On the problem of generating Chow groups by smooth subvarieties

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Fix two integers $c, d \geq 1$. In this talk, we will consider the following classical question [5], as well as variants of it in real algebraic geometry.

Question 1 (Borel, Haefliger). *Let X be a smooth projective variety of dimension $c+d$ over $\mathbb{C}$. Is the Chow group $\mathrm{CH}_d(X) = \mathrm{CH}^c(X)$ generated by classes of d-dimensional smooth closed subvarieties $Y \subset X$?*

It will be useful to keep the following heuristic in mind. In differential geometry, Whitney has shown that a generic $\mathcal{C}^\infty$ map $f : Y^d \to X^{c+d}$ of differentiable manifolds is an embedding if $d < c$. If $d \geq c$, this is not the case anymore, as Y will typically self-intersect in X. In particular, if $d = c$ one expects finitely many transverse self-intersections. It is thus easy to construct submanifolds of X if $d < c$, but hard if $d \geq c$ in general. By analogy, one may expect that Question 1 has a positive answer if $d < c$, but not necessarily so if $d \geq c$.

Let us now review what is known about the question of Borel and Haefliger. Positive results are due to Hironaka [8] and Kleiman [10].

Theorem 1 (Hironaka). *Question 1 has a positive answer if $d < c$ and $d \leq 3$.*

Theorem 2 (Kleiman). *Question 1 has a positive answer if $c = 2$ and $d \in \{2, 3\}$.*

Of these two theorems, the most important for us is Theorem 1. The bound stemming from the Whitney heuristic appears clearly in its statement. The principle of its proof is the following. Given a possibly singular subvariety $Y \subset X$, Hironaka embeds a resolution of singularities $\widetilde{Y}$ of Y in a relative projective space $\pi : \mathbb{P}^N \times X \to X$. Using linkage, he then moves $\widetilde{Y}$ in $\mathbb{P}^N \times X$ to put it in general position. That $d \leq 3$ ensures that the linkage process does not create singularities. Once $\widetilde{Y}$ has been put in general position, the Whitney-type bound $d \leq c$ guarantees that the projection $\pi|_{\widetilde{Y}}$ is an embedding, hence that $\pi(\widetilde{Y})$ is smooth.

The first counterexamples to Question 1 were found by Harthorne, Rees and Thomas [7] for $c = 2$ and $d \geq 7$. A different construction due to Debarre [6] works for $c = 2$ and $d \geq 5$.

Theorem 3 (Hartshorne, Rees, Thomas, Debarre). *Question 1 has a negative answer in general if $c = 2$ and $d \geq 5$.*

The argument of Harthorne, Rees and Thomas would give counterexamples for higher values of c as well. Debarre's method yields the lowest-dimensional known counterexamples. Their works do not allow to reach the Whitney threshold $d = c$. Our first theorem remediates this situation. To formulate it, we let $\alpha(n)$ denote the number of 1's in the dyadic expansion of n.

Theorem 4 ([2]). *Question 1 has a negative answer in general if $d \geq c$ and $\alpha(c+1) \geq 3$.*

In the crucial case where $d = c$, our proof is inspired by the Whitney heuristic. We construct a smooth projective variety X of dimension $2d$ over $\mathbb{C}$ (a well-chosen fixed point free quotient of an abelian variety) and a class $\alpha \in \mathrm{CH}_d(X)$. To show, for instance, that α cannot be the class of a smooth subvariety $Y \subset X$, we compute the number of self-intersections of Y in X. By means of a double point formula due to Fulton, and of divisibility results for Chern numbers due to Rees and Thomas, we show that the number of these self-intersections is odd, hence nonzero. This contradicts the smoothness of Y.

We will now successively consider three questions in real algebraic geometry that are related to Question 1, and we will state positive and negative results about these questions that are inspired by Theorems 1 and 4.

The first question is motivated by the fact that, in real algebraic geometry, many questions concerning real loci are insensitive to singularities at non-real points.

