

On Levi's Problem and the Imbedding of Real-Analytic Manifolds

Author(s): Hans Grauert

Source: *Annals of Mathematics*, Vol. 68, No. 2 (Sep., 1958), pp. 460-472

Published by: Mathematics Department, Princeton University

Stable URL: <https://www.jstor.org/stable/1970257>

Accessed: 13-10-2025 13:37 UTC

JSTOR is a not-for-profit service that helps scholars, researchers, and students discover, use, and build upon a wide range of content in a trusted digital archive. We use information technology and tools to increase productivity and facilitate new forms of scholarship. For more information about JSTOR, please contact support@jstor.org.

Your use of the JSTOR archive indicates your acceptance of the Terms & Conditions of Use, available at <https://about.jstor.org/terms>



JSTOR

Mathematics Department, Princeton University is collaborating with JSTOR to digitize, preserve and extend access to *Annals of Mathematics*

ON LEVI'S PROBLEM AND THE IMBEDDING OF REAL-ANALYTIC MANIFOLDS

BY HANS GRAUERT

(Received March 19, 1958)

Introduction

In 1911 E. E. Levi [18] showed that the boundary of a domain of holomorphy is not arbitrary. It satisfies certain condition of convexity and therefore is called pseudoconvex.¹ The pseudoconvexity is a local property. To prove that a domain with twice differentiable boundary is pseudoconvex it is only necessary to verify that some differential inequalities are satisfied (see [3]).

For more than forty years it was an open problem of the theory of several complex variables whether the Levi conditions are sufficient for the domains of holomorphy. At first the problem was solved for special domains. After refuting a counter-example of Blumenthal, H. Behnke proved that Levi's conjecture is true for (complex) 2-dimensional circular domains [1]. The first general result, however, was not obtained until 1942 by K. Oka [22]. Oka showed: *each pseudoconvex domain G of the 2-dimensional complex number space C^2 is a domain of holomorphy.* The problem was solved for the case of dimension $n > 2$ by K. Oka [23], H. Bremermann [5] and F. Norguet [21] in 1954. K. Oka [22] even proved that every unbranched (not necessarily schlicht) pseudoconvex domain over the n -dimensional complex number space C^n is a (holomorphically convex) domain of holomorphy. For branched domains and domains in the closed C^n , that is, in the n -dimensional complex projective space P^n , the problem is still unsolved.²

Using plurisubharmonic functions it can easily be proved that each pseudoconvex domain $G \subset C^n$ can be exhausted by strongly pseudoconvex domains $G_\nu \subset \subset G$ (see [23]). By a theorem of H. Behnke and K. Stein [2], the limit of domains of holomorphy is again a domain of holomorphy. So Levi's conjecture has to be verified only for strongly pseudoconvex domains (for definitions see § 1).

In this short paper relatively compact, *strongly* pseudoconvex subdomains G of complex manifolds \mathfrak{M} are considered (without any assump-

¹ There are several definitions of pseudoconvexity. See [17].

² K. Oka has announced that the answer is also "yes" for these cases.

tion on \mathfrak{M}). It is proved that these domains are holomorphically convex (§ 2, Theorem 1). Because the domains of holomorphy $G \subset C^n$ are exactly the holomorphically convex domains, Theorem 1 solves Levi's problem for complex manifolds. It may be of interest that Theorem 1 is wrong for general pseudoconvex domains $G \subset \mathfrak{M}$ (see the example in [15]).

In § 2 conditions are also given in order that G is a Stein manifold (Theorem 2). In § 3, Theorem 2 is applied to real-analytic manifolds to solve a problem of H. Whitney. This problem arises from the following theorem of Whitney :

If \mathfrak{R} is a real-analytic manifold of dimension n , which has countable topology, then there exists³ a regularly imbedded n -dimensional real-analytic surface S in a $2n$ -dimensional euclidean space R^{2n} and an isomorphism $\alpha: \mathfrak{R} \longleftrightarrow S$ which is differentiable of class C^∞ .

The open question was : can α be chosen as a real-analytic mapping ? As C. B. Morrey [19] proved in September last year, the answer is yes, if \mathfrak{R} is compact. Morrey's proof for this statement uses the theory of partial differential equations and difficult estimates. In § 3 a short proof of the statement is given which, however, is based on deep results of the sheaf theory on Stein manifolds. Moreover, Whitney's conjecture is verified also for non-compact real-analytic manifolds with countable topology. An extension to real-analytic spaces is possible (for definitions see [9]) : Theorems 1 and 2 can be proved *mutatis mutandis* by the same method for complex spaces. This will be done in a subsequent paper by the author (see [15]).

1. Preliminaries

1.1. \mathfrak{M} will always denote a (connected) n -dimensional complex manifold and G will denote an open relatively compact subset of \mathfrak{M} which has *piece-wise twice differentiable boundary*. This means : to each boundary point $x_0 \in \partial G$ there exists a neighborhood T of x_0 and in U finitely many twice continuously differentiable, real-valued functions $\varphi_1, \dots, \varphi_k$ such that the exterior differentials $d\varphi_\nu$ do not vanish in U and $G \cap U = \{x \in U, \varphi_\nu(x) < 0, \nu = 1, \dots, k\}$. \mathfrak{M} induces a complex structure in G ; therefore G can be considered as a complex manifold. We state the following definitions :

(1) *Let M be a subset of a complex manifold \mathfrak{M} . Then the holomorphically convex envelope \hat{M} of M is the set : $\{x \in \mathfrak{M}, |f(x)| \leq \sup |f(M)| \text{ for all holomorphic functions } f \text{ in } \mathfrak{M}\}$.*

³ H. Whitney has proved this theorem *mutatis mutandis* also for differentiable manifolds.

\hat{M} is always a closed subset of \mathfrak{M} .

