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VECTOR FIELDS ON SPHERES

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1. Results

The question of vector fields on spheres arises in homotopy theory and in the theory of fibre bundles, and it presents a classical problem, which may be explained as follows. For each n, let S^{n-1} be the unit sphere in euclidean n-space R^n. A vector field on S^{n-1} is a continuous function v assigning to each point x of S^{n-1} a vector $v(x)$ tangent to S^{n-1} at x. Given r such fields $v_1, v_2, \cdots, v_r$, we say that they are linearly independent if the vectors $v_1(x), v_2(x), \cdots, v_r(x)$ are linearly independent for all x. The problem, then, is the following: for each n, what is the maximum number r of linearly independent vector fields on S^{n-1}? For previous work and background material on this problem, we refer the reader to [1, 10, 11, 12, 13, 14, 15, 16]. In particular, we recall that if we are given r linearly independent vector fields $v_i(x)$, then by orthogonalisation it is easy to construct r fields $w_i(x)$ such that $w_1(x), w_2(x), \cdots, w_r(x)$ are orthonormal for each x. These r fields constitute a cross-section of the appropriate Stiefel fibering.

The strongest known positive result about the problem derives from the Hurwitz-Radon-Eckmann theorem in linear algebra [8]. It may be stated as follows (cf. James [13]). Let us write $n = (2a+1)2^b$ and $b = c + 4d$, where a, b, c and d are integers and $0 \leqq c \leqq 3$; let us define $\rho(n) = 2^c + 8d$. Then there exist $\rho(n) - 1$ linearly independent vector fields on S^{n-1}.

It is the object of the present paper to prove that the positive result stated above is best possible.

THEOREM 1.1. *If $\rho(n)$ is as defined above, then there do not exist $\rho(n)$ linearly independent vector fields on S^{n-1}.*

Heuristically, it is plausible that the "depth" of this result increases with b (where $n = (2a+1)2^b$, as above). For $b \leqq 3$, the result is due to Steenrod and Whitehead [15]. For $b \leqq 10$, the result is due to Toda [16].

The theorem, as stated, belongs properly to the theory of fibre bundles. However, we shall utilise a known reduction of the problem to one in homotopy theory, concerning real projective spaces. We write RP^q for real projective q-space, although this notation is not consistent with that employed by James and Atiyah [12,1]. If $p < q$, then RP^p is imbedded in RP^q, and we write RP^q/RP^p for the quotient space obtained from RP^q by identifying RP^p to a single point. Our main task is to prove the following theorem.

THEOREM 1.2. *$RP^{m+\rho(m)}/RP^{m-1}$ is not co-reducible; that is, there is no map*

$$f\colon RP^{m+\rho(m)}/RP^{m-1} \longrightarrow S^m$$

such that the composite

$$S^m = RP^m/RP^{m-1} \xrightarrow{i} RP^{m+\rho(m)}/RP^{m-1} \xrightarrow{f} S^m$$

has degree 1.

Theorem 1.1 has the following corollary in homotopy theory.

COROLLARY 1.3. *The Whitehead product $[\iota_{n-1}, \iota_{n-1}]$ in $\pi_{2n-3}(S^{n-1})$ is a $(\rho(n) - 1)$-fold suspension but not a $\rho(n)$-fold suspension.*

It is also more or less well known that Theorem 1.1 is relevant to the study of the stable J-homomorphism (cf. [16]). More precisely, one should consider the map

$$J \otimes Z_2\colon \pi_r(\mathrm{SO}) \otimes Z_2 \longrightarrow \pi_{n+r}(S^n) \otimes Z_2 \qquad (n - 1 > r > 0) .$$

One should deduce from Theorem 1.1 or Theorem 1.2 that $J \otimes Z_2$ is monomorphic. However, it appears to the author that one can obtain much better results on the J-homomorphism by using the methods, rather than the results, of the present paper. On these grounds, it seems best to postpone discussion of the J-homomorphism to a subsequent paper.

A summary of the present paper will be found at the end of § 2.

2. Methods

The proof of Theorem 1.2 will be formulated in terms of the "extraordinary cohomology theory" $K(X)$ of Grothendieck, Atiyah and Hirzebruch [2, 3]. We propose to introduce "cohomology operations" into the "cohomology theory" K; these operations will be functions from $K(X)$ to $K(X)$ which are natural for maps of X. If the space X is reducible or co-reducible, then the corresponding group $K(X)$ will split as a direct sum, and the homomorphisms of the splitting will commute with our operations. We shall find that $K(RP^{m+\rho(m)}/RP^{m-1})$ does not admit any splitting of the sort required.

The author hopes that this line of proof is self-justifying; however, a few historical remarks may serve to put it in perspective. The author's original approach to the present problem was directly inspired by the work of Steenrod and Whitehead [15], and consisted of an attempt to replace the Steenrod squares used in [15] by cohomology operations of higher kinds. This attempt is reasonable, but it involves several difficulties; the first of these is the selection of cohomology operations well-adapted to the solution of this particular problem. The author's work on this topic may be left in decent obscurity, like the bottom nine-tenths of an iceberg. However, it led to the following conclusions.

(1) The required operations should be constructed from universal examples.

(2) The universal examples should be fiberings induced by certain hypothetical maps $f\colon \mathrm{BO} \to \mathrm{BO}$. (Here BO denotes the classifying space of the infinite orthogonal group.)

(3) The hypothetical maps f should satisfy certain stringent algebraic specifications.

At this point the advisability of reformulating matters in terms in the K-theory became evident. The hypothetical maps f led immediately to the notion of cohomology operations in the K-theory. The algebraic conditions mentioned in (3) led easily to the correct operations.

The remainder of this paper is organised as follows. In § 3 we define the ring $K_\Lambda(X)$. Since our cohomology operations are defined with the aid of group representations, we also define the ring $K'_\Lambda(G)$ of virtual representations of G. The remainder of the section is devoted to necessary preliminaries. In § 4 we define and study the virtual representations which we need; in § 5 they are applied to construct our cohomology operations. In § 6 we present further material on $K_\Lambda(X)$, needed for § 7. In § 7 we compute the values of our operations in projective spaces. In § 8 we complete the proof of Theorem 1.2, by the method indicated above. In § 9 we deduce Theorem 1.1 and Corollary 1.3, by citing appropriate references.

3. The ring $K_\Lambda(X)$

In this section we shall define the cohomology ring $K_\Lambda(X)$ and the representation ring $K'_\Lambda(G)$. We proceed to discuss composition, and this leads to the basic lemma which will enable us to define operations in K_Λ. This lemma is stated as Lemma 3.8, near the end of the section.

We begin by defining $K_\Lambda(X)$. (Throughout this paper, the symbols Λ, Λ' will denote either the real field R or the complex field C.) Suppose given a finite CW-complex X; we consider the Λ-vector-bundles over X. (That is, we consider real vector bundles or complex vector bundles according to the choice of Λ. It is immaterial whether the group of our bundles is the full linear group $\mathrm{GL}(n, \Lambda)$ or a compact subgroup $\mathrm{O}(n)$ or $\mathrm{U}(n)$; but for definiteness we suppose it is $\mathrm{GL}(n, \Lambda)$. If X were not connected, we would allow our bundles to have fibres of different dimensions over the different components of X; however, for our purposes it will suffice to consider only connected complexes X.) We divide the Λ-vector bundles ξ over X into equivalence classes $\{\xi\}$, and take these classes as generators for a free abelian group $F_\Lambda(X)$. For each pair of bundles ξ, η over X we form the element $t = \{\xi \oplus \eta\} - \{\xi\} - \{\eta\}$, where $\oplus$ denotes the Whitney sum. We write $T_\Lambda(X)$ for the subgroup of $F_\Lambda(X)$ generated by such elements t; we define $K_\Lambda(X)$ to be the quotient group $F_\Lambda(X)/T_\Lambda(X)$.

We proceed to define $K'_\Lambda(G)$ in a closely analogous way. Suppose given a topological group G. A representation α of G (of degree n, over Λ) is a continuous function $\alpha: G \to \mathrm{GL}(n, \Lambda)$ which preserves products. Two such representations are equivalent if they coincide up to an inner automorphism of $\mathrm{GL}(n, \Lambda)$. We divide the representations α of G over Λ into equivalence classes $\{\alpha\}$, and take these classes as generators for a free abelian group $F'_\Lambda(G)$. For each pair of representations α, β we form the element $t = \{\alpha \oplus \beta\} - \{\alpha\} - \{\beta\}$, where $\oplus$ denotes the direct sum of representations. We write $T'_\Lambda(G)$ for the subgroup of $F'_\Lambda(G)$ generated by such elements t. We define $K'_\Lambda(G)$ to be the quotient group $F'_\Lambda(G)/T'_\Lambda(G)$. An element of $K'_\Lambda(G)$ is called a virtual representation (of G, over Λ).

It is clear that we can define a homomorphism from $F'_\Lambda(G)$ to the integers which assigns to each representation its degree. This homomorphism passes to the quotient, and defines the virtual degree of a virtual representation.

We next discuss composition. It will shorten explanations if we adopt a convention. The letters f, g, h will denote maps of complexes such as X. The letters ξ, η, ζ will denote bundles, and the letters κ, λ, μ will denote elements of $K_\Lambda(X)$. The letters α, β, γ will denote representations, and the letters θ, φ, ψ will denote virtual representations.

The basic sorts of composition are easily enumerated if we interpret a bundle ξ as a classifying map $\xi: X \to \mathrm{BGL}(n, \Lambda)$ and a representation α as a map of classifying spaces. We have to define compositions

$$\beta \cdot \alpha, \ \alpha \cdot \xi, \ \xi \cdot f, \ f \cdot g \ .$$

(Composition will be written with a dot, to distinguish it from any other product.) The formal definitions are as follows.

If $\alpha: G \to \mathrm{GL}(n, \Lambda)$ and $\beta: \mathrm{GL}(n, \Lambda) \to \mathrm{GL}(n', \Lambda')$ are representations, then $\beta \cdot \alpha$ is their composite in the usual sense. If ξ is a bundle over X with group $\mathrm{GL}(n, \Lambda)$ and $\alpha: \mathrm{GL}(n, \Lambda) \to \mathrm{GL}(n', \Lambda')$ is a representation, then $\alpha \cdot \xi$ is the induced bundle, defined by using the same coordinate neighbourhoods and applying α to the coordinate transformation functions. If ξ is a bundle over Y and $f: X \to Y$ is a map, then $\xi \cdot f$ is the induced bundle over X, defined by applying f^{-1} to the coordinate neighbourhoods and composing the coordinate transformation functions with f. If $f\colon X \to Y$ and $g\colon Y \to Z$ are maps, then $g \cdot f$ is their composite in the usual sense.

We next wish to linearize over the first factor.