Question 2. *Let X be a smooth projective variety of dimension $c + d$ over $\mathbb{R}$. Is the Chow group $\mathrm{CH}_d(X)$ generated by classes of subvarieties $Y \subset X$ that are smooth along their real locus $Y(\mathbb{R})$?*

Theorem 5 ([2])**.** *Question 2 has a positive answer if $d < c$.*

Theorem 6 ([2])**.** *Question 2 has a negative answer in general if $d \geq c$ and $\alpha(c + 1) \geq 3$.*

The proof of Theorem 5 builds on Hironaka's smoothing by linkage technique. A notable feature is the removal of the hypothesis $d \leq 3$ appearing in the statement of Theorem 1. As the linkage process is bound to create singularities if $d > 3$, proving Theorem 5 requires to control these singularities, and to ensure that they are not real. Our main tool to do so is a study of linkage in families, for which we rely on results of Peskine and Szpiro and of Huneke and Ulrich.

The proof of Theorem 6 follows the same method of counting self-intersections as that of Theorem 4.

The second question investigates the subgroup of the Chow group generated by subvarieties with no real points. To state it, we recall that for any smooth variety X over $\mathbb{R}$, the real cycle class map $\mathrm{cl}_{\mathbb{R}} : \mathrm{CH}_*(X) \to H_*(X(\mathbb{R}), \mathbb{Z}/2)$ associates with a subvariety $Y \subset X$ the homology class $[Y(\mathbb{R})]$ of its real locus.

Question 3. *Let X be a smooth projective variety of dimension $c + d$ over $\mathbb{R}$. Is $\mathrm{Ker}\left(\mathrm{cl}_{\mathbb{R}} : \mathrm{CH}_d(X) \to H_d(X(\mathbb{R}), \mathbb{Z}/2)\right)$ generated by classes of subvarieties $Y \subset X$ whose real locus $Y(\mathbb{R})$ is empty?*

Theorem 7 ([2])**.** *Question 3 has a positive answer if $d < c$.*

Theorem 8 ([2])**.** *Question 3 has a negative answer in general if $d \geq c$ and $\alpha(c + 1) \geq 2$.*

Kucharz [11] had already proved Theorem 7 if $d = 1$ and $c = 2$, and Theorem 8 if c is even. The proofs of Theorems 7 and 8 follow again, respectively, Hironaka's smoothing by linkage technique (combined with Ischebeck and Schülting's description of $\mathrm{Ker}(\mathrm{cl}_{\mathbb{R}})$ [9]), and the method of counting self-intersections.

The last question that we examine concerns algebraic approximation of $\mathcal{C}^\infty$ submanifolds. If X is a smooth projective variety over $\mathbb{R}$, and if $\iota : M \hookrightarrow X(\mathbb{R})$ is a $\mathcal{C}^\infty$ submanifold of its real locus, we consider the following property:

(i) For all neighbourhoods $\mathcal{U} \subset \mathcal{C}^\infty(M, X(\mathbb{R}))$ of ι, there exist $j \in \mathcal{U}$ and a subvariety $Y \subset X$ smooth along $Y(\mathbb{R})$ such that $j(M) = Y(\mathbb{R})$.

There are homological obstructions to the validity of (i). For instance, assertion (i) obviously implies that $\iota_*[M] \in \mathrm{Im}(\mathrm{cl}_\mathbb{R})$. More generally, (i) implies the algebraicity in $X(\mathbb{R})$ of products of Stiefel–Whitney classes of M:

(ii) For all integers $i_1, \ldots, i_r$, one has $\iota_*(w_{i_1}(M) \ldots w_{i_r}(M)) \in \mathrm{Im}(\mathrm{cl}_\mathbb{R})$.

It is then natural to ask:

Question 4. *Let X be a smooth projective variety of dimension $c + d$ over $\mathbb{R}$ and let $\iota : M \hookrightarrow X(\mathbb{R})$ be a d-dimensional $\mathcal{C}^\infty$ submanifold of its real locus. Are properties (i) and (ii) equivalent?*

Theorem 9 ([2]). *Question 4 has a positive answer if $d < c$.*

Theorem 10 ([2]). *Question 4 has a negative answer in general if $d \geq c$ and $\alpha(c + 1) = 2$.*

Theorem 9 for $d = 1$ had already been proven by Bochnak and Kucharz [4] when $c = 2$ and by Wittenberg and myself [3] in general. The proofs of Theorems 9 and 10 are further applications of Hironaka's smoothing by linkage technique (combined with the relative Nash–Tognoli theorem of Akbulut and King [1]), and of the method of counting self-intersections.

References

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