(2) \mathfrak{M} is holomorphically convex if, for every compact subset $M \subset \mathfrak{M}$, the envelope \hat{M} is compact.

(3) \mathfrak{M} is K -complete⁴ if, to each point $x_0 \in \mathfrak{M}$, there exist finitely many holomorphic functions f_1, \dots, f_k in \mathfrak{M} such that x_0 is an isolated point of the set $A = \{x \in \mathfrak{M}, f_\nu(x) = f_\nu(x_0), \nu = 1, \dots, k\}$.

“ K -complete” means that there are many holomorphic functions on \mathfrak{M} . A compact complex manifold is never K -complete, but every Riemann domain over the n -dimensional complex number space C^n is K -complete. The number k which appears in (3) cannot be chosen smaller than n .

(4) \mathfrak{M} is a Stein manifold if \mathfrak{M} is holomorphically convex and K -complete.

1.2. In [13] it is proved that the definition (4) is equivalent to the old definition of Stein manifolds given by K. Stein and H. Cartan [7]. Therefore we have the following statements (\mathfrak{M} is always a Stein manifold):

(α) \mathfrak{M} has countable topology.

(β) Let $x, y \in \mathfrak{M}$ be distinct points. Then there is always a holomorphic function f in \mathfrak{M} such that $f(x) \neq f(y)$.

(γ) To each point $x_0 \in \mathfrak{M}$ there exist holomorphic functions f_1, \dots, f_n in \mathfrak{M} such that the determinant $\partial(f_1, \dots, f_n)/\partial(z_1, \dots, z_n)$ does not vanish in x_0 (where z_1, \dots, z_n are holomorphic coordinates⁵ in a neighborhood of x_0).

Let us denote by B an arbitrary open subset of \mathfrak{M} (or of a differentiable manifold \mathfrak{R}). We call a holomorphic (differentiable) mapping $\alpha: B \rightarrow C^k$ regular if α is one-to-one and if the functional matrix $(\partial\alpha/\partial\zeta_j)$, $\zeta = (z_1, \dots, z_n)$, (or $(\partial\alpha/\partial\zeta)$, $\zeta = (x_1, \dots, x_n)$), has the rank n in all points $b \in B$. Then $\alpha(B)$ is an n -dimensional analytic (differentiable) surface S in C^k . S has no “singularities”. From the statements (α)-(γ) follows immediately:

PROPOSITION 1. If \mathfrak{M} is a Stein manifold and B , relatively compact, is contained in \mathfrak{M} then there exist a natural number k and a regular holomorphic mapping $\alpha: B \rightarrow C^k$.

Far beyond this R. Remmert [24] has proved:

IMBEDDING THEOREM. Let \mathfrak{M} be an n -dimensional Stein manifold. Then there exist a number $k = k(n)$ and a regular, proper,⁶ holomorphic mapping $\alpha: \mathfrak{M} \rightarrow C^k$.

Because α is proper and regular, $\alpha(\mathfrak{M})$ is an n -dimensional (closed) ana-

⁴ In [3] the definition of K -completeness differs somewhat from (3). R. Remmert has proved that (3) is equivalent to the definition in [13].

⁵ By holomorphic (real-analytic, etc.) coordinates, we always understand coordinates which belong to the complex (real-analytic) structure (of \mathfrak{M}).

⁶ “Proper” in the sense of N. Bourbaki.

lytic subset of C^k . $\alpha(\mathfrak{M})$ contains only regular (= gewöhnliche) points.⁷

1.3. Let $G \subset \subset \mathfrak{M}$ denote the domain of Section 1.1. We define :

(5) G is pseudoconvex if, to each point $x_0 \in \partial G$, there exist a neighborhood $U(x_0) \subset \mathfrak{M}$ and in U finitely many twice continuously differentiable, real valued functions $\varphi_1, \dots, \varphi_k$, such that $d\varphi_\kappa \neq 0$, $G \cap U = \{x \in U, \varphi_\kappa(x) < 0, \kappa = 1, \dots, k\}$, $L(\varphi_\kappa) = \sum (\partial^2 \varphi_\kappa / \partial z_\nu \partial \bar{z}_\mu) a_\nu \bar{a}_\mu \leq 0$ on $\partial G \cap U$ for all vectors (a_1, \dots, a_n) which satisfy $\sum (\partial \varphi_\kappa / \partial z_\nu) a_\nu = 0, (\kappa = 1, \dots, k)$. G is called strongly pseudoconvex, if for all $x_0 \in \partial G$, we can find the neighborhood $U(x_0)$ and the functions $\varphi_1, \dots, \varphi_k$ such that $L(\varphi_\kappa) > 0$ in U for all $(a_1, \dots, a_n) \neq 0$ with $\sum (\partial \varphi_\kappa / \partial z_\nu) a_\nu = 0 (\kappa = 1, \dots, k)$.

It is known that the property $L(\varphi_\kappa) \geq 0$ or $L(\varphi_\kappa) > 0$ does not depend on the choice of the functions $\varphi_1, \dots, \varphi_k$. It is a local property of the boundary ∂G itself. If G is holomorphically convex, then G is also pseudoconvex (see[3]).

Levi and Krzoska have proved :⁸

(δ) Let G be strongly pseudoconvex. Then, if $x \in \partial G$ is an arbitrary point and $U(x_0)$ a sufficiently small holomorphically convex neighborhood, $U(x_0) \cap G$ is a domain of holomorphy. If $U \cap G = \{\varphi_\kappa(x) < 0, \kappa = 1, \dots, k\}$ with φ_κ twice continuously differentiable in (a neighborhood of) \bar{U} , $d\varphi_\kappa \neq 0$, and if $h_\kappa(x)$ are twice differentiable functions in \bar{U} whose values and first and second derivatives are sufficiently small, then $U \cap \{\varphi_\kappa - h_\kappa < 0, \kappa = 1, \dots, k\}$ is again a domain of holomorphy.