LEMMA 3.1. *It is possible to define composites of the form $\varphi \cdot \alpha$, $\theta \cdot \xi$ and $\kappa \cdot f$, so that $\varphi \cdot \alpha$ lies in the appropriate group $K'_\Lambda(G)$, $\theta \cdot \xi$ and $\kappa \cdot f$ lie in appropriate groups $K_\Lambda(X)$, and they have the following properties.*

(i) *Each composite is linear in its first factor.*

(ii) *If we replace φ, θ or κ by β, α or ξ (respectively), then these composites reduce to those considered above.*

(iii) *The following associativity formulae hold.*

$$(\psi \cdot \beta) \cdot \alpha = \psi \cdot (\beta \cdot \alpha) \,, \qquad (\varphi \cdot \alpha) \cdot \xi = \varphi \cdot (\alpha \cdot \xi) \,,$$
$$(\theta \cdot \xi) \cdot f = \theta \cdot (\xi \cdot f) \,, \qquad (\kappa \cdot g) \cdot f = \kappa \cdot (g \cdot f) \,.$$

(iv) *If $\alpha = 1$, then $\varphi \cdot \alpha = \varphi$. If $f = 1$, then $\kappa \cdot f = \kappa$.*

PROOF. The required composites are defined by (i) and (ii), and it is easy to check that they are well defined. The formulae (iii), (iv) follow by linearity from those that hold before linearizing.

We next wish to linearize over the second factor. For this purpose we require a first factor which can act on $\mathrm{GL}(n, \Lambda)$ for any n. We therefore introduce the notion of a sequence $\Theta = (\theta_n)$, where, for each n, θ_n is a virtual representation of $\mathrm{GL}(n, \Lambda)$ over Λ'. We reserve the letters Θ, Φ, Ψ for such sequences. In order to linearize over the second factor, we require a linearity condition on the first factor. In order to state this condition, we write

$$\pi\colon \mathrm{GL}(n, \Lambda) \times \mathrm{GL}(m, \Lambda) \longrightarrow \mathrm{GL}(n, \Lambda) \,,$$
$$\varpi\colon \mathrm{GL}(n, \Lambda) \times \mathrm{GL}(m, \Lambda) \longrightarrow \mathrm{GL}(m, \Lambda)$$

for the projections of $\mathrm{GL}(n, \Lambda) \times \mathrm{GL}(m, \Lambda)$ onto its two factors. These projections are representations.

DEFINITION 3.2. The sequence $\Theta = (\theta_n)$ is additive if we have

$$\theta_{n+m} \cdot (\pi \oplus \varpi) = (\theta_n \cdot \pi) + (\theta_m \cdot \varpi)$$

for all n, m.

LEMMA 3.3. *Suppose Θ is additive. Then for any two representations $\alpha\colon G \to \mathrm{GL}(n, \Lambda)$, $\beta\colon G \to \mathrm{GL}(m, \Lambda)$ we have*

$$\theta_{n+m} \cdot (\alpha \oplus \beta) = (\theta_n \cdot \alpha) + (\theta_m \cdot \beta) \,.$$

Moreover, for any two bundles ξ, η over X with groups $\mathrm{GL}(n, \Lambda)$, $\mathrm{GL}(m, \Lambda)$ we have

$$\theta_{n+m} \cdot (\xi \oplus \eta) = (\theta_n \cdot \xi) + (\theta_m \cdot \eta) \,.$$

We postpone the proof for a few lines.

LEMMA 3.4. *If Θ, Φ, Ψ run over additive sequences, then it is possible to define composites of the form $\Phi\cdot\Theta$, $\Phi\cdot\theta$ and $\Theta\cdot\kappa$ so that $\Phi\cdot\Theta$ is an additive sequence, $\Phi\cdot\theta$ lies in the appropriate group $K'_\Lambda(G)$, $\Theta\cdot\kappa$ lies in the appropriate group $K_\Lambda(X)$, and they have the following properties.*

(i) *Each composite is bilinear in its factors.*

(ii) $(\Phi\cdot\Theta)_n = \Phi\cdot\theta_n$. *If $\alpha\colon G\to \mathrm{GL}(n,\Lambda)$ is a representation then $\Phi\cdot\alpha = \varphi_n\cdot\alpha$ (in the sense of Lemma 3.1). If ξ is a $\mathrm{GL}(n,\Lambda)$-bundle over X then $\Theta\cdot\xi = \theta_n\cdot\xi$ (in the sense of Lemma 3.1).*

(iii) *The following associativity formulae hold.*

$$(\Psi\cdot\Phi)\cdot\Theta = \Psi\cdot(\Phi\cdot\Theta)\ , \qquad (\Phi\cdot\Theta)\cdot\kappa = \Phi\cdot(\Theta\cdot\kappa)\ ,$$
$$(\Theta\cdot\kappa)\cdot f = \Theta\cdot(\kappa\cdot f)\ .$$

(iv) *If 1 denotes the additive sequence of identity maps $1_n\colon \mathrm{GL}(n,\Lambda)\to \mathrm{GL}(n,\Lambda)$, then*

$$1\cdot\Theta = \Theta\ , \qquad \Phi\cdot 1 = \Phi\ , \qquad 1\cdot\kappa = \kappa\ .$$

PROOF OF LEMMA 3.3. Suppose given two bundles ξ, η over X, with groups $\mathrm{GL}(n,\Lambda)$, $\mathrm{GL}(m,\Lambda)$. Then we can define a bundle $\xi\times\eta$ over X, with group $\mathrm{GL}(n,\Lambda)\times\mathrm{GL}(m,\Lambda)$. (The coordinate neighbourhoods are the intersections of those in ξ and those in η; the coordinate transformation functions are obtained by lumping together those in ξ and those in η.) We have $(\pi\oplus\varpi)\cdot(\xi\times\eta) = \xi\oplus\eta$, $\pi\cdot(\xi\times\eta)=\xi$, $\varpi\cdot(\xi\times\eta)=\eta$. If Θ is additive we have

$$\theta_{n+m}\cdot(\pi\oplus\varpi)\cdot(\xi\times\eta) = (\theta_n\cdot\pi + \theta_m\cdot\varpi)\cdot(\xi\times\eta)\ ,$$

that is,

$$\theta_{n+m}\cdot(\xi\oplus\eta) = \theta_n\cdot\xi + \theta_m\cdot\eta\ .$$

The proof for representations is analogous, but slightly more elementary.

PROOF OF LEMMA 3.4. The required composites are defined by (i) and (ii), and it is trivial to check that they are well defined, given the conclusion of Lemma 3.3. The associative laws are preserved at each step of the construction; finally, conclusion (iv) is trivial.

We next recall that the tensor product of bundles defines a product in $K_\Lambda(X)$ (cf. [2, 3]). Similarly, the tensor product of representations defines a product in $K'_\Lambda(G)$; thus $K_\Lambda(X)$ and $K'_\Lambda(G)$ become commutative rings with unit. Composition behaves well for tensor products of the first factor, as is shown by the following formulae.

$$(\varphi \otimes \theta) \cdot \alpha = (\varphi \cdot \alpha) \otimes (\theta \cdot \alpha) \ ,$$
$$(\varphi \otimes \theta) \cdot \xi = (\varphi \cdot \xi) \otimes (\theta \cdot \xi) \ ,$$
$$(\kappa \otimes \lambda) \cdot f = (\kappa \cdot f) \otimes (\lambda \cdot f) \ .$$

Here, for example, the third formula states that the products in $K_\Lambda(X)$ are natural for maps of X. These formulae are deduced by linearity from the corresponding ones for representations and bundles.

In order to ensure that composition behaves well for tensor products of the second factor, we require a condition on the first factor. In order to state this condition, we re-adopt the notation of Definition 3.2.

DEFINITION 3.5. The sequence $\Theta = (\theta_n)$ is multiplicative if we have

$$\theta_{nm} \cdot (\pi \otimes \varpi) = (\theta_n \cdot \pi) \otimes (\theta_m \cdot \varpi)$$

for all n, m.

LEMMA 3.6. *Suppose Θ is multiplicative. Then for any two representations $\alpha: G \to \mathrm{GL}(n, \Lambda)$, $\beta: G \to \mathrm{GL}(m, \Lambda)$ we have*

$$\theta_{nm} \cdot (\alpha \otimes \beta) = (\theta_n \cdot \alpha) \otimes (\theta_m \cdot \beta) \ .$$

Moreover, for any two bundles ξ, η over X with groups $\mathrm{GL}(n, \Lambda)$, $\mathrm{GL}(m, \Lambda)$ we have

$$\theta_{nm} \cdot (\xi \otimes \eta) = (\theta_n \cdot \xi) \otimes (\theta_m \cdot \eta) \ .$$

The proof is closely similar to that of Lemma 3.3.

LEMMA 3.7. *If Ψ is both additive and multiplicative, then we have*

$$\Psi \cdot (\theta \otimes \varphi) = (\Psi \cdot \theta) \otimes (\Psi \cdot \varphi) \ ,$$
$$\Psi \cdot (\kappa \otimes \lambda) = (\Psi \cdot \kappa) \otimes (\Psi \cdot \lambda) \ .$$

This follows from Lemma 3.6 by linearity.

We now restate our main results in one omnibus lemma.

LEMMA 3.8. *Suppose given an additive sequence $\Theta = (\theta_n)$, where $\theta_n \in K'_{\Lambda'}(\mathrm{GL}(n, \Lambda))$. Then the function $\Theta \cdot \kappa$ of κ gives (for each X) a group homomorphism*

$$\Theta: K_\Lambda(X) \to K_{\Lambda'}(X)$$

with the following properties.

(i) *Θ is natural for maps of X; that is, if $f: X \to Y$ is a map, then the following diagram is commutative.*

$$\begin{array}{ccc} K_\Lambda(Y) & \xrightarrow{\Theta} & K_{\Lambda'}(Y) \\ {\scriptstyle f^*}\downarrow & & \downarrow{\scriptstyle f^*} \\ K_\Lambda(X) & \xrightarrow{\Theta} & K_{\Lambda'}(X) . \end{array}$$

(ii) *If the sequence Θ is multiplicative as well as additive, then*

$$\Theta: K_\Lambda(X) \longrightarrow K_{\Lambda'}(X)$$

preserves products.

(iii) *If θ_1 has virtual degree* 1, *then*

$$\Theta: K_\Lambda(X) \longrightarrow K_{\Lambda'}(X)$$

maps the unit in $K_\Lambda(X)$ into the unit in $K_{\Lambda'}(X)$.

PROOF. Except for (iii), this is merely a restatement of what has been said above; thus, (i) is the associativity law $(\Theta \cdot \kappa) \cdot f = \Theta \cdot (\kappa \cdot f)$ of Lemma 3.4, and (ii) is contained in Lemma 3.7. As for (iii), the unit in $K_\Lambda(X)$ is the trivial bundle with fibres of dimension 1. Any representation of $\mathrm{GL}(1, \Lambda)$ will map this into a trivial bundle of the appropriate dimension; hence $\theta_1 \cdot 1 = d$, where d is the virtual degree of θ_1. This completes the proof.

As a first application of Lemma 3.8 (which, however, is hardly necessary in so trivial a case) we consider the following sequences.

(i) The sequence $c = (c_n)$, where

$$c_n: \mathrm{GL}(n, R) \longrightarrow \mathrm{GL}(n, C)$$

is the standard injection. (The letter "c" for "complexification" is chosen to avoid confusion with other injections.)

(ii) The sequence $r = (r_n)$, where

$$r_n: \mathrm{GL}(n, C) \longrightarrow \mathrm{GL}(2n, R)$$

is the standard injection.

