 2 coefficients that come out of the theory of quadratic forms. The class $\{-1\} \in H^1(\mathbb{R}, \mathbb{Z}/2\mathbb{Z})$ lifts uniquely to a class $\omega \in H^1(\mathbb{R}, \mathbb{Z}_2(1))$, and it turns out that f being a sum of 2^n-1 squares is also equivalent to $\omega^n \in H^n(X, \mathbb{Z}_2(n))$ being of coniveau 1. Equivalently, we need to show that ω^n vanishes in the unramified cohomology group $H^n_{\mathrm{nr}}(X, \mathbb{Z}_2(n))$.

The main tool we use to prove it is Bloch-Ogus theory [3].

Suppose first that X is smooth. Then, an important point in the analysis is the vanishing of the unramified cohomology group $H^n_{\mathrm{nr}}(X_{\mathbb{C}}, \mathbb{Z}_2)$ [6, Proposition 3.3]. More precisely, that it has no torsion is a consequence of the Milnor conjecture (this is the argument for which it is crucial to work with integral coefficients) and it is torsion by decomposition of the diagonal (this uses that $X_{\mathbb{C}}$ is rationally connected for $d \leq 2n$: it is the only place where this degree hypothesis is used).

Together with explicit computations for the 2-adic cohomology of X, that $H^n_{\mathrm{nr}}(X_{\mathbb{C}}, \mathbb{Z}_2) = 0$ allows us to obtain the required vanishing of $\omega^n \in H^n_{\mathrm{nr}}(X, \mathbb{Z}_2(n))$.

Finally, to deal with the case where X is singular, we reduce to the case where X is smooth using a degeneration argument. To implement this argument, it is necessary to run the whole proof over an arbitrary real closed field, and not only over the field $\mathbb{R}$ of real numbers.

References

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