We can assume that $U \cap G$ is contained in a local coordinate system of \mathfrak{M} . Because every domain $G^* \subset C^n$ is a Stein manifold if and only if G^* is a domain of holomorphy, we see that $U \cap G$ is a Stein manifold. Another result of Levi and Krzoska is :

(ε) Let G be strongly pseudoconvex. Then, to each point $x_0 \in \partial G$, there exist a neighborhood $U(x_0)$ and holomorphic function f in (a neighborhood of) \bar{U} with $df \neq 0$ and $\{x \in \bar{U}, f(x) = 0\} \cap \bar{G} = x_0$.

(ε) is used to prove (δ).

1.4. A twice continuously differentiable real valued function $h(x)$ in \mathfrak{M} is called plurisubharmonic if the form $L(h) = \sum (\partial^2 h / \partial z_\nu \partial \bar{z}_\mu) a_\nu \bar{a}_\mu$ is positive semidefinite everywhere in \mathfrak{M} . If $L(h)$ is positive definite everywhere in \mathfrak{M} , then h is said to be a strongly plurisubharmonic function. The following statement can easily be verified :

(*) Let h be a strongly plurisubharmonic function in \mathfrak{M} , $A \subset \mathfrak{M}$ an analytic set of dimension $d > 0$. Then the restriction $h|_A$ is not constant.

⁷ The theory of analytic sets is developed in [25]. See [25] for definitions.

⁸ See [3]. In [3] only the first statement of (δ) is proved. But the second statement follows by the same argument.

The concept of a twice differentiable plurisubharmonic function is a special case of the concept of a *general plurisubharmonic function*. General plurisubharmonic functions are not differentiable but only upper semi-continuous. They can also be defined on complex spaces.⁹ Plurisubharmonic functions always satisfy the “maximum principle”: if they are not constant, they do not take their maximum in the interior of their existence domains. Every plurisubharmonic function on a connected compact complex space is therefore constant.

Now let $h(x)$ be a strongly plurisubharmonic twice differentiable function in \mathfrak{M} , $A \subset \mathfrak{M}$ a compact analytic set of dimension $d > 0$. We take a connected component¹⁰ $A^* \subset A$, which has dimension d . A^* is a connected compact complex space (see [12]); the restriction $h^* = h|A$ is again a plurisubharmonic function (see [11, p. 180]). Because A^* is compact, h^* is constant in contradiction to (*). So we have proved the following:

PROPOSITION 2. *If a strongly plurisubharmonic function exists on \mathfrak{M} , then \mathfrak{M} contains no compact analytic subsets A of dimension $d \geq 1$.*

2. Levi's problem

In Levi's problem one asks: is every pseudoconvex domain $G \subset C^n$ a domain of holomorphy? Because the domains of holomorphy are exactly the holomorphically convex domains, one can also ask: is every pseudoconvex domain $G \subset C^n$ holomorphically convex? In this form the problem can be carried over to complex manifolds. We shall prove:

THEOREM 1. *If $G \subset \subset \mathfrak{M}$ is strongly pseudoconvex, then G is holomorphically convex.*

The proof is divided into several steps.

2.1. Let Ω be the sheaf of germs of holomorphic functions on \mathfrak{M} , $G' \supset \supset G$ an open subset of \mathfrak{M} . The restriction generates a homomorphism r^* of the cohomology groups $H^\nu(G', \Omega)$ into $H^\nu(G, \Omega)$ ($\nu = 0, 1, 2, \dots$).

PROPOSITION 3. *If G is strongly pseudoconvex and G' , with $G \subset \subset G' \subset \subset \mathfrak{M}$, is sufficiently near to G , then r^* is surjective for $\nu > 0$.*

PROOF. If $\mathfrak{U} = \{U_i, i = 0, \dots, i\}$ is a finite open covering of \overline{G} , we

⁹ In [11] plurisubharmonic functions are defined on normal complex spaces. Complex spaces, which occur here, are general complex spaces in the sense of Cartan-Serre (β -spaces; see [12]). The definition of plurisubharmonic functions can be carried over so that the maximum principle is valid.

¹⁰ For the partition of an analytic set A in connected and irreducible components see [25]. The irreducible (and also the connected components) form a closed, locally finite covering of A .

denote by $*U$ the set $U_0 - \overline{\cup_{i \neq 0} U_i}$. Because \overline{G} is compact and the statement (δ) of § 1 is valid, there exist finitely many Stein coverings (=coverings whose elements are Stein manifolds) $\mathfrak{U}^{(\kappa)} = \{U_i^{(\kappa)}, i = 0, \dots, i_\kappa\}$, $\kappa = 1, \dots, k$, of \overline{G} , such that $\cup_\kappa *U^{(\kappa)} \supset \partial G$ and $W_i^{(\kappa)} = U_i^{(\kappa)} \cap G$ is a Stein manifold. Furthermore we can assume that all the $U_i^{(\kappa)}$ are so small that $U_i^{(\kappa)} \cap \partial G$ can be given by twice differentiable equations $\varphi_\nu = 0$ with φ_ν defined in $\overline{U_i^{(\kappa)}}$ and that the statement (δ) is valid. From this it follows easily that there exists a sequence $G_\nu \subset \subset \mathfrak{M}$, $\nu = 0, 1, \dots, k$, of strongly pseudoconvex domains with twice differentiable boundary, such that $G_0 = G$, $G_\kappa \subset G_{\kappa+1}$, $G_{\kappa+1} \cap U_i^{(\kappa+1)} = G_\kappa \cap U_i^{(\kappa+1)}$ for $i \neq 0$, $\partial G_\kappa \cap \partial G_{\kappa+1} \cap *U^{(\kappa+1)} = 0$, $G_\nu \subset \cup_i U_i^{(\kappa)}$ and $W_{i\nu}^{(\kappa)} = G_\nu \cap U_i^{(\kappa)}$ is always a Stein manifold (see statement (δ)).