(iii) The sequence $t = (t_n)$, where

$$t_n: \mathrm{GL}(n, C) \longrightarrow \mathrm{GL}(n, C)$$

is defined by $t_n(M) = \bar{M}$, and $\bar{M}$ is the complex conjugate of the matrix M.

All these sequences are additive, while c and t are multiplicative. We therefore obtain the following natural group-homomorphisms:

$$\begin{aligned} c&: K_R(X) \longrightarrow K_C(X) , \\ r&: K_C(X) \longrightarrow K_R(X) , \\ t&: K_C(X) \longrightarrow K_C(X) . \end{aligned}$$

The functions c and t are homomorphisms of rings.

LEMMA 3.9. *We have*

$$rc = 2 \quad : K_R(X) \longrightarrow K_R(X) ,$$
$$cr = 1 + t: K_C(X) \longrightarrow K_C(X) .$$

This follows immediately from the corresponding fact for representations. (Cf. [7], Proposition 3.1.)

4. Certain virtual representations

In this section we shall define and study the virtual representations which we need. It is a pleasure to acknowledge at this point helpful conversations with A. Borel and Harish-Chandra; the former kindly read a draft of this section.

The result which we require is stated as Theorem 4.1; the rest of the section is devoted to proving it.

THEOREM 4.1. *For each integer k (positive, negative or zero) and for $\Lambda = R$ or C, there is a sequence Ψ^k_Λ such that this system of sequences has the following properties.*

(i) *$\psi^k_{\Lambda,n}$ is a virtual representation of* $\mathrm{GL}(n, \Lambda)$ *over Λ, with virtual degree n.*

(ii) *The sequence Ψ^k_Λ is both additive and multiplicative (in the sense of* § 3).

(iii) *$\psi^k_{\Lambda,1}$ is the k^{th} power of the identity representation of* $\mathrm{GL}(1, \Lambda)$. *(For $k \geqq 0$, the k^{th} power is taken in the sense of the tensor product. The k^{th} power also makes sense for $k < 0$, since* 1*-dimensional representations are invertible.)*

(iv) *If c is the sequence of injections c_n:* $\mathrm{GL}(n, R) \to \mathrm{GL}(n, C)$ *(as in* § 3) *then*

$$\Psi^k_C \cdot c = c \cdot \Psi^k_R .$$

(v) $\Psi^k_\Lambda \cdot \Psi^l_\Lambda = \Psi^{kl}_\Lambda$.

(vi) *Let G be a topological group (with typical element g) and let θ be a virtual representation of G over Λ; then the following formula holds for the characters χ.*

$$\chi(\Psi^k_\Lambda \cdot \theta)g = \chi(\theta)g^k .$$

(vii) *$\psi^1_{\Lambda,n}$ is the identity representation. $\psi^0_{\Lambda,n}$ is the trivial representation of degree n. $\psi^{-1}_{\Lambda,n}$ is the representation defined by*

$$\psi^{-1}_{\Lambda,n}(M) = ({}^T M)^{-1} ,$$

where ${}^T M$ is the transpose of the matrix M.

PROOF. We begin by recalling the definition of the r^{th} exterior power.

If V is a vector space over Λ, then the r^{th} exterior power $E^r(V)$ is a vector space over Λ given by generators and relations. The generators are symbols $v_1 \wedge v_2 \wedge \cdots \wedge v_r (v_i \in V)$; the relations state that these symbols are multilinear and anti-symmetric in their arguments. Since $E^r(V)$ is a covariant functor, any automorphism of V induces one of $E^r(V)$. Let us choose a base $v_1, v_2, \cdots, v_n$ in V and take as our base in $E^r(V)$ the elements $v_{i_1} \wedge v_{i_2} \wedge \cdots \wedge v_{i_r} (i_1 < i_2 < \cdots < i_r)$; we obtain a definite representation

$$E^r_\Lambda \colon \mathrm{GL}(n, \Lambda) \longrightarrow \mathrm{GL}(m, \Lambda) ,$$

where $m = n!/(r!(n-r)!)$. These representations are evidently compatible with "complexification", in the sense that

$$E^r_C \cdot c_n = c_m \cdot E^r_R .$$

We next consider the polynomial $\sum_{1 \leqq i \leqq n} (x_i)^k$ in the variables $x_1, x_2, \cdots, x_n$. Since this polynomial is symmetric, it can be written as a polynomial in the elementary symmetric functions $\sigma_1, \sigma_2, \cdots, \sigma_n$ of $x_1, x_2, \cdots, x_n$; say

$$\sum_{1 \leqq i \leqq n} (x_i)^k = Q^k_n(\sigma_1, \sigma_2, \cdots, \sigma_n) .$$

For $k \geqq 0$, we now define

$$\psi^k_{\Lambda, n} = Q^k_n(E^1_\Lambda, E^2_\Lambda, \cdots, E^n_\Lambda) .$$

(The polynomial is evaluated in the ring $K'_\Lambda(\mathrm{GL}(n, \Lambda))$.) To obtain the virtual degree of our representations, we substitute $x_1 = 1, x_2 = 1, \cdots, x_n = 1$; we find that the virtual degree of $\psi^k_{\Lambda, n}$ is n.

As trivial cases, we see that Ψ^1_Λ and Ψ^0_Λ are as described in conclusion (vii), while $\psi^k_{\Lambda, 1}$ is as described in conclusion (iii) for $k \geqq 0$.

We next define Ψ^{-1}_Λ to be as described in conclusion (vii); that is, $\psi^{-1}_{\Lambda, n}$ is the representation defined by

$$\psi^{-1}_{\Lambda, n}(M) = ({}^T M)^{-1} .$$

We define ψ^{-k}_Λ for $k > 1$ by setting

$$\psi^{-k}_{\Lambda, n} = \psi^k_{\Lambda, n} \cdot \psi^{-1}_{\Lambda, n} .$$

It is clear that conclusion (iii) holds for $k < 0$.

PROPOSITION 4.2. $\psi^k_{C, n} \cdot c_n = c \cdot \psi^k_{R, n}$.

PROOF. Since "complexification" commutes with exterior powers as well as with sums and products, this is obvious for $k \geqq 0$. It is clear for $k = -1$, and the case $k < -1$ follows.

PROPOSITION 4.3. *For each matrix M in* $\mathrm{GL}(n, \Lambda)$, *we have*

$$\chi(\psi^k_{\Lambda,n})M = \mathrm{Tr}(M^k) ,$$

where $\mathrm{Tr}(M^k)$ *denotes the trace of* M^k.

PROOF. We begin by recalling the basic facts about characters. If α is a representation of G, then its character $\chi(\alpha)$ is defined by

$$\chi(\alpha)g = \mathrm{Tr}(\alpha g) \qquad (g \in G) .$$

We have

$$\begin{aligned}\chi(\alpha \oplus \beta) &= \chi(\alpha) + \chi(\beta) ,\\ \chi(\alpha \otimes \beta) &= \chi(\alpha)\chi(\beta) .\end{aligned}$$

By linearity, one defines the character $\chi(\theta)$ of a virtual representation θ.

We next remark that, in proving Proposition 4.3, it is sufficient to consider the case $\Lambda = C$. In fact, if α is a real representation it is clear that $\chi(c \cdot \alpha) = \chi(\alpha)$; hence if θ is a virtual representation we have $\chi(c \cdot \theta) = \chi(\theta)$. This remark, together with Proposition 4.2, enables one to deduce the case $\Lambda = R$ from the case $\Lambda = C$.

Let us suppose (to begin with) that M is a diagonal matrix with non-zero complex entries $x_1, x_2, \cdots, x_n$. Then $E^r_C(M)$ is a diagonal matrix with entries $x_{i_1}x_{i_2}\cdots x_{i_r}(i_1 < i_2 < \cdots < i_r)$. Hence $\chi(E^r_C)M$ is the r^{th} elementary symmetric function σ_r of the x_i. It follows that if Q is any polynomial in n variables, we have

$$\chi\big(Q(E^1_C, E^2_C, \cdots, E^n_C)\big)M = Q(\sigma_1, \sigma_2, \cdots, \sigma_n) .$$

Substituting $Q = Q^k_n$, we find

$$\chi(\psi^k_{C,n})M = \textstyle\sum_{1 \leqq i \leqq n}(x_i)^k = \mathrm{Tr}(M^k) \qquad (k \geqq 0) .$$

This result depends only on the conjugacy class of M in $\mathrm{GL}(n, C)$, and is therefore true for any M conjugate to a diagonal matrix. But such M are everywhere dense in $\mathrm{GL}(n, C)$, and both sides of the equation are continuous in M; therefore the result holds for all M in $\mathrm{GL}(n, C)$, at least if $k \geqq 0$. It remains only to note that

$$\begin{aligned}\chi(\psi^{-k}_{C,n})M &= \chi(\psi^k_{C,n})({}^TM)^{-1}\\ &= \mathrm{Tr}({}^TM)^{-k}\\ &= \mathrm{Tr}(M^{-k}) .\end{aligned}$$

The proof is complete.

PROPOSITION 4.4. *For each representation* $\alpha\colon G \to \mathrm{GL}(n, \Lambda)$ *and each* $g \in G$ *we have*

$$\chi(\psi^k_{\Lambda,n} \cdot \alpha)g = \chi(\alpha)g^k .$$

PROOF. Substitute $M = \alpha g$ in Proposition 4.3.

PROPOSITION 4.5. *The sequence Ψ^k_Λ is additive and multiplicative.*

PROOF. Let

$$\pi\colon \mathrm{GL}(n, \Lambda) \times \mathrm{GL}(m, \Lambda) \longrightarrow \mathrm{GL}(n, \Lambda)\ ,$$
$$\varpi\colon \mathrm{GL}(n, \Lambda) \times \mathrm{GL}(m, \Lambda) \longrightarrow \mathrm{GL}(m, \Lambda)$$

be the projections of $\mathrm{GL}(n, \Lambda) \times \mathrm{GL}(m, \Lambda)$ onto its two factors, as in § 3. We have to prove

(i) $\psi^k_{\Lambda,n+m}\cdot(\pi \oplus \varpi) = (\psi^k_{\Lambda,n}\cdot\pi) + (\psi^k_{\Lambda,m}\cdot\varpi)$,

(ii) $\psi^k_{\Lambda,nm}\cdot(\pi \otimes \varpi) = (\psi^k_{\Lambda,n}\cdot\pi) \otimes (\psi^k_{\Lambda,m}\cdot\varpi)$.

We begin by checking that in each equation, the characters of the two sides agree. An obvious calculation, based on Proposition 4.4, shows that

$$\begin{aligned}\chi[\psi^k_{\Lambda,n+m}\cdot(\pi \oplus \varpi)]g &= (\chi(\pi) + \chi(\varpi))g^k \\ &= \chi[(\psi^k_{\Lambda,n}\cdot\pi) + (\psi^k_{\Lambda,m}\cdot\varpi)]g\ .\end{aligned}$$

Similarly for equation (ii), the common answer being $(\chi(\pi)\chi(\varpi))g^k$.

