COMPLEX ORIENTATIONS OF REAL ALGEBRAIC CURVES

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In this paper we study the orientations obtainable from the complexification of a plane M-curve of even degree and their influence on the topology of the curve.

1. Preliminary Survey. A curve of degree m in the present paper means a nonsingular real plane projective algebraic curve of degree m, i.e., a set in the real projective plane RP^2, free of singularities, defined by an equation of the form

$$P(x_0, x_1, x_2) = 0, \tag{1}$$

where P is a homogeneous real polynomial of degree m and x_0, x_1, x_2 are homogeneous coordinates. The components of such a set are homeomorphic to circles, and if m is even, then they are all two-sided in RP^2, but if m is odd, then there is a one-sided component. Two-sided components are called ovals. The number of components does not exceed g + 1, where g = (m−1) (m−2)/2, but if Eq. (1) has singularities (real or imaginary), then the number of components of the set defined by it in RP^2 is less than g + 1. Curves of degree m with g + 1 components exist for any m and are called M-curves. Details and references to the literature can be found in a survey by D. A. Gudkov [1].

The fundamental problem of the topology of real plane algebraic curves consists of enumerating the isotopy types of M-curves. At the present time this problem is solved only for M-curves of degree m ≤ 6 (see [1]). For m > 6, evidently both our information about the topological properties of M-curves and the proposed methods of constructing them are insufficient: the known isotopy types of M-curves are very few (see [1]), and the topological properties of M-curves found in the literature amount to an inequality of Petrovsky [2] (see also [3]), an inequality of Arnol'd [3], a congruence of Gudkov [4], and several obvious corollaries of an inequality of Bez according to which a curve of degree m_1 situated in general position with respect to a curve of degree m can have no more than mm_1 points in common with it. Petrovsky's theorem asserts that for a curve of even degree 2 k (with any number of ovals), we have

$$|2(p-n)-1| \leqslant 3k^2 - 3k + 1, \tag{2}$$

where p is the number of even ovals, i.e., ovals lying inside of an even number of other ovals, and n is the number of odd, i.e., remaining, ovals. Arnol'd's theorem asserts that, moreover,

$$p^- + p^0 \leqslant \frac{(k-1)(k-2)}{2}, \quad n^- + n^0 \leqslant \frac{(k-1)(k-2)}{2}, \tag{3}$$

$$p^- \geqslant n - \frac{3k(k-1)}{2}, \quad n^- \geqslant p - \frac{3k(k-1)}{2} - 1, \tag{4}$$

where p^- is the number of even ovals bounding the outside of a component of the complement of the curve (in RP^2) with negative Euler characteristic, p^0 is the number of even ovals bounding the outside of a component of the complement with zero Euler characteristic, and n^- and n^0 are the same numbers relative to odd ovals.* Gudkov's congruence asserts that for M-curves of even degree 2k

*In Arnol'd's paper this theorem is formulated more cautiously, namely, inequalities (3) and (4) are accompanied by additional homological conditions without which it is asserted only that

$$p^- \leqslant \frac{(k-1)(k-2)}{2}, \quad n^- \leqslant \frac{(k-1)(k-2)}{2}, \quad p^- + p^0 \geqslant n - \frac{3k(k-1)}{2}, \quad n^- + n^0 \geqslant p - \frac{3k(k-1)}{2} - 1.$$

Actually, as is easy to establish with the aid of the Smith Z_2-sequence written out for the branched covering $Y \to CP^2$ considered by Arnol'd, his homological condition is always satisfied.

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$$p - n \equiv k^2 \bmod 8. \tag{5}$$

The simplest topological corollary of Bez's inequality (obtained for $m_1 = 1$) consists of the fact that if $C_1, \ldots, C_r$; $C'_1, \ldots, C'_s$ are pairwise disjoint curves of degree m and C_i lies inside of C_{i-1} for $i = 2, \ldots, r$ and C'_j lies inside of C'_{j-1} for $j = 2, \ldots, s$, then $2(r + s) \leq m$. Let us add that inequalities (2) and (4) are trivial corollaries of a conjecture that had already been formulated in 1906 by V. Rogsdale [5], and neither proved nor disproved since then, according to which for a curve of even degree 2k (with any number of ovals)

$$p \leqslant \frac{3k(k-1)}{2} + 1, \quad n \leqslant \frac{3k(k-1)}{2}, \tag{6}$$

and that a weakened version

$$p - n \equiv k^2 \bmod 4 \tag{7}$$

of congruence (5) was originally proved by V. I. Arnol'd [3].