Let us denote by $\mathfrak{W}_\nu^{(\kappa)}$ the covering $\{W_{i\nu}^{(\kappa)}, i = 0, \dots, i_\kappa\}$ of G_ν . We have for the cycles: $Z^\nu(\mathfrak{W}_\kappa^{(\kappa+1)}, \Omega) = Z^\nu(\mathfrak{W}_{\kappa+1}^{(\kappa+1)}, \Omega)$ if $\nu > 0$. Therefore the restriction generates an epimorphism $H^\nu(\mathfrak{W}_{\kappa+1}^{(\kappa+1)}, \Omega) \rightarrow H^\nu(\mathfrak{W}_\kappa^{(\kappa+1)}, \Omega)$. By the Leray theorem $H^\nu(\mathfrak{W}_\mu^{(\kappa)}, \Omega) = H^\nu(G_\mu, \Omega)$. Therefore we obtain by restriction an epimorphism $H^\nu(G_{\kappa+1}, \Omega) \rightarrow H^\nu(G_\kappa, \Omega)$. This proves that $r^*: H^\nu(G_\kappa, \Omega) \rightarrow H^\nu(G, \Omega)$, $\nu > 0$, is surjective. If G' is a domain with $G \subset G' \subset G_\kappa$, then $r^*: H^\nu(G', \Omega) \rightarrow H^\nu(G, \Omega)$ also is surjective (if $\nu > 0$). Since $G \subset \subset G_\kappa$, Proposition 3 is proved.

2.2. Let $\{U_i, i = 0, 1, \dots, k\}$ and $\{U'_i, i = 0, 1, \dots, k\}$ be two Stein coverings of \overline{G} with $U_i \subset \subset U'_i$ and $W_i = U_i \cap G$, $U'_i \cap G =$ Stein manifolds. If $\{U'_i\}$ is sufficiently fine, then as in Section 2.1, we can find a strongly pseudoconvex domain $G' = G_{k'}$ such that $G \subset \subset G', G' \subset \cup U'_i$ and $W'_i = G' \cap U'_i$ are Stein manifolds. We denote the Stein covering $\{W_i\}$ of G by \mathfrak{W} and the Stein covering $\{W'_i\}$ of G' by \mathfrak{W}' and set

$$W_{i_0 \dots i_s} = W_{i_0} \cap W_{i_1} \cap \dots \cap W_{i_s}, W'_{i_0 \dots i_s} = W'_{i_0} \cap W'_{i_1} \cap \dots \cap W'_{i_s}.$$

$\Gamma(W_{i_0 \dots i_s}, \Omega) = H^0(W_{i_0 \dots i_s}, \Omega)$ and $\Gamma(W'_{i_0 \dots i_s}, \Omega)$ can be considered as the complex vector spaces of holomorphic functions on $W_{i_0 \dots i_s}$ (or on $W'_{i_0 \dots i_s}$). Provided with the topology of compact convergence they are Fréchet spaces (with countable topology).¹¹ Therefore the spaces of cochains $C^s(\mathfrak{W}, \Omega) = \bigoplus \Gamma(W_{i_0 \dots i_s}, \Omega)$, and the space of cocycles $Z^s(\mathfrak{W}, \Omega)$, $Z^s(\mathfrak{W}', \Omega)$ are Fréchet spaces. We set $H_1 = Z^s(\mathfrak{W}', \Omega) \oplus C^{s-1}(\mathfrak{W}, \Omega)$, $H_2 = Z^s(\mathfrak{W}, \Omega)$, $s > 0$, and define (continuous and linear) mappings $u: H_1 \rightarrow H_2$ and $v: H_1 \rightarrow H_2$ by setting $u: z \oplus c \rightarrow r(z) + \delta(c)$ and $v: z \oplus c \rightarrow -r(z)$. Here $z \in Z^s(\mathfrak{W}', \Omega)$, $c \in C^{s-1}(\mathfrak{W}, \Omega)$, and r denotes the restriction, δ the coboundary operation. Since $H^s(\mathfrak{W}', \Omega) = H^s(G', \Omega)$,

¹¹ The concepts which occur here are taken from [4].

$H^s(\mathfrak{B}, \Omega) = H^s(G, \Omega)$, the mapping u is an epimorphism by Proposition 3.

Let F_1, F_2 be two Fréchet spaces, α a continuous linear mapping $F_1 \rightarrow F_2$. Then α is called completely continuous if a neighborhood $U(0)$, $0 \in F_1$, exists, such that $\alpha(U) \subset F_2$ is relatively compact. Since $W_{i_0 \dots i_s} \subset \subset W'_{i_0 \dots i_s}$, it follows from the theorem of Montel on normal families that the restriction generates a completely continuous mapping $\Gamma(W'_{i_0 \dots i_s}, \Omega) \rightarrow \Gamma(W_{i_0 \dots i_s}, \Omega)$. Therefore v is also completely continuous.

L. Schwartz has proved the following theorem:¹²

Let F_1, F_2 be two Fréchet spaces (with countable topology), $u : F_1 \rightarrow F_2$ a continuous epimorphism, $v : F_1 \rightarrow F_2$ a completely continuous linear mapping. Then the vector space $F_2/u + v(F_1)$ has finite dimension.