Consider the case $\Lambda = C$, and examine the subgroup $\mathrm{U}(n) \times \mathrm{U}(m) \subset \mathrm{GL}(n, C) \times \mathrm{GL}(m, C)$. This subgroup is compact, and therefore two virtual representations coincide on it if and only if they have the same characters. By transporting negative terms to the opposite side of our equations, we now face the following situation: two representations are defined on $\mathrm{GL}(n, C) \times \mathrm{GL}(m, C)$ and agree on $\mathrm{U}(n) \times \mathrm{U}(m)$; we wish to show that they agree on $\mathrm{GL}(n, C) \times \mathrm{GL}(m, C)$. Now, it is a theorem that two analytic representations which are defined on $\mathrm{GL}(n, C)$ and agree on $\mathrm{U}(n)$ agree also on $\mathrm{GL}(n, C)$. (Such analytic representations define C-linear maps of Lie algebras. The Lie algebra of $\mathrm{GL}(n, C)$ is the space of all $n \times n$ complex matrices; the Lie algebra of $\mathrm{U}(n)$ is the space of skew-hermitian matrices; two C-linear maps which agree on the latter agree on the former. The map of the Lie algebra determines the map of the Lie group.) The same argument clearly applies to the subgroup $\mathrm{U}(n) \times \mathrm{U}(m)$ in $\mathrm{GL}(n, C) \times \mathrm{GL}(m, C)$. Moreover, all our representations are clearly analytic. This completes the proof in the case $\Lambda = C$.

We now consider the case $\Lambda = R$. We face the following situation. Two real representations are given over $\mathrm{GL}(n, R) \times \mathrm{GL}(m, R)$; it has been proved that after composing with c ("complexifying") they become equivalent. We wish to show that the real representations are equivalent. Now, it is a theorem that if two real representations α, β of G are equivalent over C, then they are equivalent over R. (Suppose given a complex non-singular matrix P such that $P\alpha(g) = \beta(g)P$ for all $g \in G$. Then for any complex number λ, the matrix $Q = \lambda P + \bar{\lambda}\bar{P}$ is real and such that $Q\alpha(g) = \beta(g)Q$ for all $g \in G$. In order to ensure the non-singularity of $Q = P(\lambda\bar{\lambda}^{-1}I + P^{-1}\bar{P})\bar{\lambda}$, it is sufficient to ensure that $-\lambda\bar{\lambda}^{-1}$ is not an eigenvalue of $P^{-1}\bar{P}$.) This completes the proof of Proposition 4.5.

Since the sequences Ψ^k_Λ have now been shown to be additive, the various compositions written in Theorem 4.1 are well-defined. Conclusion (iv) is a restatement of Proposition 4.2, and conclusion (vi) follows from Proposition 4.4 by linearity. It remains only to prove the following.

PROPOSITION 4.6. $\Psi^k_\Lambda \cdot \psi^l_{\Lambda,n} = \psi^{kl}_{\Lambda,n}$.

PROOF. We begin by checking that the characters of the two sides agree. An obvious calculation, based on Proposition 4.3 and conclusion (vi), shows that

$$\begin{aligned}\chi[\Psi^k_\Lambda \cdot \psi^l_{\Lambda,n}]M &= \operatorname{Tr}(M^{kl}) \\ &= \chi(\psi^{kl}_{\Lambda,n})M\ .\end{aligned}$$

The proof is completed as for Proposition 4.5.

This completes the proof of Theorem 4.1.

REMARK. Grothendieck has considered abstract rings which admit "exterior power" operations λ^i. It is evidently possible to define operations Ψ^k (for $k \geqq 0$) in such rings.

5. Cohomology operations in $K_\Lambda(X)$

In this section we shall use the results of §§ 3, 4 to construct and study certain natural cohomology operations defined in $K_\Lambda(X)$. It would perhaps be interesting to determine the set of all such operations (as defined by some suitable set of axioms); but for our present purposes this is not necessary.

By applying Lemma 3.8 to the sequences of Theorem 4.1, we obtain operations

$$\Psi^k_\Lambda\colon K_\Lambda(X) \longrightarrow K_\Lambda(X)\ ,$$

where k is any integer (positive, negative or zero) and $\Lambda = R$ or C.

THEOREM 5.1. *These operations enjoy the following properties.*

(i) Ψ^k_Λ *is natural for maps of* X.

(ii) Ψ^k_Λ *is a homomorphism of rings with unit.*

(iii) *If* ξ *is a line bundle over* X, *then* $\Psi^k_\Lambda\xi = \xi^k$.

(A line bundle is a bundle with fibres of dimension 1. For $k \geqq 0$ the k^{th} power ξ^k is taken in the sense of the tensor product. The k^{th} power also makes sense for $k < 0$, since line bundles are invertible.)

(iv) *The following diagram is commutative.*

$$\begin{array}{ccc} K_R(X) & \xrightarrow{\Psi_R^k} & K_R(X) \\ {\scriptstyle c}\downarrow & & \downarrow{\scriptstyle c} \\ K_C(X) & \xrightarrow{\Psi_C^k} & K_C(X) \end{array} .$$

(v) $\Psi_\Lambda^k(\Psi_\Lambda^l(K)) = \Psi_\Lambda^{kl}(K)$.

(vi) *If* $\kappa \in K_C(X)$ *and* $\mathrm{ch}^q\kappa$ *denotes the* $2q$*-dimensional component of the Chern character* $\mathrm{ch}(\kappa)$ [2, 3, 4], *then*

$$\mathrm{ch}^q(\Psi_C^k\kappa) = k^q\mathrm{ch}^q(\kappa) .$$

(vii) Ψ_Λ^1 *and* Ψ_R^{-1} *are identity functions.* Ψ_Λ^0 *is the function which assigns to each bundle over* X *the trivial bundle with fibres of the same dimension.* Ψ_C^{-1} *coincides with the operation* t *considered in* § 3.

PROOF. Parts (i) and (ii) of the theorem follow directly from Lemma 3.8 and Theorem 4.1 (parts (i), (ii)). By using the results of § 3 where necessary, parts (iii), (iv), (v) and (vii) of the theorem follow from the correspondingly-numbered parts of Theorem 4.1, except that it remains to identify Ψ_Λ^{-1}. If $\Lambda = R$, then any n-plane bundle is equivalent to one with structural group $\mathrm{O}(n)$, and for $M \in \mathrm{O}(n)$ we have $({}^TM)^{-1} = M$. If $\Lambda = C$, then any n-plane bundle is equivalent to one with structural group $\mathrm{U}(n)$, and for $M \in \mathrm{U}(n)$ we have $({}^TM)^{-1} = \bar{M}$. This completes the identification of Ψ_Λ^{-1}.

It remains to prove (vi). We first recall the basic facts about the Chern character. If $\Lambda = C$ and ξ is a bundle over X, then $\mathrm{ch}^q(\xi)$ is a characteristic class of ξ lying in $H^{2q}(X; Q)$ (where Q denotes the rationals). The main properties of $\mathrm{ch} = \sum_{q=0}^{\infty} \mathrm{ch}^q$ are as follows.

(i) ch defines a ring homomorphism from $K_C(X)$ to $H^*(X; Q)$.

(ii) ch is natural for maps of X.

(iii) If ξ is the canonical line bundle over CP^n, then $\mathrm{ch}\,\xi = e^{-x}$, where x is the generator of $H^2(CP^n; Z)$ and e^{-x} is interpreted as a power series.

We now turn to the proof. Let $\mathrm{T} \subset \mathrm{U}(n)$ be a (maximal) torus consisting of the diagonal matrices with diagonal elements of unit modulus. The classifying space BT is a product of complex projective spaces CP^∞. Let $Y \subset \mathrm{BT}$ be the corresponding product of complex projective spaces CP^N, where $N \geqq q$; let $x_1, x_2, \cdots, x_n$ be the cohomology generators. We may evidently imbed Y in a finite CW-complex X, and extend the inclusion $i\colon Y \to \mathrm{BU}(n)$ to a map $f\colon X \to \mathrm{BU}(n)$, so that f is an equivalence up to any required dimension. We see (by naturality and linearity) that it is sufficient to prove (vi) when the space concerned is X and the element κ is the bundle ξ over X induced by f from the canonical $\mathrm{U}(n)$-bundle over $\mathrm{BU}(n)$.

Let $i\colon Y \to X$ be the inclusion. According to Borel [5], the map

$$i^*\colon H^{2q}(X; Q) \longrightarrow H^{2q}(Y; Q)$$

is a monomorphism. Moreover,

$$i^*\xi = \xi_1 \oplus \xi_2 \oplus \cdots \oplus \xi_n ,$$

where $\xi_1, \xi_2, \cdots, \xi_n$ are line-bundles induced from the canonical line-bundles over the factors of Y. We now have

$$\begin{aligned} i^* \operatorname{ch} (\Psi^k_C \xi) &= \operatorname{ch} \Psi^k_C (i^*\xi) \\ &= \operatorname{ch} \Psi^k_C (\xi_1 \oplus \xi_2 \oplus \cdots \oplus \xi_n) \\ &= \operatorname{ch} (\Psi^k_C \xi_1 + \Psi^k_C \xi_2 + \cdots + \Psi^k_C \xi_n) \\ &= \operatorname{ch} \big((\xi_1)^k + (\xi_2)^k + \cdots + (\xi_n)^k\big) \\ &= (\operatorname{ch} \xi_1)^k + (\operatorname{ch} \xi_2)^k + \cdots + (\operatorname{ch} \xi_n)^k \\ &= e^{-kx_1} + e^{-kx_2} + \cdots + e^{-kx_n} . \end{aligned}$$

Similarly,

$$\begin{aligned} i^* \operatorname{ch} \xi &= \operatorname{ch} i^*\xi \\ &= \operatorname{ch} (\xi_1 \oplus \xi_2 \oplus \cdots \oplus \xi_n) \\ &= e^{-x_1} + e^{-x_2} + \cdots + e^{-x_n} . \end{aligned}$$

Comparing the components in dimension $2q$, we find the required result.

This completes the proof of Theorem 5.1.

In order to state the next corollary, we recall some notation [3]. Let P denote a point. Then, for any X, $K_\Lambda(P)$ is a direct summand of $K_\Lambda(X)$. We write $\widetilde{K}_\Lambda(X)$ for the complementary direct summand. Evidently our operations Ψ^k_Λ act on $\widetilde{K}_\Lambda(X)$. (Actually $K_\Lambda(P) = Z$, and $\Psi^k_\Lambda\colon K_\Lambda(P) \longrightarrow K_\Lambda(P)$ is the identity.)

COROLLARY 5.2. *The operations*

$$\begin{aligned} &\Psi^k_C\colon \widetilde{K}_C(S^{2q}) \longrightarrow \widetilde{K}_C(S^{2q}) , \\ &\Psi^k_R\colon \widetilde{K}_R(S^{2q}) \longrightarrow \widetilde{K}_R(S^{2q}) \qquad (q \text{ even}) \end{aligned}$$

are given by

$$\Psi^k_\Lambda(\kappa) = k^q \kappa .$$

PROOF. It is a well-known corollary of Bott's work that

$$\operatorname{ch}^q\colon \widetilde{K}_C(S^{2q}) \longrightarrow H^{2q}(S^{2q}; Q)$$

maps $\widetilde{K}_C(S^{2q})$ isomorphically onto the image of $H^{2q}(S^{2q}; Z)$. (See, for example, [4] Proposition 2.2.) The result for $\Lambda = C$ now follows from Theorem 5.1 (vi).