2. Main Formula. In this section an M-curve A of degree m is assumed given. Since its equation does not have singularities in CP^2, the set CA defined by the equation in CP^2 represents a submanifold of the plane CP^2 in the complex-analytic sense diffeomorphic to a sphere with g handles. It is well known that A separates CA into two halves diffeomorphic to spheres with g + 1 holes and taken into each other by complex conjugation conj: $CP^2 \to CP^2$ with inverted orientation. We denote these halves by X and Y, orient them in accordance with the natural orientation of the manifold CA, and orient A like ∂X or ∂Y. Further, we call an injective pair of ovals of A, i.e., a pair of ovals one of which lies inside of the other, positive if the orientations of the ovals induce an orientation of the annulus bounded by them in RP^2, and negative in the opposite case, and we denote the number of positive pairs by Π^+ and the number of negative pairs by Π^-. It turns out that for an even number m = 2k, we have

$$\Pi^+ - \Pi^- = \frac{(k-1)(k-2)}{2}. \tag{8}$$

Proof. Denote by B_C the disc bounded by the oval C in RP^2 and complete X to a sphere Σ by adding nonintersecting copies of the disk B_C. Let T be the sphere obtained from Y by the same procedure, and let φ: $\Sigma \to CP^2$ and ψ: $T \to CP^2$ be mappings fixed on X and Y and superimposing copies of B_C onto these dikes. Further, let ξ and η be elements of the (integral) homology group $H_2(CP^2)$ determined by the mappings φ and ψ and the natural orientations of the spheres Σ and T (i.e., the orientations obtained from X and Y). We shall establish Eq. (8) by computing the intersection index $\xi\eta$ by two procedures.

The first procedure is based on the fact that $\xi\eta$ can be interpreted as the algebraic number of points in the intersection of the oriented singular spheres φ: $\Sigma \to CP^2$ and ψ: $T \to CP^2$. This number cannot be determined directly, since the intersection consists of whole disks, and we begin by applying a deformation to φ, making the intersection more regular. Let u be some tangent vector field on RP^2 with a finite number of zeros, not having zeros on A and normal to A on A. Since the field iu is normal to RP^2 in CP^2 and normal to CA on A, it can be normally extended to some field v on $RP^2 \cup X$ (the latter, of course, will have zeros inside of X); let γ: $RP^2 \cup X \to CP^2$ be a geodesic translation defined by the field δv, where δ is a sufficiently small positive number, and φ': $\Sigma \to CP^2$ be the mapping defined by the formula $\varphi'(x) = \gamma(\varphi(x))$. For φ' the algebraic number of points of intersection with ψ is determined directly and can be found in the following way. Since the sum index of the singularities of u in each of the disks B_C is equal to 1 and multiplication by i anti-isomorphically maps the tangent bundle of RP^2 onto its normal bundle in CP^2, the sum index of v on each of the disks B_C is equal to -1. Consequently, the contribution added by the pair of disks B_C and $B_{C'}$ to the algebraic number of intersection points that we are interested in is equal to +1 if C = C', equal to +2 if the pair C,C' is negative, and equal to -2 if the pair C,C' is positive, and this number itself is equal to $g + 1 + 2(\Pi^- - \Pi^+)$. Since φ' is homotopic to φ, the index $\xi\eta$ is also like that, and thus

$$\xi\eta = 2k^2 - 3k + 2 + 2(\Pi^- - \Pi^+). \tag{9}$$

The second procedure reduces to two remarks. First, the class of $\xi + \eta$ is realized by the surface CA and therefore coincides with $2k\alpha$, where α is a natural generator of the group $H_2(CP^2)$. Second, since the homomorphism conj_*: $H_2(CP^2) \to H_2(CP^2)$ represents multiplication by -1 and takes ξ to $-\eta$, we have $\xi = \eta$. From these remarks it follows that $\xi = k\alpha$, $\eta = k\alpha$, and $\xi\eta = k^2$. Comparing the last equation with (9), we obtain (8).

3. Real Corollaries. Since the numbers Π^+ and Π^- depend on the orientations received by the ovals of A from a complex region, the information contained in Eq. (8) directly relates not to the relative disposition of these ovals in RP^2, which is a subject of the topology of real algebraic curves, but to the relative disposition of the surfaces CA and RP^2 in CP^2. At the same time we can draw purely real corollaries from Eq. (8).

In order to do this we note that the definitions of the numbers Π^+ and Π^- can be repeated assuming as a basis a perfectly arbitrary orientation of A. This leads to the idea of calling the orientation of an M-curve of degree 2k quasi-complex if the corresponding numbers Π^+ and Π^- satisfy relation (8). The exhaustive real information included in Eq. (8) is contained in the fact that an M-curve of even degree possesses a quasi-complex orientation.