In our case $F_2/u + v(F_1) = H_2/u + v(H_1)$ is equal to $H^s(\mathfrak{B}, \Omega) \cong H^s(G, \Omega)$. So we have proved :

PROPOSITION 4. *If $G \subset \subset \mathfrak{M}$ is strongly pseudoconvex, then the complex vector spaces $H^\nu(G, \Omega)$, $\nu > 0$, have finite dimension.*

2.3. Now it is possible to prove Theorem 1. Let $x_0 \in \partial G$ be an arbitrary point, $U(x_0)$ a sufficiently small holomorphically convex neighborhood of x_0 . Then by (δ) the open set $U \cap G$ is a Stein manifold ; by (ϵ) a holomorphic function exists in \bar{U} , such that $df \neq 0$, $\bar{G} \cap S = x_0$, where $S = \{x \in \bar{U}, f(x) = 0\}$. As in Sections 1 and 2 we construct a strongly pseudoconvex domain G' with $G \subset \subset G'$, $W' = G' \cap U =$ Stein manifold, such that G' is so near to G that $S' = S \cap G' \cap U$ is a closed subset of G' . S' , as a closed analytic submanifold of the Stein manifold W' , is a Stein manifold.

S' can be regarded as a positive divisor in G' . Therefore to S' belongs a complex-analytic line bundle F' which has a canonical holomorphic cross-section h vanishing only on S' of the first order.¹³ Let F^k denote the tensor product $F \otimes \dots \otimes F$ (k factors) and \underline{F}^k the sheaf of germs of holomorphic sections of the complex-analytic line bundle F^k . We set $\underline{F}^0 = \Omega$, $F^k(S') =$ restriction of F^k to S' , $\underline{F}^k(S') =$ sheaf of germs of holomorphic sections of $F^k(S')$ over S' . We consider, as usual, cross-sections in \underline{F}^k (and $\underline{F}^k(S')$) as holomorphic cross-sections in F^k (or in $F^k(S')$).

The correspondence $s_x \rightarrow s_x \otimes h_x$ defines a sheaf homomorphism

¹² See [8], the corollary in Exposé XVI (by Serre). The ideas which occur in this proof are taken from Exposé XVII (by H. Cartan).

¹³ For definitions of complex line bundles and their relationship to divisors, see [16, pp. 40-53 and 110-113].

$\alpha_k: \underline{F}^k \rightarrow \underline{F}^{k+1}$. The quotient sheaf $\underline{F}^{k+1}/\alpha_k(\underline{F}^k)$ is the trivial extension $*\underline{F}^{k+1}$ of $\underline{F}^{k+1}(S')$ to G' . We have the exact sequence :

$$0 \rightarrow \underline{F}^k \rightarrow \underline{F}^{k+1} \rightarrow *\underline{F}^{k+1} \rightarrow 0$$

to which corresponds an exact cohomology sequence :

$$0 \rightarrow H^0(G', \underline{F}^k) \rightarrow H^0(G', \underline{F}^{k+1}) \rightarrow H^0(G', *\underline{F}^{k+1}) \rightarrow H^1(G', \underline{F}^k) \rightarrow H^1(G', \underline{F}^{k+1}) \rightarrow H^1(G', *\underline{F}^{k+1}) \rightarrow \dots$$

Because $H^v(G', *\underline{F}^{k+1}) = H^v(S', \underline{F}^{k+1}(S'))$ and $H^1(S', \underline{F}^{k+1}(S')) = 0$ (see [7, Théorème B]), we obtain the exact sequence :

$$H^0(G', \underline{F}^{k+1}) \rightarrow H^0(S', \underline{F}^{k+1}(S')) \xrightarrow{\beta_*} H^1(G', \underline{F}^k) \xrightarrow{\alpha_*} H^1(G', \underline{F}^{k+1}) \rightarrow 0.$$

It follows that : $d_{k+1} \leq d_k$, where $d_k = \dim H^1(G', \underline{F}^k)$. By Proposition 4, $d_0 = \dim H^1(G', \Omega)$ is finite. Therefore all the dimensions d_k are finite and, for $k > k_0$, we have $d_{k+1} = d_k$; thus α_* is an isomorphism, and the image of β_* is zero. We obtain the exact sequence :

$$H^0(G', \underline{F}^{k+1}) \xrightarrow{q_*} H^0(S', \underline{F}^{k+1}(S')) \longrightarrow 0.$$

Because q_* is nothing else but the restriction of the cross-sections of F^{k+1} over G' to cross-sections of F^{k+1} over S' , each holomorphic cross-section s of F^{k+1} over S' can be continued to a holomorphic cross-section \hat{s} of F^{k+1} over G' .

By [7, Théorème A], there exists a holomorphic cross-section s in F^{k+1} which does not vanish at x_0 . Let \hat{s} be its continuation to G' . F^{k+1} has a canonical holomorphic cross-section h^{k+1} which vanishes exactly on S' of order $k + 1$. The quotient $f = \hat{s}/h^{k+1}$ is a meromorphic function in G' . f has exactly S' for polar surface, is holomorphic in G and does not have at $x_0 \in S'$ a point of indeterminacy : $\lim_{x \rightarrow x_0} |f(x)| = +\infty$.

Because we can construct such a function f for each point $x_0 \in \partial G$, the axiom in Definition (2) of § 1 is satisfied : G is holomorphically convex.

2.4. We prove :

PROPOSITION 5. *Let \mathfrak{M} be a holomorphically convex complex manifold. Then \mathfrak{M} is K -complete if and only if \mathfrak{M} contains no compact analytic subsets of dimension greater than zero.*

PROOF. Let B be a compact analytic subset of \mathfrak{M} , $\dim B > 0$, and let B^* be a connected component of B , $\dim B^* > 0$. Furthermore, let, $x_0 \in B^*$ be an arbitrary point. Since every holomorphic function f is constant on

the compact complex space B^* , every set $A = \{x \in \mathfrak{M}, f_\nu(x) = f_\nu(x_0), \nu = 1, \dots, k\}$ contains B^* : \mathfrak{M} is not K -complete.