It also follows from Bott's work that if q is even, then

$$c\colon Z = \widetilde{K}_R(S^{2q}) \longrightarrow \widetilde{K}_C(S^{2q}) = Z$$

is monomorphic. (In fact, $\operatorname{Im} c = Z$ if $q \equiv 0 \bmod 4$, while $\operatorname{Im} c = 2Z$ if $q \equiv 2 \bmod 4$. This can be derived from the corresponding result for the homomorphism

$$c_*\colon \pi_{2q-1}(\mathrm{O}) \longrightarrow \pi_{2q-1}(\mathrm{U}) \ ;$$

and this in turn can be obtained from the results on U/O given in [6, p. 315].) The case $\Lambda = R$ therefore follows from the case $\Lambda = C$.

In order to state our next corollary, we recall that according to [7, Theorem 1] we have the following isomorphisms.

$$I\colon \widetilde{K}_C(X) \xrightarrow{\cong} \widetilde{K}_C(S^2X) \ ,$$
$$J\colon \widetilde{K}_R(X) \xrightarrow{\cong} \widetilde{K}_R(S^8X) \ .$$

Here S^rX denotes the r^{th} suspension of X, and I, J are defined as follows. The groups $\widetilde{K}_C(S^2X)$, $\widetilde{K}_R(X^8X)$ are represented as direct summands in $\widetilde{K}_C(S^2 \times X)$, $\widetilde{K}_R(S^8 \times X)$. We now define

$$I(\kappa) = \pi^*\lambda \otimes \varpi^*\kappa \ ,$$
$$J(\kappa) = \pi^*\mu \otimes \varpi^*\kappa \ .$$

Here π, ϖ denote the projections of $S^2 \times X$, resp. $S^8 \times X$ on its factors, and the elements $\lambda \in \widetilde{K}_C(S^2)$, $\mu \in \widetilde{K}_R(S^8)$ are generators.

COROLLARY 5.3. *The following diagrams are* NOT *commutative.*

$$\begin{array}{ccc} \widetilde{K}_C(X) & \xrightarrow{I} & \widetilde{K}_C(S^2X) \\ {\scriptstyle \Psi_C^k}\downarrow & & \downarrow{\scriptstyle \Psi_C^k} \\ \widetilde{K}_C(X) & \xrightarrow{I} & \widetilde{K}_C(S^2X) \ , \end{array} \qquad \begin{array}{ccc} \widetilde{K}_R(X) & \xrightarrow{J} & \widetilde{K}_R(S^8X) \\ {\scriptstyle \Psi_R^k}\downarrow & & \downarrow{\scriptstyle \Psi_R^k} \\ \widetilde{K}_R(X) & \xrightarrow{J} & \widetilde{K}_R(S^8X) \ . \end{array}$$

In fact, we have

$$\Psi_C^k \cdot I = kI \cdot \Psi_C^k \ , \qquad \Psi_R^k \cdot J = k^4 J \cdot \Psi_R^k \ .$$

PROOF. Consider the case $\Lambda = C$. Using the previous corollary, we have

$$\begin{aligned} \Psi_C^k \cdot I(\kappa) &= \Psi_C^k(\pi^*\lambda \otimes \varpi^*\kappa) \\ &= (\pi^*\Psi_C^k\lambda) \otimes (\varpi^*\Psi_C^k\kappa) \\ &= k(\pi^*\lambda) \otimes (\varpi^*\Psi_C^k\kappa) \\ &= kI \cdot \Psi_C^k(\kappa) \ . \end{aligned}$$

Similarly for the case $\Lambda = R$.

6. A spectral sequence

In this section we recall certain extra material on the groups $K(X)$.

To begin with, recall from [3] that one can define the groups of a "cohomology theory" as follows. Let Y be a subcomplex of X, and let X/Y be the space obtained by identifying Y with a newly-introduced base-point. Define

$$K_\Lambda^{-n}(X, Y) = \widetilde{K}_\Lambda(S^n(X/Y)) .$$

If Y is empty we have $K_\Lambda^0(X, \varphi) \cong K_\Lambda(X)$.

Using the Bott periodicity (as at the end of § 5) one shows that

$$K_C^{-n-2}(X, Y) \cong K_C^{-n}(X, Y) ,$$
$$K_R^{-n-8}(X, Y) \cong K_R^{-n}(X, Y) .$$

One may use these equations to define the abelian groups $K_\Lambda^n(X, Y)$ for positive values of n.

Nota bene. Owing to the state of affairs revealed in Corollary 5.3, we shall be most careful not to identify $K_C^{-n-2}(X, Y)$ with $K_C^{-n}(X, Y)$ or $K_R^{-n-8}(X, Y)$ with $K_R^{-n}(X, Y)$. We therefore regard $K_\Lambda^n(X, Y)$ as graded over Z, not over Z_2 or Z_8. Given this precaution one can define operations Ψ_Λ^k in $K_\Lambda^n(X, Y)$ for $n \leqq 0$; however, we shall not need such operations. We shall use operations Ψ_Λ^k only in $K_\Lambda(X)$ and $\widetilde{K}_\Lambda(X)$ (that is, in dimension $n = 0$); the groups $K_\Lambda^n(X, Y)$ with $n \neq 0$ will be used only to help in calculating the additive structure of $K_\Lambda^0(X, Y)$. This will avoid any confusion.

We next recall from [3] that one can define induced maps and coboundary maps between the groups $K^n(X, Y)$, so that these groups verify all the Eilenberg-Steenrod axioms [9] except for the dimension axiom. (If one chose to introduce operations Ψ_Λ^k into the groups $K_\Lambda^n(X, Y)$ for $n \leqq 0$, then these operations would commute with induced maps and coboundary maps, because both are defined in terms of induced maps of $\widetilde{K}_\Lambda$.)

We next recall from [3, § 2] the existence of a certain spectral sequence. Let X be a finite CW-complex, and let X^p denote its p-skeleton. Then each pair X^p, X^q yields an exact sequence of groups K_Λ^n. These exact sequences yield a spectral sequence. The E_∞ term of the spectral sequence is obtained by filtering $K_\Lambda^*(X) = \sum_{n=-\infty}^{+\infty} K_\Lambda^n(X)$. The E_1 and E_2 terms of the spectral sequence are given by

$$E_1^{p,q} \cong C^p(X; K_\Lambda^q(P)) , \qquad E_2^{p,q} \cong H^p(X; K_\Lambda^q(P)) ,$$

where P is a point. The values of $K_\Lambda^q(P)$ are given by the homotopy groups of BO or BU; they are as follows, by [6, p. 315].

$q \equiv 0,$	-1	-2	-3	-4	-5	-6	-7	-8	mod 8
$K_C^q(P) = Z$	0	Z	0	Z	0	Z	0	Z	
$K_R^q(P) = Z$	Z_2	Z_2	0	Z	0	0	0	Z.	

In what follows it will sometimes be useful to know that the spectral sequence is defined in this particular way. For example, $\tilde{K}_\Lambda(X)$ is filtered by the images of the groups $\tilde{K}_\Lambda(X/X^{p-1})$; and if we are given an explicit element κ in $\tilde{K}_\Lambda(X/X^{p-1})$, then we can take the image of κ in $\tilde{K}_\Lambda(X^{p-1+r}/X^{p-1})$, and so, by passing to quotients, obtain an explicit element κ_r in $E_r^{p,-p}$ (for $1 \leqq r \leqq \infty$), so that $d_r\kappa_r = 0$ and the homology class of κ_r is κ_{r+1}, while κ_∞ is the class containing κ. Again, an element κ in $\tilde{K}_C(X^p/X^{p-1})$ gives an element in $E_1^{p,-p}$; the space X^p/X^{p-1} is a wedge-sum of spheres S^p, and we can tell whether κ is a generator or not by examining ch κ (cf. the proof of Lemma 5.2).

On the other hand, the work that follows has been so arranged that we do not need any theorem concerning the identification of the differentials in the spectral sequence in terms of cohomology operations.

7. Computations for projective spaces

It is a pleasure to acknowledge at this point my indebtedness to J. Milnor, who read a draft of the following section and suggested several improvements.

In this section we shall calculate the various rings $K_\Lambda(X)$ which we require, together with their operations Ψ_Λ^k. Our plan is to obtain our results in the following order.

(i) Results on complex projective spaces for $\Lambda = C$.

(ii) Results on real projective spaces for $\Lambda = C$.

(iii) Results on real projective spaces for $\Lambda = R$.

These results are stated as Theorems 7.2, 7.3 and 7.4. For completeness, every stunted projective space is considered, whether or not it arises in the applications. The theorems are preceded by one lemma, which we need in order to specify generators in our rings.

LEMMA 7.1. *Let ξ be the canonical real line-bundle over RP^{2n-1}; let η be the canonical complex line-bundle over CP^{n-1}; let π: $RP^{2n-1} \to CP^{n-1}$ be the standard projection. Then we have*

$$c\xi = \pi^*\eta .$$

PROOF. Complex line bundles are classified by their first Chern class c_1. In our case this lies in $H^2(RP^{2n-1}; Z)$, which is Z_2, at least if $n > 1$ (the case $n = 1$ being trivial). We have $c_1\pi^*\eta = \pi^*c_1\eta \neq 0$. It is therefore sufficient to show that the bundle $c\xi$ is non-trivial. Let w denote the total Stiefel-Whitney class, let x be the generator of $H^1(RP^{2n-1}; Z_2)$, and let r be as in Lemma 3.9. Then we have $rc\xi = \xi \oplus \xi$ and $w(rc\xi) = 1 + x^2$. This shows that $c\xi$ is non-trivial, and completes the proof.

In terms of the canonical line-bundles we introduce the following elements λ, μ, ν.

$$\begin{aligned} \lambda &= \xi - 1 \in \widetilde{K}_R(RP^n) , \\ \mu &= \eta - 1 \in \widetilde{K}_C(CP^n) , \\ \nu &= c\lambda = \pi^*\mu \in \widetilde{K}_C(RP^n) . \end{aligned}$$

In terms of these elements we may write polynomials $Q(\lambda)$, $Q(\mu)$, $Q(\nu)$.

THEOREM 7.2. *$K_C(CP^n)$ is a truncated polynomial ring (over the integers) with one generator μ and one relation $\mu^{n+1} = 0$. The operations are given by*

$$\Psi_C^k \cdot \mu^s = ((1 + \mu)^k - 1)^s .$$

The projection $CP^n \to CP^n/CP^m$ maps $\widetilde{K}_C(CP^n/CP^m)$ isomorphically onto the subgroup of $K_C(CP^n)$ generated by $\mu^{m+1}, \mu^{m+2}, \cdots, \mu^n$.

Note. If k is negative, $(1 + \mu)^k$ may be interpreted by means of the binomial expansion

$$1 + \kappa\mu + \frac{k(k-1)}{2!}\mu^2 + \cdots .$$

This expansion terminates because $\mu^{n+1} = 0$.

