A real period-index theorem

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A field K is said to be C_i if every degree d hypersurface $X \subset \mathbb{P}^N_K$ with $d^i \leq N$ has a K-point. A key example of C_i fields is given by the Tsen–Lang theorem [5].

Theorem 1 (Tsen–Lang). *If B is an integral complex variety of dimension i, the field $\mathbb{C}(B)$ is C_i.*

Geometrically, this means that hypersurface fibrations over complex varieties have a rational section, if the degree of the hypersurfaces is low enough. When the base B is a curve, the inequality $d \leq N$ exactly means that the hypersurfaces are in the Fano range. This suggests that one might expect a more general statement, for rationally connected fibrations: this is the Graber–Harris–Starr theorem [4].

Theorem 2 (Graber–Harris–Starr). *If C is an integral complex curve, every rationally connected variety X over $\mathbb{C}(C)$ has a $\mathbb{C}(C)$-point.*

Both theorems fail badly if $\mathbb{C}$ is replaced by the field $\mathbb{R}$ of real numbers, as some real points of the base might not lift to real points of the total space of the fibration. For instance, the hypersurface $X := \{X_0^2 + \cdots + X_N^2 = 0\} \subset \mathbb{P}^N_{\mathbb{R}}$ over $B = \mathrm{Spec}(\mathbb{R})$ has no $\mathbb{R}$-point. Lang suggested in [6, p. 379] that one might still obtain correct statements if the base has no real points (thus expressing the hope that real varieties with no real points behave as complex varieties).

Conjecture 1 (Lang). *If B is an integral real variety of dimension i such that $B(\mathbb{R}) = \emptyset$, the field $\mathbb{R}(B)$ is C_i.*

By analogy with the complex situation, when B is a curve, it is natural to formulate a real variant of the Graber–Harris–Starr theorem:

Conjecture 2 (Manin, Kollár). *If C is an integral real curve with $C(\mathbb{R}) = \emptyset$, every rationally connected variety X over $\mathbb{R}(C)$ has a $\mathbb{R}(C)$-point.*

Applying Conjecture 2 when the base C is the real conic with no real points, and when X is defined over $\mathbb{R}$ (i.e. when the fibration is trivial) would answer positively the following question of Kollár:

Conjecture 3 (Kollár). *Every rationally connected variety over $\mathbb{R}$ contains a geometrically integral curve of geometric genus* 0.

Very little is known concerning these conjectures. Lang had answered positively Conjecture 1 for odd degree hypersurfaces [6, p. 390], mimicking the proof of Theorem 1 and taking advantage of the fact real polynomials of odd degree have a real root. Conjecture 2 has been answered positively by Steinberg [8] for compactifications of varieties that are homogeneous under the action of a connected linear algebraic group. The case of conics (that is Conjecture 1 for $i = 1$ and $d = 2$) was already known to Witt [9, Satz 22]. Steinberg's theorem has nothing to do with real algebraic geometry: it remains valid if one replaces the function field of a real curve with no real points with any field of cohomological dimension 1.

Our goal is to solve new cases of Conjectures 1 and 2, providing evidence for their validity beyond Lang's and Steinberg's results.

Theorem 3. *Let S be a real surface such that $S(\mathbb{R}) = \emptyset$. Then every quadric of dimension ≥ 3 over $\mathbb{R}(S)$ has a $\mathbb{R}(S)$-point.*

Theorem 4. *Let C be a real curve such that $C(\mathbb{R}) = \emptyset$. Then every degree 4 del Pezzo surface over $\mathbb{R}(C)$ has a $\mathbb{R}(C)$-point.*

Theorem 3 is Conjecture 1 for $i = d = 2$. Theorem 4 follows at once from Theorem 3, by applying the Amer-Brumer theorem [1, Théorème 1]: a degree 4 del Pezzo surface over K has a rational point if and only if the pencil of quadrics that defines it, viewed as a quadric over $K(t)$, has a rational point.

It has been understood by Elman, Lam and Pfister (see [7, Proposition 9]) that Theorem 3 would be a consequence of the following real period–index theorem:

Theorem 5. *Let S be a smooth integral surface over $\mathbb{R}$, and let $\alpha \in \mathrm{Br}(S) \subset \mathrm{Br}(\mathbb{R}(S))$ be such that $\alpha|_x = 0 \in \mathrm{Br}(\mathbb{R})$ for every $x \in S(\mathbb{R})$. Then $\mathrm{ind}(\alpha) = \mathrm{per}(\alpha)$.*

Over the complex numbers, Theorem 5 is the celebrated period–index theorem of de Jong [3]. Only the particular case where $S(\mathbb{R}) = \emptyset$ is needed to prove Theorem 3. The finer hypothesis that α vanish in restriction to real points was put forward by Pfister in [7].

De Jong's proof of the period–index theorem does not adapt over $\mathbb{R}$. The argument given in [2] to prove Theorem 5 uses a different strategy, relying on Hodge theory. The talk was devoted to explaining the principle of this strategy.

Let us just mention here how Hodge theory enters the picture. One has a short exact sequence $0 \to \mathrm{Pic}(S)/n \to H^2_{\text{ét}}(S, \mu_n) \to \mathrm{Br}(S)[n] \to 0$. To show that the Brauer class associated to $\beta \in H^2_{\text{ét}}(S, \mu_n)$ has index dividing n, one has to find a degree n ramified cover $p : T \to S$ such that $p^*\beta \in H^2_{\text{ét}}(T, \mu_n)$ is algebraic. This is only possible if T carries enough algebraic cycles. To ensure this, we choose T in an appropriate Noether-Lefschetz locus, using Green's infinitesimal criterion.

References

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