Now let \mathfrak{M} contain no compact analytic sets B , $\dim B > 0$. We denote by $x_0 \in \mathfrak{M}$ an arbitrary point and by M the holomorphically convex envelope $\{\hat{x}_0\}$. Since M is compact, we can find an open subset D with $M \subset \subset D \subset \subset \mathfrak{M}$. To each point $x \in \partial D$ there exists a holomorphic function f in \mathfrak{M} with $|f(x)| > \sup |f(x_0)|$. We can find finitely many holomorphic functions $f_\nu(x)$, $\nu = 1, \dots, k$, in \mathfrak{M} such that $|f_\nu(x)| > |f_\nu(x_0)|$ for at least one ν in each point $x \in \partial D$. Let A be the analytic set $\{x \in \mathfrak{M}, f_\nu(x) = f_\nu(x_0), \nu = 1, \dots, k\}$. Then the intersection $A^* = A \cap D$ is contained in D as a relatively compact subset. A^* is a compact analytic set and consists of isolated points, because $\dim A^* = 0$. Therefore x_0 is an isolated point of A^* and also of A : \mathfrak{M} is K -complete.

From Proposition 2 follows :

THEOREM 2. *Let $G \subset \subset \mathfrak{M}$ be a strongly pseudoconvex domain and $p(x)$ a strongly plurisubharmonic function in G . Then G is a Stein manifold.*

If G is strongly pseudoconvex and contains compact analytic subsets, one can prove that G is a modification of a holomorphically complete space. This will be shown in a subsequent paper by the author (see [15]).

3. Real-analytic manifolds

3.1. F. Bruhat and H. Whitney [6] have proved :

(η) *Let \mathfrak{R} be a real-analytic paracompact (connected) manifold of (real) dimension n . Then there exist a paracompact (complex) n -dimensional complex manifold \mathfrak{M} , a real-analytic (real) n -dimensional closed surface $\mathfrak{R}' \subset \mathfrak{M}$ and a bi-real-analytic mapping $\pi: \mathfrak{R} \rightarrow \mathfrak{R}'$. \mathfrak{M} has a covering with local coordinate patches $(U_\iota, z_1^{(\iota)}, \dots, z_n^{(\iota)})$, $\iota \in I$, such that $U_\iota \cap \mathfrak{R}' = \{z_\nu^{(\iota)} \in U_\iota, z_\nu^{(\iota)} = \text{real}, \nu = 1, \dots, n\}$, $\iota \in I$.*

Since \mathfrak{M} is paracompact and locally compact, we can assume that the covering $\{U_\iota, \iota \in I\}$ is locally finite and that all $U_\iota \subset \mathfrak{M}$ are contained relatively compact. The paracompact (connected) manifolds are exactly the manifolds with countable topology. Therefore we can also assume that the index set $I = \{0, 1, 2, \dots\}$. In general, if \mathfrak{M} has no countable topology, the statement (η) is not true. The Alexandroff line, which carries a natural real-analytic structure, provides a counter-example.¹⁴

3.2. We denote by $p(x)$ the function $- [(z_1^{(\iota)} - \bar{z}_1^{(\iota)})^2 + \dots + (z_n^{(\iota)} - \bar{z}_n^{(\iota)})^2]$. $p(x)$ is strongly plurisubharmonic in U_ι and $p(x) = 0, dp(x) = 0$, if

¹⁴ Lecture of H. Kneser in the conference on complex manifolds in August, 1953, in Oberwolfach, Germany.

$x \in \mathfrak{R}'$; $p_i(x) > 0$ if $x \in U_i - \mathfrak{R}'$.

We call a function $p(x)$, defined in an open subset $U \subset \mathfrak{M}$, a (strong) p -function if $p(x)$ is differentiable of class C^∞ and satisfies the following conditions: (1) $p(x) \geq 0$, (2) $p(x) = dp(x) = 0$ on $U \cap \mathfrak{R}'$, (3) $p(x)$ is (strongly) plurisubharmonic on $U \cap \mathfrak{R}'$ (i.e., satisfies the differential inequalities). As we saw, all the functions $p_i(x)$ are strong p -functions. If $p_1(x)$ and $p_2(x)$ are p -functions and $\alpha(x) > 0$ is a differentiable function of class C^∞ in U , then $\alpha(x) \cdot p_1(x)$, $p(x) = p_1(x) + p_2(x)$ are also p -functions. If $p_1(x)$ is a strong p -function, then $p(x)$ is also a strong p -function. From these properties we obtain immediately, using a partition of unity:

PROPOSITION 6. *Let $r(x)$ be a real twice continuously differentiable function in \mathfrak{M} . Then there exists a strong p -function $p(x)$ in \mathfrak{M} , such that $p(x) + r(x)$ is strongly plurisubharmonic on \mathfrak{R}' .*