PROOF. So far as the additive and multiplicative structures go, this result is due to Atiyah and Todd; see [4, Propositions 2.3, 3.1 and 3.3]. In any case, it is almost evident. The spectral sequence of § 6 shows that $K_C^s(CP^n, CP^m)$ is zero for s odd, and free abelian on $n - m$ generators for s even. Let us examine matters more closely, and suppose as an inductive hypothesis that $K_C(CP^{n-1})$ is as stated (this is trivial for $n = 1$). Then the elements $1, \mu, \mu^2, \cdots, \mu^{n-1}$ in $K_C(CP^n)$ project into a Z-base for $K_C(CP^{n-1})$. Moreover, the element μ^n in $K_C(CP^n)$ projects into zero in $K_C(CP^{n-1})$, so it must come from $\widetilde{K}_C(CP^n/CP^{n-1})$. Let $y \in H^2(CP^n; Z)$ be the cohomology generator; then $\mathrm{ch}\,\mu = y + y^2/2 + \cdots$ and $\mathrm{ch}\,\mu^n = y^n$; hence μ^n comes from a generator of $\widetilde{K}_C(CP^n/CP^{n-1}) = Z$. Using the exact sequence of the pair CP^n, CP^{n-1}, we see that $1, \mu, \mu^2, \cdots, \mu^n$ form a Z-base for $K_C(CP^n)$. It is now clear that

$$\mathrm{ch}\colon K_C(CP^n) \longrightarrow H^*(CP^n; Q)$$

is monomorphic. (This also follows from a general theorem; see [3, § 2.5], [4, Prop. 2.3].) Since $\mathrm{ch}(\mu^{n+1}) = 0$, we have $\mu^{n+1} = 0$. This completes the induction, and establishes the result about $K_C(CP^n)$. The result about

$\widetilde{K}_C(CP^n/CP^m)$ follows immediately from the exact sequence of the pair CP^n, CP^m.

It remains to calculate the operations. According to Theorem 5.1 (iii), we have $\Psi_C^k(\eta) = \eta^k$; that is

$$\Psi_C^k(1 + \mu) = (1 + \mu)^k .$$

Hence

$$\Psi_C^k(\mu) = (1 + \mu)^k - 1$$

and

$$\Psi_C^k(\mu^s) = \big((1 + \mu)^k - 1\big)^s .$$

This completes the proof.

In order to state the next theorem, we define certain generators. We write $\mu^{(s+1)}$ for the element in $\widetilde{K}_C(CP^N/CP^s)$ which maps into μ^{s+1} in $K_C(CP^N)$ (see Theorem 7.2). (It is clear that as we alter N the resulting elements $\mu^{(s+1)}$ map into one another; this justifies us in not displaying N in the notation.)

The standard projection π: $RP^{2N+1} \to CP^N$ factors to give

$$\bar{\varpi}: RP^{2N+1}/RP^{2s+1} \longrightarrow CP^N/CP^s ,$$
$$\varpi: RP^{2N+1}/RP^{2s} \longrightarrow CP^N/CP^s .$$

We write $\bar{\nu}^{(s+1)} = \bar{\varpi}^*\mu^{(s+1)}$, $\nu^{(s+1)} = \varpi^*\mu^{(s+1)}$. It is clear that $\bar{\nu}^{(s+1)}$ maps into $\nu^{(s+1)}$, and $\nu^{(s+1)}$ in turn maps into the element ν^{s+1} in $K_C(RP^{2N+1})$; this explains the notation. (As above, the dependence of these elements on N is negligible.)

THEOREM 7.3. *Assume $m = 2t$. Then we have $\widetilde{K}_C(RP^n/RP^m) = Z_{2^f}$, where f is the integer part of $\frac{1}{2}(n - m)$. If $m = 0$ then $K_C(RP^n)$ may be described by the generator ν and the two relations*

$$\nu^2 = -2\nu , \qquad \nu^{f+1} = 0$$

(so that $2^f\nu = 0$). Otherwise $\widetilde{K}_C(RP^n/RP^m)$ is generated by $\nu^{(t+1)}$ (where $t = \frac{1}{2}m$, as above); and the projection $RP^n \to RP^n/RP^m$ maps $\widetilde{K}_C(RP^n/RP^m)$ isomorphically onto the subgroup of $K_C(RP^n)$ generated by ν^{t+1}.

In the case when m is odd, we have

$$\widetilde{K}_C(RP^n/RP^{2t-1}) = Z + \widetilde{K}_C(RP^n/RP^{2t}) ,$$

where the first summand is generated by $\bar{\nu}^{(t)}$, and the second is imbedded by the projection $RP^n/RP^{2t-1} \to RP^n/RP^{2t}$.

The operations are given by the following formulae.

$$\text{(i)}\quad \Psi_C^k \nu^{(t+1)} = \begin{cases} 0 & (k \text{ even}) \\ \nu^{(t+1)} & (k \text{ odd}) , \end{cases}$$

$$\text{(ii)}\quad \Psi_C^k \bar{\nu}^{(t)} = k^t \bar{\nu}^{(t)} + \begin{cases} \frac{1}{2} k^t \nu^{(t+1)} & (k \text{ even}) \\ \frac{1}{2}(k^t - 1)\nu^{(t+1)} & (k \text{ odd}) . \end{cases}$$

Note 1. As usual, the symbol Z denotes a cyclic infinite group, and the symbol Z_{2^f} denotes a cyclic group of order 2^f.

Note 2. So far as the additive structure of $K_C(RP^n)$ goes, the result is due to J. Milnor (unpublished).

Note 3. The factor $\frac{1}{2}$ in the final formula will be vitally important in what follows; the reader is advised to satisfy himself as to its correctness.

PROOF. We begin by establishing the relation $\nu^2 = -2\nu$ in $K_C(RP^n)$; for this purpose we begin work in $K_R(RP^n)$. A real line-bundle is equivalent to one with structural group $\mathrm{O}(1) = \{+1, -1\}$; it is therefore directly obvious that, for any real line-bundle ξ, we have $\xi \otimes \xi = 1$. (Alternatively, this may be deduced from the fact that real line-bundles are characterized by their first Stiefel-Whitney class w_1.) Taking ξ to be the canonical real line-bundle over RP^n, we have $\xi^2 = 1$, that is, $(1 + \lambda)^2 = 1$ or $\lambda^2 = -2\lambda$. Applying c, we find $(\pi^*\eta)^2 = (c\xi)^2 = 1$ and $\nu^2 = -2\nu$. (Alternatively, the former equation may be deduced from the fact that complex line-bundles are characterized by their first Chern class.)

The relation $\nu^{f+1} = 0$ follows from the fact that ν^{f+1} is the image of $\bar{\nu}^{f+1} \in \tilde{K}_C(RP^N/RP^{2f+1})$ and $2f + 1 \geqq m$.

We now apply the spectral sequence of § 6 to the space $X = RP^n/RP^m$. If n and m are even the group $H^p(X; Z)$ is Z_2 for even p such that $m < p \leqq n$; otherwise it is zero. If $m = 2t - 1$ we obtain an extra group $H^{2t}(X; Z) = Z$. If n is odd we obtain an extra group $H^n(X; Z) = Z$. Let f be the integral part of $\frac{1}{2}(n - 2t)$, where $m = 2t$ or $2t - 1$. Then the elements $\bar{\nu}^{(t+i)} \in \tilde{K}_C(RP^n/RP^{2t+2i-1})$ $(i = 1, 2, \cdots, f)$ yield generators for the f groups Z_2 in our E_2 term, and survive to E_∞ (as explained in § 6). Again, if $m = 2t - 1$, the element $\bar{\nu}^{(t)}$ yields a generator for the corresponding group Z in our E_2 term; this also survives to E_∞. If n is odd the group $H^n(X; Z) = Z$ has odd total degree, and all differentials vanish on it for dimensional reasons. Our spectral sequence is therefore trivial. This leads to the following conclusions.

(i) If $m = 2t$, $\tilde{K}_C(X)$ can be filtered so that the successive quotients are f copies of Z_2, whose generators are the images of $\bar{\nu}^{(t+1)}, \bar{\nu}^{(t+2)}, \cdots, \bar{\nu}^{(t+f)}$.

(ii) If $m = 2t - 1$, we have an exact sequence

$$0 \longleftarrow Z = \widetilde{K}_O(RP^{2t}/RP^{2t-1}) \longleftarrow \widetilde{K}_O(RP^n/RP^{2t-1}) \longleftarrow \widetilde{K}_O(RP^n/RP^{2t}) \longleftarrow 0 ,$$

in which $\bar{\nu}^{(t)}$ maps to a generator of Z.

It is now evident that $\widetilde{K}_O(RP^n/RP^{2t})$ is monomorphically imbedded in $\widetilde{K}_O(RP^n)$.

We have next to determine the group extensions involved in (i) above, in the case $t = 0$. If $t = 0$ then the generators of the successive quotients become $\nu, \nu^2, \cdots, \nu^f$, and the relation $\nu^2 = -2\nu$ resolves the problem; the extension is a cyclic group Z_{2^f} generated by ν.

We have now done all that is needed to determine the additive and multiplicative structures of our groups; it remains to calculate the operations Ψ^k_O.

According to Theorem 5.1 (iii) we have $\Psi^k_O\xi = \xi^k$ for a line-bundle ξ. As remarked above, the line-bundle $\pi^*\eta$ over RP^n satisfies $(\pi^*\eta)^2 = 1$. Therefore

$$\Psi^k_O(\pi^*\eta) = \begin{cases} 1 & (k \text{ even}) \\ \pi^*\eta & (k \text{ odd}) . \end{cases}$$

That is,

$$\Psi^k_O(1 + \nu) = \begin{cases} 1 & (k \text{ even}) \\ 1 + \nu & (k \text{ odd}) . \end{cases}$$

Therefore

$$\Psi^k_O(\nu) = \begin{cases} 0 & (k \text{ even}) \\ \nu & (k \text{ odd}) , \end{cases}$$

and

$$\Psi^k_O(\nu^s) = \begin{cases} 0 & (k \text{ even}) \\ \nu^s & (k \text{ odd}) . \end{cases}$$

Since $\widetilde{K}_O(RP^n/RP^{2t})$ is monomorphically imbedded in $K_O(RP^n)$, the result about $\Psi^k_O\nu^{(t+1)}$ follows.

We necessarily have

$$\Psi^k_O\bar{\nu}^{(t)} = a\bar{\nu}^{(t)} + b\nu^{(t+1)}$$

for some coefficients a, b; our problem is to determine them. By using the injection $RP^{2t}/RP^{2t-1} \to RP^n/RP^{2t-1}$ and Corollary 5.2 for $RP^{2t}/RP^{2t-1} = S^{2t}$, we see that $a = k^t$. Now project into RP^n/RP^{2t-2}; $\bar{\nu}^{(t)}$ maps into $\nu^{(t)}$ and $\nu^{(t+1)}$ into $-2\nu^{(t)}$, and we see that

$$\Psi^k_O\nu^{(t)} \equiv a\nu^{(t)} - 2b\nu^{(t)} \qquad \bmod 2^{f+1} .$$

Therefore

$$b \equiv \tfrac{1}{2}(k^t - \varepsilon) \qquad \bmod 2^f ,$$

where $\varepsilon = 0$ or 1 according as k is even or odd. This completes the proof.