It turns out that this information is not new with respect to the topological properties of M-curves enumerated in Sec. 1. More precisely, let a topological M-curve of degree 2k mean an arbitrary collection of $2k^2-3k+2$ pairwise disjoint circles lying two-sided in RP^2, and let us extend to the nonalgebraic case the definitions of even and odd ovals, the notation p, n, . . ., Π^+, Π^-, and the definition of quasi-complex orientation; it turns out that every topological M-curve of degree 2k satisfying the right half of inequality (4) and congruence (7) possesses a quasi-complex orientation.

For the proof we first provide the curve with an alternating orientation, i.e., an orientation with respect to which an injective pair of ovals is positive if and only if one oval is even and the other is odd. It is clear that with this orientation $\Pi^+-\Pi^-=n$, which in particular settles the case where $n=(k-1)(k-2)/2$. In the general case, we take into consideration the difference $\delta=n-((k-1)(k-2)/2)$ and note that congruence (7) is equivalent to δ being even, and the right halves of (3) and (4) are equivalent to the inequalities $\delta \le n^+$ and $-\delta \le n^-$, where n^+ is the number of empty odd ovals. If $\delta>0$, then we invert the orientation of $\delta/2$ empty odd ovals; this reduces $\Pi^+-\Pi^-$ by δ and converts the alternating orientation of the curve to a quasi-complex orientation. If $\delta<0$, then the alternating orientation is transformed into a quasi-complex one in two steps. First, we invert the orientation of every oval except the outermost and the empty ones that are odd; this enlarges the difference $\Pi^+-\Pi^-$ by $2(q-n^--n^0)$, where q is the number of even ovals that are not outermost, and makes it equal to $n+2(q-n^--n^0)$, which is larger than $(k-1)(k-2)/2$ since $q-n^--n^0 \ge n^-$ (for every finite collection of pairwise disjoint two-sided circles in RP^2) and $2n^- > n^- \ge -\delta$. Second, we invert the orientation of π empty even ovals that are not outermost and ν empty odd ovals; this reduces $\Pi^+-\Pi^-$ by $2(2\pi+\nu)$ and makes the orientation of the curve quasi-complex if $2\pi+\nu=(\delta/2)+q-n^--n^0$. The possibility of choosing such a π and ν follows from the fact that it is only subject to the inequalities $\pi \le q^+$ and $\nu \le n^+$, where q^+ is the number of empty even ovals that are not outermost, and the fact that $2q^+ + n^+$ is known to exceed $(\delta/2)+q-n^--n^0$; the latter is obvious if we take into consideration the fact that $2q^+ + n^+ \geqslant q^+ + n^+ = q^+ + n - n^- - n^0$, that $q^+ + n \ge q$ (for every finite collection of pairwise disjoint two-sided circles in RP^2), and that $\delta<0$.

Two Special Corollaries. Two special corollaries of the existence of quasi-complex orientations deserve attention: the inequality

$$\Pi \geqslant \frac{(k-1)(k-2)}{2}, \tag{10}$$

where Π is the total number of injective pairs of ovals, and the congruence

$$\Pi \equiv \frac{(k-1)(k-2)}{2} \mod 2. \tag{11}$$

To derive them from Eq. (8) it is sufficient to note that $\Pi=\Pi^+ + \Pi^-$.

Inequality (10) is also a consequence of the right half of inequality (4) and the inequality $\Pi \ge n+n^-$, which is satisfied for every finite collection of pairwise disjoint two-sided circles in RP^2, and for topological M-curves of degree 2k congruence (11) is equivalent to congruence (7), since the latter is equivalent to the congruence $n \equiv (k-1)(k-2)/2 \mod 2$, and n and Π are connected by the congruence $\Pi \equiv n \mod 2$.

4. A New Interpretation of Congruence (5). An odd oval of an oriented M-curve of even degree is called disoriented if it forms a negative pair with the innermost of the ovals outside of it. The number of disoriented ovals is denoted by d, the number of positive pairs with disoriented outer oval by D^+, and the number of negative pairs with disoriented outer oval by D^-. Obviously

$$\Pi^+ - \Pi^- = n - 2(d + D^- - D^+). \tag{12}$$

From this formula and the equation $p + n = 2k^2 - 3k + 2$ it follows that the orientation's being quasi-complex is equivalent to the equation

$$k^2 - (p - n) = 4(d + D^- - D^+). \tag{13}$$

Equation (13) sheds a new light on Arnol'd's congruence (7), Gudkov's congruence (5), and Rogsdale's conjecture (6). In particular, it shows that congruence (5) is equivalent in the case of quasi-complex orientation to the congruence

$$d + D^- - D^+ \equiv 0 \mod 2, \tag{14}$$

and the left half of (6) is equivalent to the inequality $d + D^- - D^+ \geq 0$.

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