3.3. Let $U = U(\mathfrak{R}')$ be an open neighborhood of \mathfrak{R}' . Because \mathfrak{M} is regular, we can find coverings $\{V_i, i = 0, 1, 2, 3, \dots\}$, $\{W_i, i = 0, 1, 2, 3, \dots\}$ of \mathfrak{M} such that $W_i \subset \subset V_i \subset \subset U$. Denote by d_i the distance of \overline{W}_i from ∂V_i with respect to the coordinates $z_1^{(i)}, \dots, z_n^{(i)}$. $\{V_i\}$ is locally finite. Hence there is a neighborhood $V = V(\mathfrak{R}') \subset U$, such that, if $x \in V_i \cap V$, $[(y_1^{(i)})^2 + \dots + (y_n^{(i)})^2]^{1/2} < 1/2d_i$ (if $x = (z_1^{(i)}, \dots, z_n^{(i)}) = (x_1^{(i)} + iy_1^{(i)}, \dots, x_n^{(i)} + iy_n^{(i)})$). Let $a = (a_1^{(i)}, \dots, a_n^{(i)})$ be an arbitrary point of $W_i \cap \mathfrak{R}'$. Then in $U_i \cap V$ the boundary of the cone $K_a^{(i)} = \{z^{(i)} \in V_i \cap V, \tilde{p}_a^{(i)}(x) = 2\sum_v |y_v^{(i)}|^2 - \sum_v |x_v^{(i)} - a_v^{(i)}|^2 > 0\}$ consists only of the points $\{z^{(i)} \in V_i \cap V, \tilde{p}_a^{(i)}(x) = 0\}$. $\tilde{p}_a^{(i)}(x)$ is strongly plurisubharmonic in $K_a^{(i)}$; by setting $p_a^{(i)}(x) = 0$ for $x \notin K_a^{(i)}$, $p_a^{(i)}(x) = \exp[-\tilde{p}_a^{(i)}(x)]$ for $x \in K_a^{(i)}$, we construct a p -function in $V(\mathfrak{R}')$ which is greater than 0 in $K_a^{(i)}$. There exists a sequence of points $a_v \in W_{i_v} \cap \mathfrak{R}'$, such that $\{K_{a_v}^{(i_v)}\}$ is locally finite and $\partial(V - \cup K_{(a_v)}^{(i_v)}) \cap \partial V = 0$, if V is properly chosen. Therefore, if $c_v > 0$ are sufficiently large real numbers, $p^*(x) = \sum c_v p_{a_v}^{(i_v)}$ is a p -function in $V(\mathfrak{R}')$ which is strongly plurisubharmonic wherever it is different from 0 and it has the following property: the intersection of the closure of the tube $T = \{x \in V, p^*(x) < 1\}$ with ∂V is empty. We have proved:

PROPOSITION 7. *Let $U = U(\mathfrak{R}')$ be an arbitrary open neighborhood. Then there exists a neighborhood $V = V(\mathfrak{R}') \subset U$ and in V a plurisubharmonic function $p^*(x)$, such that $p^*(x)$ is strongly plurisubharmonic wherever different from 0 and $\partial V \cap \overline{T} = 0$, where $T = \{x \in V, p^*(x) < 1\}$.*

3.4. Since \mathfrak{M} has countable topology, by a partition of unity we can construct a real function $r(x) \geq 0$ in \mathfrak{M} such that, for every real number

$K > 0$, the set $\{x \in \mathfrak{M}, r(x) < K\}$ is relatively compact in \mathfrak{M} . Let $p_1(x)$ be a p -function in the sense of Proposition 6. Then $p(x) = r(x) + p_1(x)$ is strongly plurisubharmonic in a neighborhood $U(\mathfrak{N})$. By Proposition 7 we can find a p -function $p_2(x)$ and a neighborhood $V = V(\mathfrak{N}) \subset U$, such that $\bar{T} \cap \partial V = 0$ where $T = \{x \in V, p_2(x) < 1\}$. We define the function

$$q(x) = p(x) + (1 - p_2(x))^{-1} = p(x) + 1 + p_2(x) + p_2^2(x) + \dots$$

Then $q(x)$ is strongly plurisubharmonic, every set $T_\rho = \{x \in T, q(x) < \rho\}$ is contained as a relatively compact subset in T and \mathfrak{M} . We have $\cup T_\rho = T, T_{\rho_1} \subset \subset T_{\rho_2}$ if $\rho_1 < \rho_2$. By [20], there exists a dense set D of real numbers $\rho, \inf q(x) < \rho$, such that for $\rho \in D$ the tube T_ρ has twice differentiable (smooth) boundary. By Theorem 2 all these T_ρ are Stein manifolds. In [14] the following result is proved by a generalized approximation theorem of Runge (see also [10]):

Let T be a complex manifold, D a dense set of real numbers $\rho, \rho > \rho_0, T_\rho$ a continuous family of relatively compact subdomains of T , such that $T_{\rho_1} \subset \subset T_{\rho_2}$ if $\rho_1 < \rho_2, \cup T_\rho = T$, and T_ρ is a Stein manifold if $\rho \in D$. Then T itself is a Stein manifold.

So our tube T is a Stein manifold. By the imbedding theorem of Remmert there exists a regular proper holomorphic mapping ψ of T into a complex number space C^k of sufficiently high dimension. Thus $\psi \circ \pi$ is a real-analytic, regular and proper imbedding of \mathfrak{R} into $C^k = R^{2k}$.

THEOREM 3. *Let \mathfrak{R} be a real-analytic n -dimensional manifold with countable topology. Then there exists a real-analytic, regular and proper mapping ψ of \mathfrak{R} into a euclidean space R^k of sufficiently high dimension.¹⁵*

3.5. Let us denote by G_ν a sequence of domains, which are contained as relatively compact subsets in the euclidean space R^k such that $G_\nu \subset \subset G_{\nu+1}, \cup G_\nu = R^k$. If c_ν is any sequence of positive numbers, $r(x)$ an s -times continuously differentiable real function in R^k , then there exists a real-analytic function $y(x) = \sum a_{\nu_1 \dots \nu_k} x^{\nu_1} \dots \cdot x^{\nu_k}$ such that $\sup_{x \in G_{\nu+1} - G_\nu} |r(x) - y(x)| < c_\nu$ and the same inequalities hold for the derivatives up to order s if $s < \infty$ (= Weierstrass approximation theorem).¹⁶ If $\psi: \mathfrak{R} \rightarrow R^k$ is a real-analytic, regular, proper imbedding of a real-analytic manifold \mathfrak{R} , then by a partition of unity it can easily be proved that, to every s -times continuously differentiable function $r(x)$ in \mathfrak{R} , there exists an s -times continuously differentiable function $\hat{r}(x)$ in R^k such that $r(x) =$

¹⁵ In the case that \mathfrak{R} is compact, it is not necessary to use the imbedding theorem of Remmert and the limit theorem for Stein manifolds. Proposition 1 is enough.