REMARK. An alternative method for obtaining the last formula is as follows. According to Theorem 7.2, we have in $\widetilde{K}_C(CP^N/CP^{t-1})$ the formula

$$\Psi_C^k \mu^{(t)} = k^t \mu^{(t)} + \Sigma ,$$

where Σ denotes a sum of higher terms. Applying the projection $\varpi\colon RP^{2N+1}/SP^{2t-1} \to CP^N/CP^{t-1}$, we find

$$\Psi_C^k \bar{\nu}^{(t)} = k^t \bar{\nu}^{(t)} + \varpi^*\Sigma .$$

It is therefore only necessary to evaluate $\varpi^*\Sigma$, which leads to the same result.

In order to state our next theorem, we define $\varphi(n, m)$ to be the number of integers s such that $m < s \leqq n$ and $s \equiv 0, 1, 2$ or $4 \bmod 8$.

THEOREM 7.4. *Assume $m \not\equiv -1 \bmod 4$. Then we have $\widetilde{K}_R(RP^n/RP^m) = Z_{2^f}$, where $f = \varphi(n, m)$. If $m = 0$, then $K_R(RP^n)$ may be described by the generator λ and the two relations*

$$\lambda^2 = -2\lambda , \qquad \lambda^{f+1} = 0$$

(so that $2^f\lambda = 0$). Otherwise the projection $RP^n \to RP^n/RP^m$ maps $\widetilde{K}_R(RP^n/RP^m)$ isomorphically onto the subgroup of $\widetilde{K}_R(RP^n)$ generated by λ^{g+1}, where $g = \varphi(m, 0)$. We write $\lambda^{(g+1)}$ for the element in $\widetilde{K}_R(RP^n/RP^m)$ which maps into λ^{g+1}.

In the case $m \equiv -1 \bmod 4$ we have

$$\widetilde{K}_R(RP^n/RP^{4t-1}) = Z + \widetilde{K}_R(RP^n/RP^{4t}) .$$

Here the second summand is imbedded by the projection $RP^n/RP^{4t-1} \to RP^n/RP^{4t}$, and the first is generated by an element $\bar{\lambda}^{(g)}$ which will be defined below. (We have written g for $\varphi(4t, 0)$.)

The operations are given by the following formulae.

$$\text{(i)}\quad \Psi_R^k \lambda^{(g+1)} = \begin{cases} 0 & (k \text{ even}) \\ \lambda^{(g+1)} & (k \text{ odd}) , \end{cases}$$

$$\text{(ii)}\quad \Psi_R^k \bar{\lambda}^{(g)} = k^{2t}\bar{\lambda}^{(g)} + \begin{cases} \tfrac{1}{2} k^{2t} \lambda^{(g+1)} & (k \text{ even}) \\ \tfrac{1}{2}(k^{2t} - 1)\lambda^{(g+1)} & (k \text{ odd}) . \end{cases}$$

REMARK. So far as the additive structure of $K_R(RP^n)$ goes, the result is due to R. Bott and A. Shapiro (unmimeographed notes).

PROOF. We begin by applying the spectral sequence of § 6 to the space

$X = RP^n/RP^m$. Let us recall from § 6 that

$$K^{-q}(P) = \begin{cases} Z_2 & \text{if} \quad q \equiv 1, 2 \bmod 8 \\ Z & \text{if} \quad q \equiv 0, 4 \bmod 8 \\ 0 & \text{otherwise} . \end{cases}$$

The group $H^p(X, Z_2)$ is Z_2 for $m < p \leqq n$, otherwise zero. If n and m are even the group $H^p(X; Z)$ is Z_2 for even p such that $m < p \leqq n$, otherwise zero. However, if n is odd we obtain an extra group $H^n(X; Z) = Z$, and if m is odd we obtain $H^{m+1}(X; Z) = Z$ instead of Z_2. We can now enumerate the terms $E_2^{p,q}$ in our spectral sequence which have total degree zero. If $m + 1 \not\equiv 0 \bmod 4$ we find (apart from zero groups) just $\varphi(n, m)$ copies of Z_2. If $m + 1 \equiv 0 \bmod 4$ we find $\varphi(n, m)$ groups, of which one is Z and the remainder Z_2.

LEMMA 7.5. *If* $n \equiv 6, 7$ *or* $8 \bmod 8$ *then*

$$c: \widetilde{K}_R(RP^n) \longrightarrow \widetilde{K}_C(RP^n)$$

is an isomorphism.

PROOF. The homomorphism c is always an epimorphism, because $\widetilde{K}_C(RP^n)$ is generated by ν (Theorem 7.3) and $\nu = c\lambda$ (Lemma 7.1). If $n - 8t = 6$ or 7, then $\varphi(n, 0) = 4t + 3$, so that $\widetilde{K}_R(RP^n)$ contains at most 2^{4t+3} elements. On the other hand, $\widetilde{K}_C(RP^n)$ contains exactly 2^{4t+3} elements, by Theorem 7.3. Therefore c is an isomorphism in this case. Similarly if $n = 8t + 8$ (with $4t + 3$ replaced $4t + 4$). This completes the proof.

It follows that $\widetilde{K}_R(RP^n)$ is generated by λ when $n \equiv 6, 7$ or $8 \bmod 8$.

Let us reconsider the spectral sequence for the space $X = RP^n$. We have found $\varphi(n, 0)$ copies of Z_2 with total degree zero in our E_2 term; we have shown that if $n \equiv 6, 7$ or $8 \bmod 8$ they all survive unchanged to E_∞. It follows that the same thing holds for smaller values of n. We conclude that for any n we have $\widetilde{K}_R(RP^n) = Z_{2^f}$, where $f = \varphi(n, 0)$; this group is generated by λ. We have already shown that $\lambda^2 = -2\lambda$ (see the proof of Theorem 7.3). The formula $\lambda^{f+1} = 0$ therefore follows from the fact that $2^f\lambda = 0$.

Let us now consider the exact sequence

$$\widetilde{K}_R(RP^m) \xleftarrow{i^*} \widetilde{K}_R(RP^n) \longleftarrow \widetilde{K}_R(RP^n/RP^m) .$$

The kernel of i^* has 2^f elements, where $f = \varphi(n, 0) - \varphi(m, 0) = \varphi(n, m)$. If $m \not\equiv -1 \bmod 4$ then $\widetilde{K}_R(RP^n/RP^m)$ has at most 2^f elements. It is now clear that $\widetilde{K}_R(RP^n/RP^m)$ maps isomorphically onto the subgroup of $\widetilde{K}_R(RP^n)$ generated by $\pm 2^g\lambda = \pm\lambda^{g+1}$, where $g = \varphi(m, 0)$. We write $\lambda^{(g+1)}$ for the element in $\widetilde{K}_R(RP^n/RP^m)$ which maps into λ^{g+1}. This completes our consideration of the case $m \not\equiv -1 \bmod 4$.

In the case $m \equiv -1 \bmod 4$, our first concern is to show that the following exact sequence splits.

$$Z = \widetilde{K}_R(RP^{4t}/RP^{4t-1}) \xleftarrow{\ i\ } \widetilde{K}_R(RP^n/RP^{4t-1}) \xleftarrow{\ j\ } \widetilde{K}_R(RP^n/RP^{4t}) \ .$$

It is clear that j is a monomorphism, since we have just shown that the composite

$$\widetilde{K}_R(RP^n/RP^{4t}) \xrightarrow{\ j\ } \widetilde{K}_R(RP^n/RP^{4t-1}) \longrightarrow \widetilde{K}_R(RP^n)$$

is monomorphic.

LEMMA 7.6. *The map i is an epimorphism.*

PROOF. Inspect the following commutative diagram, in which the row and columns are exact.

$$\begin{tikzcd}
 & \widetilde{K}_R(RP^{4t-1}/RP^{4t-2}) \arrow[r, phantom, "="] & \widetilde{K}_R(RP^{4t-1}/RP^{4t-2}) \\
 & \widetilde{K}_R(RP^{4t}/RP^{4t-2}) \arrow[u] & \widetilde{K}_R(RP^n/RP^{4t-2}) \arrow[l, "i_1"'] \arrow[u] \\
K_R^1(RP^n/RP^{4t}) & \widetilde{K}_R(RP^{4t}/RP^{4t-1}) \arrow[l, "\delta"'] \arrow[u, "j_1"] & \widetilde{K}_R(RP^n/RP^{4t-1}) \arrow[l, "i"'] \arrow[u, "j_2"'] \ .
\end{tikzcd}$$

We have $RP^{4t-1}/RP^{4t-2} = S^{4t-1}$ and $RP^{4t}/RP^{4t-1} = S^{4t}$; thus $\widetilde{K}_R(RP^{4t-1}/RP^{4t-2}) = 0$, and the maps j_1, j_2 are epimorphic. We have also calculated that $\widetilde{K}_R(RP^{4t}/RP^{4t-2}) = Z_2$ and i_1 is epimorphic. Hence $j_1 i$ is epimorphic. But j_1 is an epimorphism from Z to Z_2; hence $\operatorname{Im} i$ consists of the multiples of some odd number ω, and $\operatorname{Im}\delta = Z_\omega$. But it is clear from the spectral sequence that $K_R^1(RP^n/RP^{4t})$ contains no elements of odd order (except zero). Hence $\omega = 1$ and i is epimorphic. This completes the proof.

We wish next to specify a generator $\bar{\lambda}^{(g)}$. For this purpose we consider the map

$$c\colon \widetilde{K}_R(RP^n/RP^{4t-1}) \longrightarrow \widetilde{K}_C(RP^n/RP^{4t-1}) \ .$$

LEMMA 7.7. *If $n \equiv 6, 7$ or $8 \bmod 8$ then c is an isomorphism for t even, a monomorphism for t odd.*

PROOF. Inspect the following commutative diagram, in which each row is a split exact sequence.

$$\begin{tikzcd}
Z = \widetilde{K}_R(RP^{4t}/RP^{4t-1}) \arrow[d, "c_1"] & \widetilde{K}_R(RP^n/RP^{4t-1}) \arrow[l, "i"'] \arrow[d, "c"] & \widetilde{K}_R(RP^n/RP^{4t}) \arrow[l, "j"'] \arrow[d, "c_3"] \\
Z = \widetilde{K}_C(RP^{4t}/RP^{4s-1}) & \widetilde{K}_C(RP^n/RP^{4t-1}) \arrow[l] & \widetilde{K}_C(RP^n/RP^{4t}) \arrow[l] \ .
\end{tikzcd}$$

We will establish the nature of c_3. Suppose that $t = 2u$ and $n - 8v = 6, 7$ or 8. Then $\widetilde{K}_R(RP^n/RP^{4t}) = Z_{2^f}$, where

$$f - 4(v-u) = \begin{cases} 3 & (n - 8v = 6 \text{ or } 7) \\ 4 & (n - 8v = \phantom{6 \text{ or }} 8) . \end{cases}$$

This group is generated by $\lambda^{(4u+1)}$. We also have $\widetilde{K}_C(RP^n/RP^{4t}) = Z_{2^f}$, generated by $\nu^{(4u+1)}$. We have $c_3\lambda^{(4u+1)} = \nu^{(4u+1)}$, so c_3 is an isomorphism if t is even. Next suppose that $t = 2u + 1$ and $n - 8v = 6, 7$ or 8. Then $\widetilde{K}_R(RP^n/RP^{4t}) = Z_{2^f}$, where

$$f - 4(v-u) = \begin{cases} 0 & (n - 8v = 6 \text{ or } 7) \\ 1 & (n - 8v = \phantom{6 \text{ or }} 8) . \end{cases}$$

This group is generated by $\lambda^{(4u+4)}$. We also have $\widetilde{K}_C(RP^n/RP^{4t}) = Z_{2^{f+1}}$, generated by $\nu^{(4u+3)}$. We have $c_3\lambda^{(4u+4)} = -2\nu^{(4u+3)}$, so c_3 is a monomorphism if t is odd.

According to the results of Bott (as explained during the proof of Corollary 5.2), the map c_1 is an isomorphism for t even and a monomorphism for t odd. The result now follows by the Five Lemma.

We next explain how to choose the generator $\bar{\lambda}^{(g)}$, assuming that $n \equiv 6, 7$ or 8. If $t = 2u$ we take $\bar{\lambda}^{(4u)}$ to be the unique element in $\widetilde{K}_R(RP^n/RP^{4t-1})$ such that $c\bar{\lambda}^{(4u)} = \bar{\nu}^{(4u)}$. If $t = 2u + 1$ we define $\bar{\lambda}^{(4u+3)} = -r\bar{\nu}^{(4u+2)}$; then

$$\begin{aligned} c\bar{\lambda}^{(4u+3)} &= -cr\bar{\nu}^{(4u+2)} \\ &= -(1 + \Psi_C^{-1})\bar{\nu}^{(4u+2)} && \text{(Lemma 3.9)} \\ &= -2\bar{\nu}^{(4u+2)} && \text{(Theorem 7.3)} . \end{aligned}$$

Since $\operatorname{Im} c_1 = 2Z$ in this case, $i\bar{\lambda}^{(4u+3)}$ is a generator, and we may take $\bar{\lambda}^{(4u+3)}$ as our generator for the summand Z in $\widetilde{K}_R(RP^n/RP^{4t-1})$.

So far we have only defined $\lambda^{(g)}$ for $n \equiv 6, 7$ or $8 \bmod 8$. However, by naturality we obtain for smaller values of n an image element, also written $\lambda^{(g)}$, with the same properties. This procedure is clearly self-consistent if we reduce n from n_1 to n_2, where both n_1 and n_2 are congruent to 6, 7, or 8 mod 8.

Whether t is odd or even, one verifies that the image of $\lambda^{(g)}$ in $\widetilde{K}_C(RP^n)$ is ν^g. Therefore the image of $\lambda^{(g)}$ in $\widetilde{K}_R(RP^n)$ is λ^g. This explains the notation.

We now turn to the operations Ψ_R^k. Their values may be obtained by either of the following methods.

(i) The argument given in proving Theorem 7.3 goes over immediately to the case $\Lambda = R$, using the fact that $\lambda = \xi - 1$ and ξ is a line-bundle.

(ii) The operations in $\widetilde{K}_C(RP^n/RP^m)$ are known by Theorem 7.3. The map

$$c\colon \widetilde{K}_R(RP^n/RP^m) \longrightarrow \widetilde{K}_C(RP^n/RP^m)$$

is known, and commutes with the operations. We can deduce the values of the operations Ψ^k_R in $\widetilde{K}_R(RP^n/RP^n)$ if $n \equiv 6, 7$ or $8 \bmod 8$, because c is then a monomorphism (this follows from Lemmas 7.5, 7.7). The results follow for smaller values of n by naturality.

REMARK. It is also possible to compute the groups $\widetilde{K}_R(RP^n/RP^m)$ directly from the spectral sequence of § 6; and this was, of course, the author's original approach. The group extensions are determined by computing

$$K^*_R(RP^n/RP^m;\, Z_2) = K^*_R\big((RP^n/RP^m) \mathbin{⨳} RP^2\big) ,$$

and examining the universal coefficient sequence. It is necessary to know the expression of certain differentials in the spectral sequence in terms of Steenrod squares; it is easy to compute these squares in $(RP^n/RP^m) \mathbin{⨳} RP^2$, using the Cartan formula. No details will be given, but the earnest student may reconstruct them.

8. Proof of Theorem 1.2

In this section we complete the proof of Theorem 1.2. Let us suppose given, then, a map

$$f\colon RP^{m+\rho(m)}/RP^{m-1} \longrightarrow S^m$$

such that the composite

$$S^m = RP^m/RP^{m-1} \xrightarrow{\ i\ } RP^{m+\rho(m)}/RP^{m-1} \xrightarrow{\ f\ } S^m$$

has degree 1. (Here, as in § 1, we have $m = (2a+1)2^b$, $b = c + 4d$ and $\rho(m) = 2^c + 8d$.)

We first remark that if $d = 0$, then the Steenrod squares suffice to contradict the existence of f (cf. [15]). In what follows, then, we may certainly assume that $m \equiv 0 \bmod 8$. This ensures that

$$\varphi\big(m + \rho(m), m\big) = b + 1 ,$$

where $\varphi(m, n)$ is the function introduced in § 7. According to Theorem 7.4, then, we have

$$\widetilde{K}_R(RP^{m+\rho(m)}/RP^{m-1}) = Z + Z_{2^{b+1}} ;$$

here the summands are generated by $\bar{\lambda}^{(g)}$ and $\lambda^{(g+1)}$, where $g = \frac{1}{2}m$. We know that $i^*\lambda^{(g+1)} = 0$ and $i^*\bar{\lambda}^{(g)}$ is a generator γ of $\widetilde{K}_R(S^m) = Z$. If we had a map f, we would have

$$f^*\gamma = \bar{\lambda}^{(g)} + N\lambda^{(g+1)}$$

for some integer N. From the equation

$$f^*\Psi_R^k\gamma = \Psi_R^k f^*\gamma$$

we obtain (using Corollary 5.2)

$$f^*(k^{m/2}\gamma) = \Psi_R^k(\bar{\lambda}^{(g)} + N\lambda^{(g+1)}) \ ;$$

that is (using Theorem 7.4),

$$k^{m/2}\bar{\lambda}^{(g)} + k^{m/2}N\lambda^{(g+1)} = k^{m/2}\bar{\lambda}^{(g)} + \tfrac{1}{2}(k^{m/2} - \varepsilon)\lambda^{(g+1)} + \varepsilon N\lambda^{(g+1)} \ ,$$

where $\varepsilon = 0$ or 1 according as k is even or odd. That is,

$$(N - \tfrac{1}{2})(k^{m/2} - \varepsilon)\lambda^{(g+1)} = 0 \ ,$$

or equivalently,

$$(N - \tfrac{1}{2})(k^{m/2} - \varepsilon) \equiv 0 \qquad \bmod 2^{b+1} \ .$$

It remains only to prove that for a suitable choice of k, we have

$$k^{m/2} - \varepsilon \equiv 2^{b+1} \qquad \bmod 2^{b+2} \ ;$$

this will establish the required contradiction. We take $k = 3$.

LEMMA 8.1. *If* $n = (2a + 1)2^f$, *then* $3^n - 1 \equiv 2^{f+2} \bmod 2^{f+3}$.

(Note that since $n = \frac{1}{2}m$ or $m = 2n$, we have $b = f + 1$.)

PROOF. We first note that since $3^2 \equiv 1 \bmod 8$, we have $3^{2n} \equiv 1 \bmod 8$ and $3^{2n} + 1 \equiv 2 \bmod 8$. We now prove by induction over f that

$$3^{(2^f)} - 1 \equiv 2^{f+2} \qquad \bmod 2^{f+4} \qquad\qquad (\text{for} \quad f \geqq 1) \ .$$

For $f = 1$ the result is true, since $3^2 - 1 = 8$. Suppose the result true for some value of f. Then we have

$$\begin{aligned} 3^{(2^{f+1})} - 1 &= (3^{2^f} - 1)(3^{2^f} + 1) \\ &= (2^{f+2} + x2^{f+4})(2 + y2^3) \\ &\equiv 2^{f+3} \qquad \bmod 2^{f+5} \ . \end{aligned}$$

This completes the induction.

We now note that, since

$$3^{2^{f+1}} \equiv 1 \qquad \bmod 2^{f+3} \ ,$$

we have

$$3^{(2a)2^f} \equiv 1 \qquad \bmod 2^{f+3} \ ,$$

and

$$3^{(2a+1)2^f} - 1 \equiv 3^{2^f} - 1 \qquad \bmod 2^{f+3}$$
$$\equiv 2^{f+2} \qquad \bmod 2^{f+3} .$$

This establishes Lemma 8.1; the proof of Theorem 1.2 is thus completed.

9. Proofs of Theorem 1.1 and Corollary 1.3

We begin with Theorem 1.1.

Suppose, for a contradiction, that there were some n for which S^{n-1} admits $\rho(n)$ linearly independent vector fields. Then it is not hard to see that for each integer p, the sphere S^{pn-1} admits at least $\rho(n)$ linearly independent vector fields; this and more is proved by James [13, Corollary 1.4]. If p is sufficiently large then the appropriate Stiefel manifold $V_{pn,\rho(n)+1}$ may be approximated by a truncated projective space, which in James's notation is called $Q_{pn,\rho(n)+1}$ [12]. From the cross-section in the Stiefel manifold, we deduce that the complex $Q_{pn,\rho(n)+1}$ is reducible, at least for $pn \geqq 2(\rho(n) + 1)$ [12, Theorem 8.2, p. 131]. According to Atiyah, $Q_{pn,\rho(n)+1}$ is S-dual to $P_{\rho(n)+1-pn,\rho(n)+1}$, and therefore the latter object is S-co-reducible (see [1, p. 299 and Theorem 6.1, p. 307]). The latter object, however, is somewhat fictitious if $\rho(n) + 1 - pn$ is negative (which is generally so); one has to interpret it as $P_{\rho(n)+1-pn+qr,\rho(n)+1}$, where r is an integer arising from Atiyah's work but not explicitly determined by him, and q is an integer sufficient to make $qr - pn$ positive (see [1, p. 307, second footnote]). In our notation $P_{\rho(n)+1-pn+qr,\rho(n)+1}$ becomes

$$RP^{qr-pn+\rho(n)}/RP^{qr-pn-1} = X , \quad \text{say} .$$

If q is chosen large enough (the precise condition being $qr \geqq pn + \rho(n) + 3$) we enter the domain of stable homotopy theory, and the complex X is S-co-reducible if and only if it is co-reducible. It remains to show how this contradicts Theorem 1.2. We may suppose that p is odd, and (by choice of q if necessary) that qr is divisible by $2n$. If we set $m = qr - pn$, we see that m is an odd multiple of n, so that $\rho(m) = \rho(n)$. We now have

$$X = RP^{m+\rho(m)}/RP^{m-1}$$

with X co-reducible. This contradiction establishes Theorem 1.1.

We turn now to the proof of Corollary 1.3. The affirmative part of the result is due to James [13, Theorem 3.1, p. 819]. Given Theorem 1.1, the negative result follows from the same theorem of James, provided we have $n - 1 > 2\rho(n)$. The only possible exceptions to this are $n = 1, 2, 3, 4, 8$ and 16. In the first five cases the result is trivially true; and in the case $n = 16$, it follows from the work of Toda, as has been remarked by

James [13, p. 819]. This completes the proof.

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