¹⁶ This statement is a stronger form of the Weierstrass approximation theorem.

$\hat{r} \circ \phi(x)$. By the Weierstrass approximation theorem we obtain :

PROPOSITION 8. *Let \mathfrak{R} be a real-analytic manifold with countable topology, $\{U_i\}$, $\{V_i\}$ open locally finite coverings of \mathfrak{R} , such that $V_i \subset \subset U_i$, and let ${}^{(i)}x_1, \dots, {}^{(i)}x_n$ be real-analytic coordinates in U_i . Then, if $r(x)$ is a real, s -times continuously differentiable function in \mathfrak{R} and c_i is a sequence of positive numbers, there exists a real-analytic function $y(x)$ in \mathfrak{R} which satisfies the inequalities :*

$$|\partial^\sigma(y(x) - r(x)) / \partial^{(\nu_1)}x_1 \dots \partial^{(\nu_n)}x_n| < c_i \quad \text{in } V_i$$

(where $0 \leq \sigma = \nu_1 + \dots + \nu_n \leq s$, $s < \infty$).

H. Whitney [26] has proved : every n -dimensional real-analytic manifold with countable topology can be differentiably, regularly, locally-properly¹⁷ imbedded in the euclidean space R^{2n} . By Proposition 2 we obtain :

THEOREM 4. *Let \mathfrak{R} be a real-analytic n -dimensional manifold with countable topology, $\phi^* : \mathfrak{R} \rightarrow R^k$ a continuously differentiable, regular, locally proper (or proper) mapping. Then there exists a real-analytic, regular, locally proper (or proper) mapping $\psi : \mathfrak{R} \rightarrow R^k$. If $k \geq 2n$, a locally proper mapping ψ exists even if we have not assumed a mapping ϕ^* .*

THE INSTITUTE FOR ADVANCED STUDY

BIBLIOGRAPHY

1. H. BEHNKE, *Über analytische Funktionen mehrerer Veränderlicher, II: Natürliche Grenzen*, Abh. Hamburg, 5 (1927), 290-312.
2. ——— and K. STEIN, *Konvergente Folgen nichtschlichter Regularitätsbereiche*, Ann. Mat. pura. appl., 28 (1949), 317-326.
3. ——— and P. THULLEN, *Theorie der Funktionen mehrerer komplexer Veränderlichen*, Erg. Math., 3, 1934.
4. N. BOURBAKI, *Espaces vectoriels topologiques*, Paris.
5. H. BREMERMAN, *Über die Äquivalenz der pseudokonvexen Gebiete und der Holomorphiegebiete im Raum von n komplexen Veränderlichen*, Math. Ann., 128 (1954), 63-91.
6. F. BRUHAT and H. WHITNEY, This paper on real-analytic manifolds is in preparation. Probably to appear in Comment. Helvet.
7. H. CARTAN, *Variétés analytiques complexes et cohomologie*, Coll. sur. les fonct. pls. variables, Bruxelles, 1953.
8. ———, Séminaire E.N.S. 1953/54, Exposés XVI and XVII (multigraphed).
9. ———, *Variétés analytiques réelles et variétés analytiques complexes*, Bull. Soc. Math. France, 85 (1957), 77-99.
10. F. DOCQUIER, Dissertation, Münster, 1958.
11. H. GRAUERT and R. REMMERT, *Plurisubharmonische Funktionen in komplexen Räu-*

¹⁷ Here the concept "locally proper" is weaker than proper. In the papers of H. Whitney it is called "proper". See Ann. of Math., 45, (1944), p. 292.

- men, Math. Zeit., 65 (1956) 175–194.
12. ———, *Komplexe Räume*, to appear, Math. Ann., 1958.
 13. H. GRAUERT, *Charakterisierung der holomorph vollständigen komplexen Räume*, Math. Ann., 129 (1955), 233–259.
 14. ———, *Levisches Problem und Rungescher Satz für Teilgebiete Steinscher Mannigfaltigkeiten*, to appear, Math. Ann.
 15. ———, *Über das Levische Problem auf komplexen Räumen*, to appear.
 16. F. HIRZEBRUCH, *Neue topologische Methoden in der algebraischen Geometrie*, Erg. Math., 9, 1956.
 17. S. HITOTUMATU, *On some conjectures concerning pseudoconvex domains*, J. Math. Soc. Japan, 6, (1954), 173–195.
 18. E. E. LEVI, *Studii sui punti singolari essenziali delle funzioni anal. di due 0 più var. compl.*, Ann. Mat. pura appl., 3 (1910), 17.
 19. C. B. MORREY, *The analytic embedding of abstract real-analytic manifolds*, Multi-graphed January, 1958.
 20. A. P. MORSE, *The behavior of a function on its critical set*, Ann. of Math., 40 (1939), 62–70.
 21. F. NORGUET, *Sur les domaines d'holomorphie des fonctions uniformes de plusieurs variables complexes*, Bull. Soc. Math. France, 82 (1954), 137–159.
 22. K. OKA, *Sur les fonctions analytiques de plusieurs variables. VI. Domaines pseudoconvexes*, Tôhoku Math. J. II, Ser. 49 (1942), 19–52.
 23. ———, *Sur les fonctions analytiques de plusieurs variables. IX. Domaines finis sans point critique intérieur*, Jap. J. Math., 23 (1953), 97–155.
 24. R. REMMERT, *Einbettung Steinscher Mannigfaltigkeiten und holomorph-vollständiger komplexer Räume*, to appear, Math. Ann. See also R. REMMERT. Habilitationsschrift, Münster, 1957.
 25. ——— and K. STEIN, *Über die wesentlichen Singularitäten analytischer Mengen*, Math. Ann., 126 (1953), 263–306.
 26. H. WHITNEY, *The self-intersections of a smooth n -manifold in $2n$ -space*, Ann. of Math. 45 (1944), 220–246.